Hermitian Forms & Hermitian Matrices

We want to look at the quadratic form:

$$p(\underline{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, \overline{x}_i \cdot x_j$$

Such a form is called <u>Hermitian</u> iff $p(\underline{x})$ is <u>real</u> for any vector $\underline{x}$ (the components of $\underline{x}$ can be arbitrary complex).

In a matrix notation:

$$p(\underline{x}) = \underline{x}^* \cdot A \cdot \underline{x}$$

conj. complex transpose.

<u>Question</u>: Which conditions exist on A to make $p(\underline{x})$ a Hermitian form?

$$p(\underline{x}) = \text{real} \iff p(\underline{x}) \equiv p^*(\underline{x})$$

$$p^*(\underline{x}) = [\underline{x}^* A \underline{x}]^* = \underline{x}^* A^* \underline{x}$$

$$\equiv p(\underline{x}) = \underline{x}^* A \underline{x}$$

for <u>any</u> $\underline{x}$

$$\Rightarrow \quad \boxed{A^* = A}$$

- A matrix with this property is called a <u>Hermitian matrix</u>.

- An important subclass of the Hermitian matrices are the symmetric real matrices.

<u>Properties of Hermitian Matrices</u>:

(1) <u>Lemma</u>: The eigenvalues of Hermitian matrices are always real.

Proof: $$A \cdot \underline{v}_i = \lambda_i \underline{v}_i$$

$$\Rightarrow \underbrace{\underline{v}_i^* \cdot A \cdot \underline{v}_i}_{\text{real}} = \lambda_i \underbrace{\underline{v}_i^* \underline{v}_i}_{\text{pos. \& real}}$$

$$\Rightarrow \underline{\underline{\lambda_i = \text{real}}}$$ q.e.d.

(2) Lemma: Hermitian matrices have always a diagonal Jordan form $\Leftrightarrow$ The modal matrix is nonsingular

Proof: Assume there exists a generalized eigenvector of grade $k > 1$

$$\Rightarrow \left| \begin{array}{l} (A - \lambda_i I)^k \cdot \underline{v}_i = \emptyset \\ (A - \lambda_i I)^{k-1} \cdot \underline{v}_i \neq \emptyset \end{array} \right|$$

$$\Rightarrow \left[\underbrace{(A-\lambda_i I)^k \underline{v}_i}_{\phi}\right]^* \cdot \left[(A-\lambda_i I)^{k-2} \cdot \underline{v}_i\right] = \phi$$

$$\Rightarrow \underline{v}_i^* \left[(A-\lambda_i I)^k\right]^* \cdot \left[(A-\lambda_i I)^{k-2} \cdot \underline{v}_i\right] = \phi$$

$$\Rightarrow \underline{v}_i^* \left[(A-\lambda_i I)^*\right]^k \cdot \left[(A-\lambda_i I)^{k-2} \cdot \underline{v}_i\right] = \phi$$

$$\Rightarrow \underline{v}_i^* \left[(A-\lambda_i I)^k\right] \cdot \left[(A-\lambda_i I)^{k-2} \cdot \underline{v}_i\right] = \phi$$

$$\Rightarrow \underline{v}_i^* \cdot (A-\lambda_i I)^{k-1} \cdot (A-\lambda_i I)^{k-1} \cdot \underline{v}_i = \phi$$

$$\Rightarrow \underline{v}_i^* \cdot \left[(A-\lambda_i I)^*\right]^{k-1} \cdot \left[(A-\lambda_i I)^{k-1} \cdot \underline{v}_i\right] = \phi$$

$$\Rightarrow \underline{v}_i^* \cdot \left[(A-\lambda_i I)^{k-1}\right]^* \cdot \left[(A-\lambda_i I)^{k-1} \cdot \underline{v}_i\right] = \phi$$

$$\Rightarrow \left[(A-\lambda_i I)^{k-1} \cdot \underline{v}_i\right]^* \cdot \left[(A-\lambda_i I)^{k-1} \cdot \underline{v}_i\right] = \phi$$

$$\Rightarrow \left\| (A-\lambda_i I)^{k-1} \cdot \underline{v}_i \right\| = \phi$$

$$\Longrightarrow (A-\lambda_i I)^{k-1} \cdot \underline{v}_i = \phi$$

$\Rightarrow$ contradiction q.e.d.

(3) Lemma: The eigenvectors of a Hermitian matrix correspon= ding to different eigenvalues are orthogonal to each other.

Proof: Given $\lambda_i \neq \lambda_j \Rightarrow \underline{v}_i \neq \underline{v}_j$

$$A\underline{v}_i = \lambda_i \underline{v}_i \quad ; \quad A\underline{v}_j = \lambda_j \underline{v}_j$$

$$\Rightarrow \underline{v}_j^* A \underline{v}_i = \lambda_i \underline{v}_j^* \underline{v}_i \; ; \; \underline{v}_i^* A \underline{v}_j = \lambda_j \underline{v}_i^* \underline{v}_j$$

$$\Downarrow$$

$$\left[\underline{v}_j^* A \underline{v}_i\right]^* = \left[\lambda_i v_j^* \underline{v}_i\right]^*$$

$$\Downarrow$$

$$\Rightarrow \underline{v}_i^* A^* \underline{v}_j = \lambda_i \underline{v}_i^* \underline{v}_j$$

$$\Downarrow$$

$$\Rightarrow \underline{v}_i^* A \underline{v}_j = \lambda_i \underline{v}_i^* \underline{v}_j \equiv \lambda_j \underline{v}_i^* \underline{v}_j$$

$$\Rightarrow \underbrace{(\lambda_i - \lambda_j)}_{\neq \phi} \underbrace{\underline{v}_i^* \underline{v}_j}_{= \phi} \equiv \phi \qquad \Longrightarrow$$

$$\Rightarrow \underline{v}_j \perp \underline{v}_i$$

q.e.d.

(4) <u>Lemma</u>: The modal matrix of a Hermitian matrix is a <u>unitary</u> <u>matrix</u>.

<u>Proof</u>: Follows directly from (3). If we have a multiple eigenvalue λ_i with multiplicity m_i $\Longrightarrow$

$$\text{Rank}\{(\lambda_i I - A)\} \equiv n - m_i$$

as there are no generalized eigenvectors.

$$\underline{v}_i' \cdot \underline{v}_j = \phi \quad , \forall i \neq j$$

$$\Longrightarrow \qquad (\lambda_i I - A)\,\underline{v}_i = \phi$$

spans a subspace in which it is possible to choose all vectors perpendicular to each other, and which as a whole is perpendicular to all other eigenvectors.

<u>Lemma</u>: The inverse of a unitary matrix is its Hermitian transpose.

<u>Proof</u>: $A = V \cdot \Lambda \cdot V^{-1}$

$$\equiv A^* = (V \cdot \Lambda \cdot V^{-1})^*$$

$$= (V^{-1})^* \cdot \underset{\uparrow \text{ diagonal \& real}}{\Lambda^*} \cdot V^*$$

$$\equiv (V^*)^{-1} \cdot \Lambda \cdot (V^*)$$

$$\Longrightarrow \boxed{V^{-1} \equiv V^*}$$

q.e.d.

Examples of Hermitian matrices:

(1) $M = A + A^*$; $A \in \mathbb{C}^{n\times n}$

e.g. $A = \begin{bmatrix} (a+jb) & (c+jd) \\ (e+jf) & (g+jh) \end{bmatrix}$

$\Rightarrow M = A + A^* = \begin{bmatrix} (2a) & [(c+e)+j(d-f \\ [(c+e)+j(f-d)] & (2g) \end{bmatrix}$

$\Rightarrow M$ is Hermitian as $M^* \equiv M$.

(2) $M = A^*A$; $A \in \mathbb{C}^{n\times m}$

e.g. $A = \begin{bmatrix} 2 & (5+j) & -7 \\ \phi & 3 & (2-3j) \end{bmatrix}$

$\Rightarrow A^*A = \begin{bmatrix} 2 & \phi \\ (5-j) & 3 \\ -7 & (2+3j) \end{bmatrix} \cdot \begin{bmatrix} 2 & (5+j) & -7 \\ \phi & 3 & (2-3j) \end{bmatrix}$

$= \begin{bmatrix} 4 & (1\phi+2j) & -14 \\ (1\phi-2j) & 35 & (-29-2j) \\ -14 & (-29+2j) & 62 \end{bmatrix}$

$\Rightarrow A^*A$ is <u>Hermitian</u>.

(3) $M = A \cdot A^*$; $A \in \mathbb{C}^{n \times m}$

$$\begin{bmatrix} 2 & (5+j) & -7 \\ \emptyset & 3 & (2-3j) \end{bmatrix} \cdot \begin{bmatrix} 2 & \emptyset \\ (5-j) & 3 \\ -7 & (2+3j) \end{bmatrix} = \begin{bmatrix} 79 & (1-18j) \\ (1+18j) & 22 \end{bmatrix}$$

$\Rightarrow A \cdot A^*$ is <u>Hermitian</u>.

(4) $M = A - A^*$; $A \in \mathbb{C}^{n \times n}$

A as in (1)

$$\Rightarrow M = A - A^* = \begin{bmatrix} (2jb) & [(c-e)+j(d+f)] \\ [(e-c)+j(d+f)] & (2jh) \end{bmatrix}$$

$$= \begin{bmatrix} (2jb) & +[(c-e)+j(d+f)] \\ -[(c-e)-j(d+f)] & (2jh) \end{bmatrix}$$

$$\Rightarrow m_{ji} = -\overline{m}_{ij}$$

$\Rightarrow A - A^*$ is <u>skew-Hermitian</u>.

Lemma: Any matrix can be split into the sum of two matrices out of which one is Hermitian, the other is skew-Hermitian:

$$A = A_H + A_{SH}$$

Proof (by construction):

$$A_H = \tfrac{1}{2}(A + A^*) \quad \text{is Hermitian}$$
$$A_{SH} = \tfrac{1}{2}(A - A^*) \quad \text{is skew-Hermiti}$$

$$A_H + A_{SH} \equiv A \qquad \text{q.e.d.}$$

Lemma: A Hermitian matrix is called positive definite

$$(A > \emptyset)$$

pos. def.

iff the quadratic form:

$$\underline{x}^* A \underline{x} > \emptyset \quad ; \quad \forall \underline{x} \neq (\underline{x} = \emptyset)$$

Example:

$M = A^* \cdot A$ is a Hermitian matrix. M is positive definite ($M > \emptyset$) since:

$$\underline{x}^* M \underline{x} = \underline{x}^* A^* A \underline{x} = (A\underline{x})^* \cdot (A\underline{x})$$
$$= \|A\underline{x}\|_2^2 > \emptyset \qquad \text{q.e.d.}$$

Of course: $\vec{M} = A \cdot A^*$ is also positive definite.

Lemma: The eigenvalues of a positive definite matrix are all positive and real.

Proof: $A \cdot \underline{v}_i = \lambda_i \underline{v}_i$

$$\Rightarrow \underbrace{\underline{v}_i^* \cdot A \cdot \underline{v}_i}_{> \emptyset} = \lambda_i \cdot \underbrace{\underline{v}_i^* \cdot \underline{v}_i}_{> \emptyset}$$

$$\Rightarrow \underline{\underline{\lambda_i > \emptyset}}$$

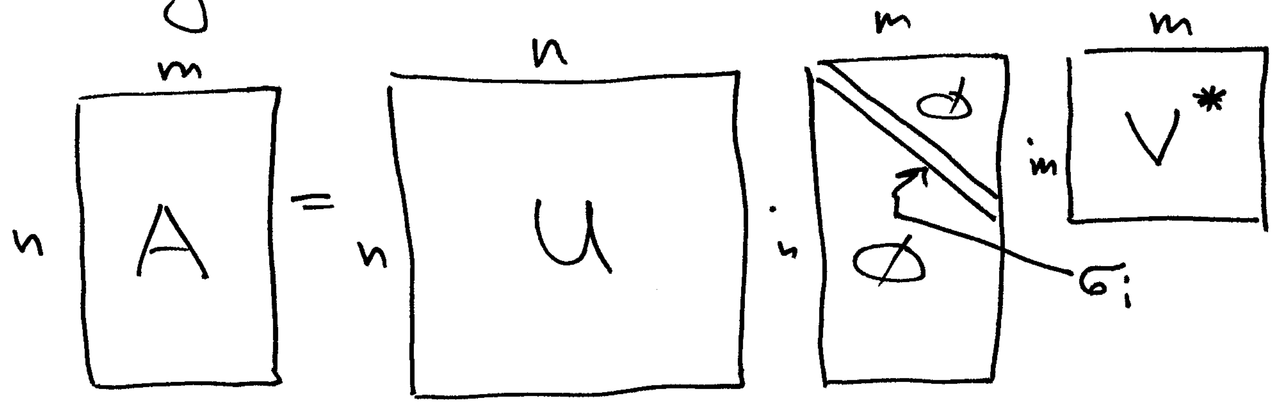
q.e.d.

The Singular Value Decomposition

Def: Any (even non-square) matrix can be decomposed into:

$$\boxed{A = U \cdot \Sigma \cdot V^*}$$

where U and V are two unitary matrices, and Σ is a diagonal matrix.



In Matlab:

s = svd (A) ; "s" is a vector containing σ_i, the singular values.

[U,S,V] = svd(A) ; returns three matrices, where s = diag(S).

Proof:

- $A \cdot A^* = (U \cdot \Sigma \cdot V^*) \cdot (V \cdot \Sigma^* \cdot U^*)$

$$= U \cdot \Sigma \cdot \underbrace{V^* \cdot V}_{I} \cdot \underbrace{\Sigma^*}_{\Sigma} \cdot U^*$$

$$\Rightarrow \boxed{A \cdot A^* = U \cdot \Sigma^2 \cdot U^*}$$

spectral decomposition of $(A \cdot A^*)$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \phi \\ & & \ddots & \\ & \phi & & \sigma_m \\ & & & \end{bmatrix} \Rightarrow \underline{\underline{\{\sigma_i\} = \sqrt{Eig(A \cdot A^*)}}}$$

U is the right modal matrix of $A \cdot A^*$.

- $A^* \cdot A = (V \cdot \Sigma^* \cdot U^*) \cdot (U \cdot \Sigma \cdot V^*)$

$$= V \cdot \underbrace{\Sigma^*}_{\Sigma} \cdot \underbrace{U^* \cdot U}_{I} \cdot \Sigma \cdot V^*$$

$$\Rightarrow \boxed{A^* A = V \cdot \Sigma^2 \cdot V^*}$$

$$\Rightarrow \sigma_i = \sqrt{\mathrm{Eig}(A \cdot A^*)} \equiv \sqrt{\mathrm{Eig}(A^* \cdot A)}$$

σ_i are positive real.

V is the right modal matrix of $A^* \cdot A$.

<u>Def</u>: The σ_i are called the <u>singular values</u> of A. They are usually sorted such that:

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$$

<u>Algorithm</u>:

```
[U,S2] = eig(A*A');
s2 = diag(S2);
s = sqrt(s2);
[V,S2] = eig(A'*A);
SS = U'*A*V;
for i = 1:n,
    if SS(i,i) < 0,
        V(:,i) = -V(:,i);
    end
end
```

$$\boxed{A = U \cdot \Sigma \cdot V^*}$$

As U, V are unitary

$\Rightarrow$ $\text{Rank}(\Sigma) \equiv \text{Rank}(A)$

$\Rightarrow$ The Rank of A equals the # of nonzero singular values of A.

$\Rightarrow$ The Nullity of A equals the # of zero singular values of A.

- As A^*A and AA^* are Hermitian, the two eigenvalue problems are very well conditioned $\Rightarrow$ the SVD-algorithm is numerically benign.

- This is the algorithm that CTRL-C uses to determine the RANK of a matrix.

$$\begin{bmatrix} A \end{bmatrix} = \left[\begin{array}{c|c} U_1 & U_2 \end{array}\right] \cdot \left[\begin{array}{c|c} \overbrace{\Sigma}^{r} & \emptyset \\ \hline \emptyset & \emptyset \end{array}\right] \cdot \left[\begin{array}{c} V_1^* \\ \hline V_2^* \end{array}\right]$$

$$\text{Rank}(A) = r < n$$

$$\Rightarrow \quad A = U \cdot (\Sigma \cdot V_1^*) = U_1 \cdot (\Sigma \cdot V_1^*) + U_2 \cdot$$

As U is unitary

$\Rightarrow$ U_1 is a column-image of A
U_2 is a column-nullspace of A

$$A^* = (U \Sigma V^*)^* = V \Sigma^* U^* = V \Sigma U^*$$

$$\begin{bmatrix} A^* \end{bmatrix} = \left[\begin{array}{c|c} V_1 & V_2 \end{array}\right] \cdot \left[\begin{array}{c|c} \overbrace{\Sigma}^{r} & \\ \hline & \end{array}\right] \cdot \left[\begin{array}{c} U_1^* \\ \hline U_2^* \end{array}\right]$$

$$= \left[\begin{array}{c|c} V_1 & V_2 \end{array}\right] \cdot \left[\begin{array}{c} \Sigma U_1^* \\ \hline \emptyset \end{array}\right]$$

As V is unitary

$\Rightarrow$ V_1^* is a row-image of A
V_2^* is a row-nullspace of A

$\Rightarrow$ One SVD gives all Images and Nullspaces at once. (This is even better, though more expensive, than QR.)

<u>Norms and Inner Product</u>:

(1) <u>Vector Norms</u>: A norm is a metric that maps vectors into scalars. It is basically a generalization of the concept of "length".

The following rules must apply:

(a) $\|\underline{x}\| = \begin{cases} > \emptyset & ; \ \underline{x} \neq \emptyset \\ = \emptyset & ; \ \underline{x} = \emptyset \end{cases}$

(b) $\|\alpha \cdot \underline{x}\| = |\alpha| \cdot \|\underline{x}\|$

(c) $\|\underline{x}_1 + \underline{x}_2\| \leq \|\underline{x}_1\| + \|\underline{x}_2\|$

(without justification!)

Certainly, the length of the vector fulfills these conditions.

$$|\underline{x}| \equiv \|\underline{x}\|_E \equiv \|\underline{x}\|_2$$

Euclidean Norm   L2-Nor

<u>Frequently used:</u>

$$\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|\underline{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \|\underline{x}\|_E$$

$$\|\underline{x}\|_\infty = \max_i |x_i| = \|\underline{x}\|_H$$

H-infinity norm

<u>In Matlab</u>:

k = norm (x, p)

$$\equiv \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

for any $p > \emptyset$, integer

k = norm (x) ≡ norm (x, 2)

k = norm (x, 'inf')

⇒ infinity norm.

(2) Matrix Norms:

It is common to define the norms of matrices through the equivalent norms of vectors:

$$\|A\| \doteq \sup_{\underline{x} \neq 0} \frac{\|A\underline{x}\|}{\|\underline{x}\|}$$

This can be done with any definition of the underlying vector-norm, yielding different results, e.g.

$$\|A\|_1 = \max_j \left(\sum_{i=1}^{n} |a_{ij}| \right)$$

$$\|A\|_\infty = \max_i \left(\sum_{j=1}^{n} |a_{ij}| \right) \equiv \|$$

$$\|A\|_2 = \sigma_1 \equiv \|A\|_L$$

other definitions:

$$\|A\|_E = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2 = Tr\,(A \cdot A^*)$$

$\uparrow$ Euclidean norm $\qquad$ $\uparrow$ "trace"

$$\left[\, Tr\,(A) = \sum_{i=1}^{n} a_{ii} = \text{sum}(\text{diag}(A)) \,\right]$$

$$\|A\|_F = \sqrt{Tr\,(A^* A)} \equiv \sqrt{Tr\,(A \cdot A^*)}$$

$\uparrow$ Frobenius norm

In Matlab:

There is no "trace"-function provided; however, this can be easily generated, since:

$$tr\,(A) \equiv \text{sum}(\text{diag}(A))$$

also:

$$tr\,(A) \equiv \text{sum}(\text{eig}(A))$$

Among the matrix norms, the following are provided in Matlab:

$$\text{norm}(A,1) \equiv \|A\|_1$$
$$\text{norm}(A,2) = \text{norm}(A) \equiv \|A\|_L$$
$$\text{norm}(A,'\text{inf}') = \|A\|_\infty$$
$$\text{norm}(A,'\text{fro}') = \|A\|_F$$

There is no Euclidean norm provided, however:

$$\|A\|_E = \text{sum}(\text{diag}(A'*A))$$

or $\|A\|_E = \text{norm}(A,'\text{fro}')\wedge 2$

(3) Inner Product:

Very often, we have operated on quadratic forms:

$$\underline{x}^*\cdot A\cdot \underline{x} \equiv \underline{x}^*\cdot(A\cdot\underline{x})$$
$$\equiv (A^*\cdot\underline{x})^*\cdot\underline{x}$$

These products of a row vector with a column-vector are frequently termed "<u>inner product</u>" and often written as:

$$\boxed{\langle \underline{x}, \underline{y} \rangle ::= \underline{x}^* \cdot \underline{y}}$$

Thus, $\underline{x}^* A \underline{x} = \langle \underline{x}, A\underline{x} \rangle$
$= \langle A^* \underline{x}, \underline{x} \rangle$

and:

$$\Rightarrow \quad \|\underline{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{\underline{x}^* \underline{x}}$$

$$= \sqrt{\langle \underline{x}, \underline{x} \rangle} \qquad \text{etc.}$$

(4) The <u>outer product</u> of two vectors is defined as:

$$\boxed{> \underline{x}, \underline{y} < \; ::= \underline{x} \cdot \underline{y}^*}$$

is a $(n \times n)$-matrix.

Of course: $\mathrm{Rank}(\underline{x} \cdot \underline{y}^*) = 1$

and: $\langle \underline{x}, \underline{y} \rangle = \mathrm{Tr}(> \underline{x}, \underline{y} <)$

Properties of Quadratic Forms:

Given a general square matrix

$$A \in \mathbb{C}^{n \times n}$$

$$\Rightarrow \quad p(\underline{x}, A) = \underline{x}^* A \underline{x} \quad \in \mathbb{C}^1$$

However: $\quad A = A_H + A_{SH}$

Question: What are $p(\underline{x}, A_H)$ and $p(\underline{x}, A_{SH})$?

$$p(\underline{x}, A_H) = \underline{x}^* A_H \underline{x} = \tfrac{1}{2}\underline{x}^*(A + A^*)\underline{x}$$

$$= \tfrac{1}{2}\underline{x}^* A \underline{x} + \tfrac{1}{2}\underline{x}^* A^* \underline{x}$$

$$= \tfrac{1}{2}\underbrace{\underline{x}^* A \underline{x}}_{\alpha + j\beta} + \tfrac{1}{2}\underbrace{(\underline{x}^* A \underline{x})^*}_{\alpha - j\beta} = \alpha$$

$$\Rightarrow \boxed{p(\underline{x}, A_H) \equiv \mathrm{Re}\{p(\underline{x}, A)\}}$$

$$p(\underline{x}, A_{SH}) = \underline{x}^* A_{SH} \underline{x} = \tfrac{1}{2}\underline{x}^*(A - A^*)\underline{x}$$

$$= \tfrac{1}{2}\underline{x}^* A \underline{x} - \tfrac{1}{2}\underline{x}^* A^* \underline{x}$$

$$= \tfrac{1}{2}\underbrace{\underline{x}^* A \underline{x}}_{\alpha + j\beta} - \tfrac{1}{2}\underbrace{(\underline{x}^* A \underline{x})^*}_{\alpha - j\beta} = j\beta$$

$$\Rightarrow \boxed{p(\underline{x}, A_{SH}) = j \cdot \mathrm{Im}\{p(\underline{x}, A)\}}$$

<u>Remark</u>: We have seen that the eigenvalues of Hermitian matrices are always real:

$$\boxed{\mathrm{Im}\{\mathrm{Eig}(A_H)\} = 0}$$

It is also true that the eigenvalues of skew-Hermitian matrices are always purely imaginary:

$$\boxed{\mathrm{Re}\{\mathrm{Eig}(A_{SH})\} = 0}$$

<u>However</u>:

$$A = A_H + A_{SH}$$

$$\mathrm{Eig}(A_H) \neq \mathrm{Re}\{\mathrm{Eig}(A)\}$$
$$-j \cdot \mathrm{Eig}(A_{SH}) \neq \mathrm{Im}\{\mathrm{Eig}(A)\}$$

as sometimes stated in text books !!! (Therefore, we do <u>not</u> define positive definiteness for arbitrary square matrices as some text books do!)