Superposition Principle:

(1)

$$u \longrightarrow \boxed{\text{System}} \longrightarrow y$$

If: $u := u_a(t) \longrightarrow y := y_a(t)$

and: $u := u_b(t) \longrightarrow y := y_b(t)$

$$\Longrightarrow u := \alpha u_a(t) + \beta u_b(t) \longrightarrow y := \alpha \cdot y_a(t) + \beta \cdot y_b(t)$$

Warning:

System: $y = a \cdot u + b$

We try the superposition principl

$$y_a(t) = a \cdot u_a(t) + b$$

$$y_b(t) = a \cdot u_b(t) + b$$

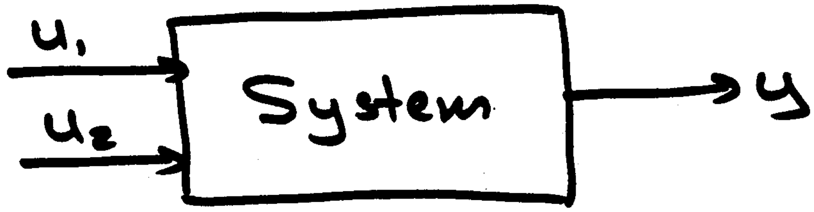
$$y_{a+b}(t) = a\left[u_a(t) + u_b(t)\right] + b$$

but: $y_a(t) + y_b(t) = a\left[u_a(t) + u_b(t)\right] + 2b$

$$\Rightarrow y_a(t) + y_b(t) \neq y_{a+b}(t)$$

$\Rightarrow$ System is **not** linear

(2)



If: $u_1 := u_a(t) \cap u_2 := \phi$

$\longrightarrow y := y_a(t)$

and: $u_1 := \phi \cap u_2 := u_b(t)$

$\longrightarrow y := y_b(t)$

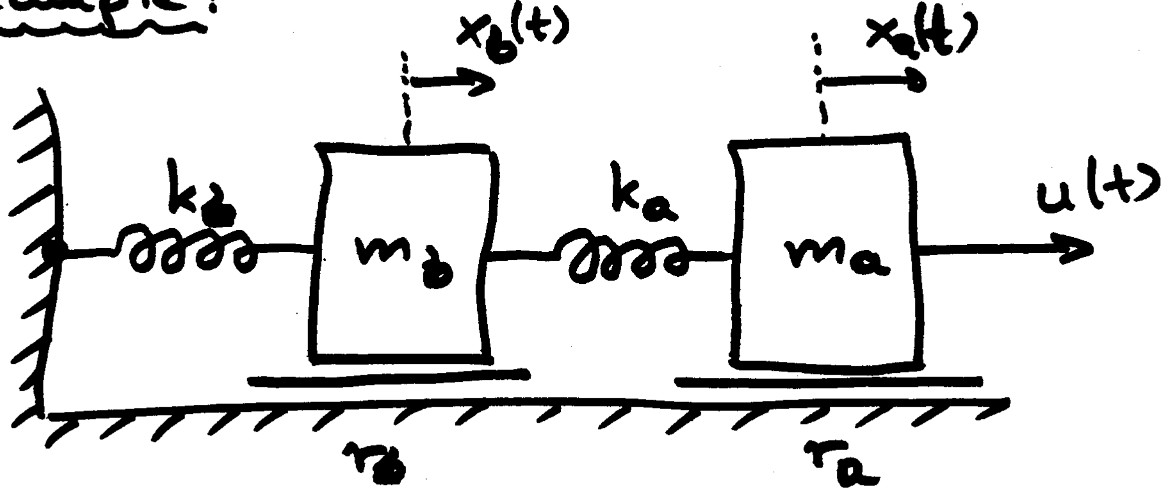
$\Longrightarrow u_1 := u_a(t) \cap u_2 := u_b(t)$

$\longrightarrow y := y_a(t) + y_b(t)$

This second facet shows that the superposition principle is also valid across several channels. This feature is often used in this class.

State-space Representation:

Example:



We "normalize" the position quantitie $x_a(t)$ and $x_b(t)$ such that in the steady-state ($u(t) = \emptyset$, $\dot{x}_a(t) = \dot{x}_b(t) = \emptyset$) $\Longrightarrow x_a(t) \equiv x_b(t) \equiv \emptyset$.

Newton's Law:

$$
\left| \begin{array}{l} m_a a_a = u(t) - r_a v_a - k_a(x_a - x_b) \\ m_b a_b = k_a(x_a - x_b) - r_b v_b - k_b x_b \end{array} \right|
$$

where:

$$
v_i = \frac{dx_i}{dt} = \dot{x}_i; \quad a_i = \frac{dv_i}{dt} = \frac{d^2 x_i}{dt^2} = \ddot{x}_i
$$

We eliminate the v_i and a_i :

$$\left| \begin{array}{l} m_a \ddot{x}_a = u(t) - r_a \dot{x}_a - k_a (x_a - x_b) \\ m_b \ddot{x}_b = k_a (x_a - x_b) - r_b \dot{x}_b - k_b x_b \end{array} \right|$$

Recipe: We solve each differential equation for the highest derivative :

$$x^{(n)} = f(x, \dot{x}, \ddot{x}, \dots, x^{(n-1)})$$

and use x and its first $(n-1)$ derivatives as "state variables" :

$$x_1 = x \; ; \; x_2 = \dot{x} \; ; \; x_3 = \ddot{x} \; ; \dots$$
$$\dots \; x_{n-1} = x^{(n-2)} \; ; \; x_n = x^{(n-1)}$$

$$\Rightarrow \left| \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_{n-2} \\ \dot{x}_n = f(x_1, x_2, x_3, \dots, x_n) \end{array} \right|$$

In this way, we can reduce all higher order differential equations to first order differential equations.

Example (continued):

$$\left|\begin{aligned} \ddot{x}_a &= -\frac{k_a}{m_a} x_a - \frac{r_a}{m_a}\dot{x}_a + \frac{k_a}{m_a} x_b + \frac{1}{m_a} u(t) \\ \ddot{x}_b &= \frac{k_a}{m_b} x_a - \frac{k_a+k_b}{m_b} x_b - \frac{r_b}{m_b}\dot{x}_b \end{aligned}\right.$$

We set: $x_1 = x_a$; $x_2 = \dot{x}_a$

$x_3 = x_b$; $x_4 = \dot{x}_b$

$$\Rightarrow \left|\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k_a}{m_a} x_1 - \frac{r_a}{m_a} x_2 + \frac{k_a}{m_a} x_3 + \frac{1}{m_a} u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{k_a}{m_b} x_1 - \frac{k_a+k_b}{m_b} x_3 - \frac{r_b}{m_b} x_4 \end{aligned}\right.$$

or written in a matrix notation

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_a}{m_a} & -\frac{r_a}{m_a} & \frac{k_a}{m_a} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_a}{m_b} & 0 & -\frac{(k_a+k_b)}{m_b} & -\frac{r_b}{m_b} \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \frac{1}{m_a} \\ 0 \\ 0 \end{bmatrix} u$$

Such a representation is calle
a <u>state-space representation</u> of
our linear system.

Usually, we select some
measurement variables as
outputs, e.g.

$$y = x_a \equiv x_1$$

$$\Rightarrow \quad y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \end{bmatrix} u$$

is the output equation.