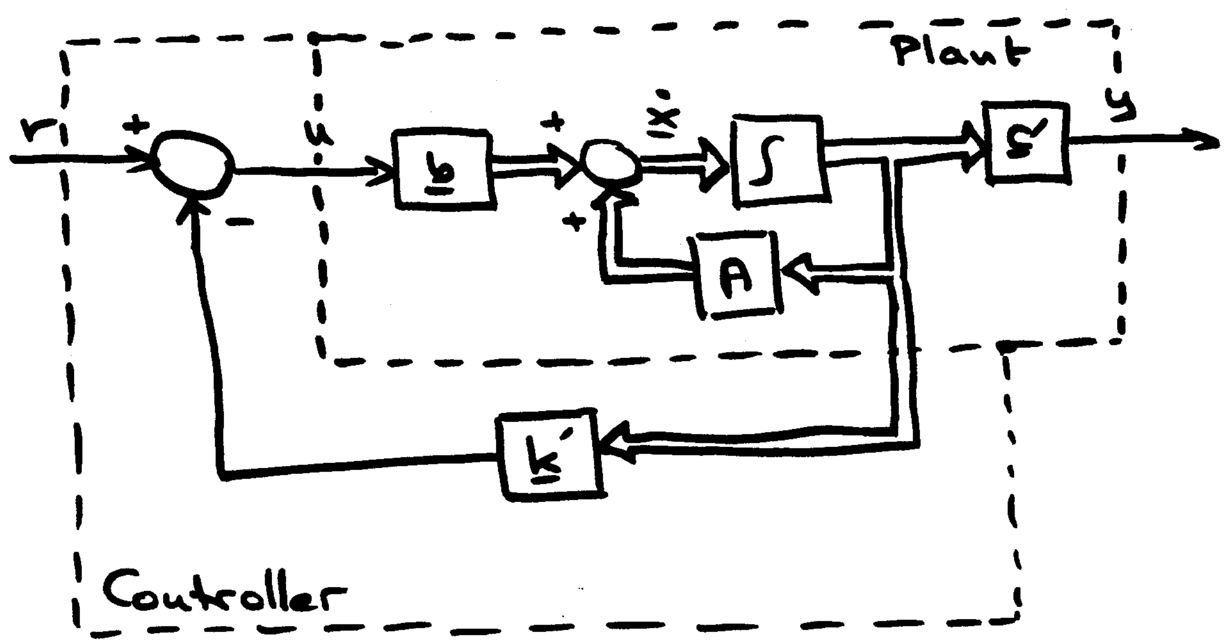
System Design Using State Feedback

I.) The Pole Placement Problem

Question: We want to design a controller which places the closed-loop system poles in a prescribed location.

Answer: This can be achieved by feeding back a linear combination of all states:

$$u = r - \underline{k}'\underline{x}$$



$$\left|\begin{aligned} \dot{\underline{x}} &= A\,\underline{x} + \underline{b}\,u \\ y &= \underline{c}'\underline{x} + d u \\ u &= -\underline{k}'\underline{x} + r \end{aligned}\right|$$

$$\Rightarrow \left|\begin{aligned} \dot{\underline{x}} &= A_{CL}\,\underline{x} + \underline{b}_{CL}\,r \\ y &= \underline{c}'_{CL}\,\underline{x} + d_{CL}\,r \end{aligned}\right|$$

$$\left|\begin{aligned} A_{CL} &= A - \underline{b}\,\underline{k}' \\ \underline{b}_{CL} &= \underline{b} \\ \underline{c}'_{CL} &= \underline{c}' - d\,\underline{k}' \\ d_{CL} &= d \end{aligned}\right|$$

Let us assume that our open-loop system was represented in a controller-canonical form

$$A = \begin{bmatrix} 0 & 1 & & & 0 \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix}; \quad \underline{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}; \quad \underline{k} = \begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_{n-1} \end{bmatrix}$$

$$\Rightarrow \underline{b}\,\underline{k}' = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \\ k_0 & k_1 & \cdots & k_{n-1} \end{bmatrix} \Rightarrow A_{CL} = \begin{bmatrix} 0 & 1 & & & 0 \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \\ -(a_0+k_0) & -(a_1+k_1) & \cdots & & \end{bmatrix}$$

$\Rightarrow$ If the open-loop system was in controller canonical form, the closed-loop system will again be in controller-canonical form.

Example:

$$G(s) = \frac{5s+7}{s^2+8s-10}$$

$\uparrow$ instable

Find a state feedback which places the poles at $\lambda_{1,2} = -1 \pm j$

$$\Rightarrow q_{cl}(s) = (s-\lambda_1)(s-\lambda_2)$$
$$= (s+1+j)(s+1-j)$$
$$= s^2+2s+2$$

$\Rightarrow a_0 = -10 \quad ; \quad a_0 + k_0 = 2 \Rightarrow \underline{\underline{k_0 = 12}}$

$a_1 = 8 \quad ; \quad a_1 + k_1 = 2 \Rightarrow \underline{\underline{k_1 = -6}}$

$$\Rightarrow \underline{\underline{\underline{k} = \begin{bmatrix} 12 \\ -6 \end{bmatrix}}}$$

$$G_{OL} = \frac{5s+7}{s^2+8s-10}$$

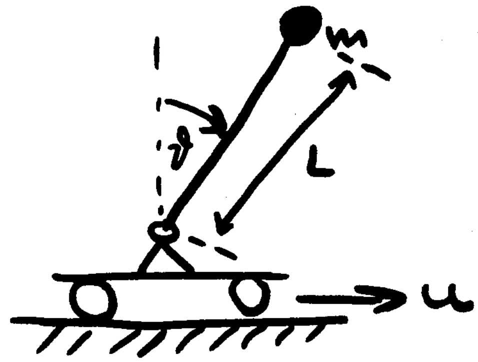
$$\Rightarrow \left| \begin{aligned} \dot{\underline{x}}_{OL} &= \begin{bmatrix} 0 & 1 \\ 10 & -8 \end{bmatrix} \underline{x}_{OL} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 7 & 5 \end{bmatrix} \underline{x} \end{aligned} \right|$$

$$\Rightarrow \left| \begin{aligned} \dot{\underline{x}}_{CL} &= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \underline{x}_{CL} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 7 & 5 \end{bmatrix} \underline{x}_{CL} \end{aligned} \right|$$

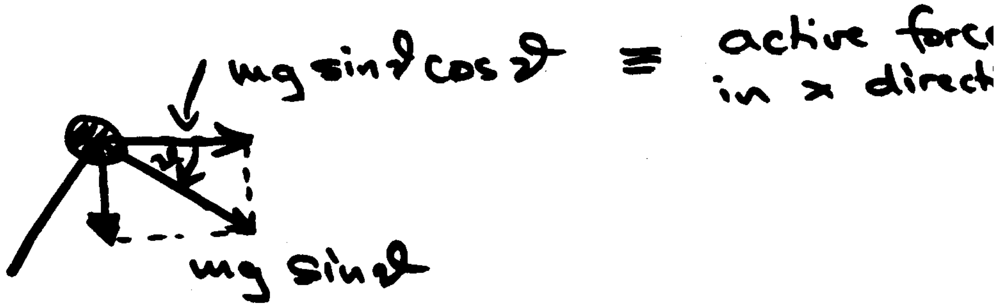
$$\Rightarrow \quad G_{CL} = \frac{5s+7}{s^2+2s+2}$$

$\Rightarrow$ The poles have been placed where desired. The zeros have not moved as the direct coupling (d) was zero.

Example (Inverted Pendulum):



Eigenmovement of the pendulum:



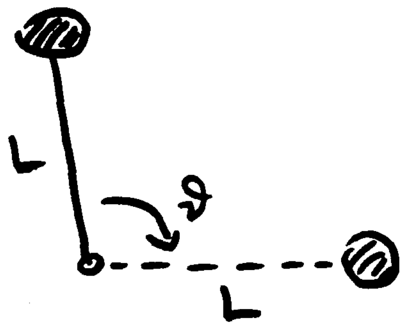
$$\Rightarrow m\ddot{x} = mg\sin\vartheta\cos\vartheta$$

$$\Rightarrow \ddot{x} = g\sin\vartheta\cos\vartheta$$

Let: $x_1 = x \quad ; \quad x_2 = \dot{x}$

$$\left| \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = g \sin\vartheta \cos\vartheta \end{array} \right|$$

<u>Relation between x and ϑ:</u>



$\vartheta = 0 \Rightarrow x = 0$;
$\vartheta = \frac{\pi}{2} \Rightarrow x = L$ <u>etc</u>:

$$\Rightarrow \quad x = L \cdot \sin\vartheta = x_1$$
$$\dot{x} = L \cdot \cos\vartheta = x_2$$

$$\Rightarrow \quad \left| \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = (g/L^2) \cdot x_1 x_2 \end{array} \right|$$

is a nonlinear system describing the motion of the pendulum without external force.

Small signal behavior:

$$\cos\vartheta \approx 1$$

$$\Rightarrow \left| \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = (g/L)\cdot x_1 \end{array} \right|$$

is a linearized model around $\underline{\underline{\vartheta = \emptyset}}$.

Goal: Make $\underline{\underline{\ddot{x} = \emptyset}}$. We apply a movement to the cart which will counter = balance the eigenmovement.

$$\Rightarrow \left| \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = (g/L)\cdot x_1 - (g/L)\,u \end{array} \right|$$

Output equation:

$$x = L\cdot \sin\vartheta \approx L\cdot\vartheta$$

$$\Rightarrow \vartheta = \left(\tfrac{1}{L}\right) x$$

but due to input:

$$\vartheta = \left(\tfrac{1}{L}\right) x - \left(\tfrac{1}{L}\right) u$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ g/L & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ -g/L \end{bmatrix} u \\ \\ y = \begin{bmatrix} 1/L & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} -1/L \end{bmatrix} u \end{array} \right|$$

is a description of the small signal behavior of the system around $\underline{\vartheta = 0}$.

- We start by transforming the system into controller-canonical form:

$$\ddot{x} = \frac{g}{L} x - \frac{g}{L} u$$

$$\Rightarrow \left(s^2 - \frac{g}{L}\right) x = -\frac{g}{L} u$$

$$\Rightarrow X(s) = \frac{-g/L}{s^2 - g/L} U(s)$$

$$y = \left(\frac{1}{L}\right) x - \left(\frac{1}{L}\right) u$$

$$\Rightarrow Y(s) = \underbrace{\left[\frac{-g/L^2}{s^2 - g/L} - \frac{1}{L} \right]}_{G(s)} U(s)$$

$$\Rightarrow \left| \begin{aligned} \dot{\underline{\xi}} &= \begin{bmatrix} \phi & 1 \\ g/L & \phi \end{bmatrix} \underline{\xi} + \begin{bmatrix} \phi \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -g/L^2 & \phi \end{bmatrix} \underline{\xi} + \begin{bmatrix} -1/L \end{bmatrix} u \end{aligned} \right|$$

is the controller - canonical representation.

- We want the controlled system to be stable and non-oscillatory, e.g.

$$\underline{\underline{\lambda_1 = -2}} \; ; \; \underline{\underline{\lambda_2 = -3}}$$

$$\Rightarrow q_{cL}(s) = (s+2)(s+3) = s^2 + 5s + 6$$

$$a_0 = -\frac{g}{L} \; ; \; a_0 + k_0 = 6 \Rightarrow \underline{\underline{k_0 = 6 + g/L}}$$

$$a_1 = \phi \; ; \; a_1 + k_1 = 5 \Rightarrow \underline{\underline{k_1 = 5}}$$

$$\Rightarrow \underline{k} = \begin{bmatrix} 6 + g/L \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad \dot{\underline{x}}_{CL} = \begin{bmatrix} \phi & 1 \\ -6 & -5 \end{bmatrix} \underline{x}_{CL} + \begin{bmatrix} \phi \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_{1_{CL}} & c_{2_{CL}} \end{bmatrix} \underline{x}_{CL} + \begin{bmatrix} -1/L \end{bmatrix} u$$

$$c_{1_{CL}} = c_{1_{OL}} - d \cdot k_0$$
$$= -\frac{g}{L^2} - \left(-\frac{1}{L}\right)\left(6 + \frac{g}{L}\right)$$
$$= 6/L$$

$$c_{2_{CL}} = c_{2_{OL}} - d \cdot k_1$$
$$= \phi - \left(-\frac{1}{L}\right) \cdot 5 = 5/L$$

$$\Rightarrow \left| \begin{aligned} \dot{\underline{x}}_{CL} &= \begin{bmatrix} \phi & 1 \\ -6 & -5 \end{bmatrix} \underline{x}_{CL} + \begin{bmatrix} \phi \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} \frac{6}{L} & \frac{5}{L} \end{bmatrix} \underline{x}_{CL} + \begin{bmatrix} -1/L \end{bmatrix} u \end{aligned} \right|$$

$$\Rightarrow G_{CL}(s) = \frac{1}{L} \cdot \frac{5s+6}{s^2+5s+6} - \frac{1}{L} \quad //$$

Here, the zeros <u>have</u> also changed due to the direct coupling between input and output.

- We have seen that the controller-canonical form is particularly appealing for the design of controllers using the pole placement technique.

<u>General algorithm</u>:

Given: $\left| \begin{array}{l} \underline{\dot{x}} = A\underline{x} + \underline{b}u \\ y = \underline{c}'\underline{x} + du \end{array} \right|$

in any representation.

(1) Find the transformation matrix to controller-canonical form: $\underline{\xi} = T_{CCF} \cdot \underline{x}$

$$\Rightarrow \left| \begin{array}{l} \underline{\dot{\xi}} = A_{CCF}\,\underline{\xi} + \underline{b}_{CCF}\,u \\ y = \underline{c}'_{CCF}\,\underline{\xi} + du \end{array} \right|$$

There is actually no need to perform the transformation
⇒ System must be controllable.

(2) Find the characteristic polynomial

$$q(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

(3) Determine the feedback vector $\underline{\hat{k}}$ for controller-canonical coordinates:

$$u = r - \underline{\hat{k}}' \cdot \underline{\xi}$$

(4) This can now easily be transformed back to the original coordinate system.
As $\underline{\xi} = T_{ccF} \cdot \underline{x}$

$$\Rightarrow \quad u = r - \underbrace{\underline{\hat{k}}' \cdot T_{ccF}}_{\underline{k}'} \cdot \underline{x}$$

$$\boxed{\underline{k}' = \underline{\hat{k}}' \cdot T_{ccF}}$$

is the feedback vector for the

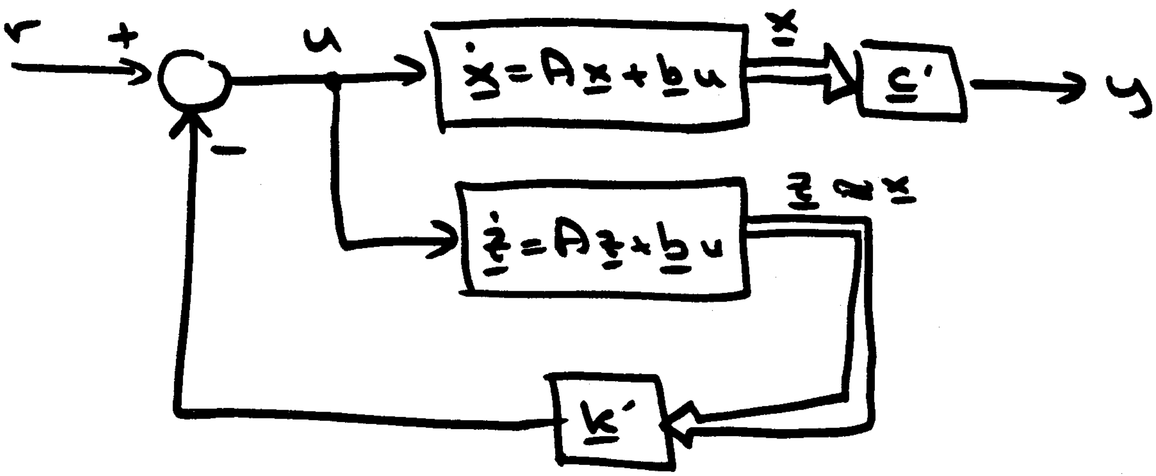
original coordinate system, where T_{CCF} is the transformation matrix from the original representation into controller-canonical coordinates.

II.) Observer Design

The principal shortcoming of all state-feedback techniques is the fact that <u>all</u> states need to be fed back and therefore measured. If the system to be controlled is "boxed up", and all I got available is the set of inputs and a limited set of outputs, I need to <u>reconstruct</u> the missing state variables from inputs and outputs.

A) Trivial Observer.

The idea is simple enough:



You build a <u>model</u> ($\dot{\underline{z}} = A\underline{z} + \underline{b}u$) of your <u>system</u> ($\dot{\underline{x}} = A\underline{x} + \underline{b}u$). If the system and input matrices are the same, and if $\underline{z}_0 \equiv \underline{x}_0$ $\Longrightarrow$ $\underline{z} = \underline{x}$, and we can use this for feedback.

<u>Disadvantage</u>: The smallest disturbance will separate the system behavior from the model behavior. We have

no means to control the error :

$$\underline{e} = \underline{z} - \underline{x}$$

Proof: $\underline{\dot{e}} = \underline{\dot{z}} - \underline{\dot{x}} = A\underline{z} + \underline{b}u - A\underline{x} - \underline{b}u$

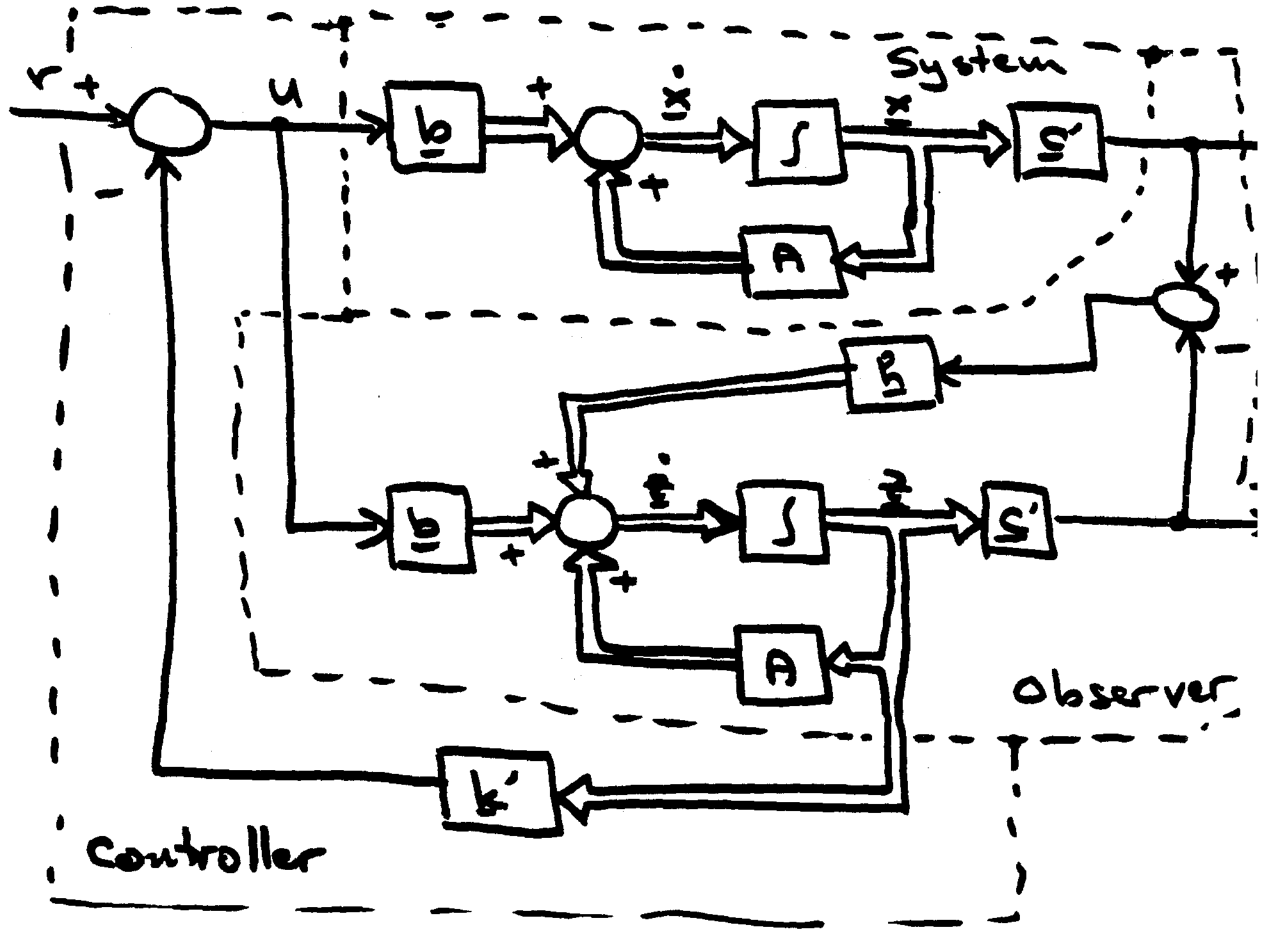
$$\Rightarrow \underline{\dot{e}} = A\underline{z} - A\underline{x} = A(\underline{z} - \underline{x})$$

$$\Rightarrow \boxed{\underline{\dot{e}} = A\underline{e}}$$

⇒ If the system (without control) was instable, the error will grow without bounds.

B) n^{th} Order Luenberger Observer

The idea is again simple. We create a model of our system, and use the difference between model output and system output in a second feedback loop to calibrate the model:



System: $\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underline{b}u \\ y = \underline{c}'\underline{x} + du \end{array} \right|$

Observer: $\left| \begin{array}{l} \dot{\underline{z}} = A\underline{z} + \underline{b}u + \underline{h}(y - \hat{y}) \\ \hat{y} = \underline{c}'\underline{z} + du \end{array} \right|$

Controller: $\left| u = r - \underline{k}' \cdot \underline{z} \right|$

Let us look into the error equation now:

$$\underline{e} = \underline{z} - \underline{x}$$

$$\Rightarrow \dot{\underline{e}} = \dot{\underline{z}} - \dot{\underline{x}} = A\underline{z} + \underline{b}u + \underline{h}(y - \hat{y}) - A\underline{x} - \underline{b}u$$

$$= A(\underline{z} - \underline{x}) + \underline{h}(\underline{c}'\underline{x} + du - \underline{c}'\underline{z} - du)$$

$$= A\underline{e} + \underline{h}\underline{c}'(\underline{x} - \underline{z})$$

$$= A\underline{e} - \underline{h}\underline{c}'\underline{e}$$

$$\Rightarrow \boxed{\dot{\underline{e}} = (A - \underline{h}\underline{c}')\underline{e}}$$

Let us look at the dual system:

$$\dot{\underline{\eta}} = (A - \underline{h}\underline{c}')'\underline{\eta}$$

$$\Rightarrow \boxed{\dot{\underline{\eta}} = (A' - \underline{c}\underline{h}')\underline{\eta}}$$

This takes the same form as our previous problem:

$$\dot{\underline{x}} = (A - \underline{b}\underline{k}')\underline{x} + \underline{b}r$$

$\Rightarrow$ By selecting $\underline{h}$ appropriately we can place our poles of the error equation anywhere, as long as the dual system is completely controllable $\Leftrightarrow$ the error system must be completely observable.

Algorithm:

Assume your system was in observer-canonical form:

$$\left|\begin{array}{l} \dot{\underline{x}} = \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & \phi & -a_1 \\ & 1 & \ddots & & \vdots \\ & \phi & \ddots & 0 & \\ & & & 1 & -a_{n-1} \end{bmatrix} \underline{x} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} u \\ \\ y = [\phi \, - - - \, \phi \,\vdots\, 1]\, \underline{x} + [d]\, u \end{array}\right|$$

$$\Rightarrow \underline{h}\underline{c}' = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{n-1} \end{bmatrix} [\phi\phi \cdots \phi 1] = \begin{bmatrix} \phi & - - - & \phi & h_0 \\ \vdots & & & \vdots \\ \phi & - - - & \phi & h_{n-1} \end{bmatrix}$$

$$\Rightarrow \quad A_{obs} = A - \underline{h}\,\underline{c}' = \begin{bmatrix} 0 & & & & -(a_0+h_0) \\ 1 & 0 & & \emptyset & -(a_1+h_1) \\ & 1 & \ddots & & \vdots \\ & \emptyset & \ddots & 0 & \vdots \\ & & & 1 & -(a_{n-1}+h_{n-1}) \end{bmatrix}$$

is again in observer-canonical form. The algorithm is completely dual to the previous one.

- If the system is not represented in observer-canonical form, we start by finding the transformation matrix to convert to observer-canonical form:

$$\underline{\xi} = T_{OCF} \cdot \underline{x} \quad ; \quad \underline{\hat{\xi}} = T_{OCF} \cdot \underline{z}$$

$$\Rightarrow \quad A_{obs} = \hat{A} - \underline{\hat{h}}\,\underline{\hat{c}}'$$

$$\dot{\underline{\hat{\xi}}} = \hat{A}\underline{\hat{\xi}} + \underline{\hat{h}}(y - \hat{y}) + \underline{\hat{b}}u$$

$$\Rightarrow T_{OCF}\,\dot{\underline{z}} = T_{OCF} A \underbrace{T_{OCF}^{-1} \cdot T_{OCF}}_{I} \cdot \underline{z} + \underline{\hat{h}}(y - \hat{y}) + T_{OCF}\,\underline{b}$$

$$\Rightarrow \dot{\underline{z}} = A \cdot \underline{z} + \underbrace{T_{OCF}^{-1} \cdot \hat{\underline{h}}}_{\underline{h}} \cdot (y - \hat{y}) + \underline{b} \cdot u$$

- If the model is not given in observer-canonical form, the correct $\underline{h}$-vector can be found by calculating the feedback vector for the observer-canonical form, and then:

$$\boxed{\underline{h} = T_{OCF}^{-1} \cdot \hat{\underline{h}}}$$

Of course, we would not have needed this algorithm due to the duality principle.

In Matlab:

Form a vector of the desired pole locations:

$$p = [\,-1\,;\; -2+j\,;\; -2-j\,]$$

then:

$$k = \mathrm{PLACE}\,(A, B, p)$$

solves the pole placement problem for single-input systems in arbitrary coordinates.

. The observer design is done through duality:

$$\left|\begin{array}{l} h = \mathrm{PLACE}\,(A', (\underline{c}')', p_2) \\ h = h' \end{array}\right|$$

will compute the observer feedback vector for single-output systems in arbitrary coordinates.

Question: How can we be sure that our observer design does not influence the controller design?

Given the system:

$$\left|\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u \\ y &= \underline{c}'\underline{x} + du \\ \dot{\underline{z}} &= A\underline{z} + \underline{b}u + \underline{\beta}(y - \hat{y}) \\ \hat{y} &= \underline{c}'\underline{z} + du \\ u &= r - \underline{k}'\cdot\underline{z} \end{aligned}\right|$$

Let us find a total state space representation choosing $\underline{x}$ and $\underline{e} = \underline{z} - \underline{x}$ as our $2n$ state variables:

$$\left|\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u \\ y &= \underline{c}'\underline{x} + du \\ \dot{\underline{e}} &= (A - \underline{\beta}\underline{c}')\underline{e} \\ u &= r - \underline{k}'(\underline{e} + \underline{x}) \end{aligned}\right|$$

$$\Rightarrow \quad \dot{\underline{x}} = A\underline{x} + \underline{b}r - \underline{b}\underline{k}'\underline{e} - \underline{b}\underline{k}'\underline{x}$$

$$\Rightarrow \quad \dot{\underline{x}} = (A - \underline{b}\underline{k}')\underline{x} - \underline{b}\underline{k}'\underline{e} + \underline{b}r$$

$$\begin{pmatrix} \dot{\underline{x}} \\ \dot{\underline{e}} \end{pmatrix} = \begin{bmatrix} (A-\underline{b}\underline{k}') & (-\underline{b}\underline{k}') \\ \phi & (A-\underline{h}\underline{c}') \end{bmatrix} \begin{pmatrix} \underline{x} \\ \underline{e} \end{pmatrix} + \begin{bmatrix} \underline{b} \\ \phi \end{bmatrix} r$$

- We notice that this is a block-triangular representation

$$\Rightarrow \quad Eig\{A_{cL}\} = \{ \underbrace{Eig\{A-\underline{b}\underline{k}'\}}_{\substack{\text{controller} \\ \text{poles}}}, \underbrace{Eig\{\ldots}_{\substack{\text{obser} \\ \text{pole}}}$$

- We also notice that <u>all</u> observer poles are uncontrollable and thus do <u>not</u> influence the input/output behavior.

$\Rightarrow$ A "minimal" realization is:

$$\dot{\underline{x}} = (A-\underline{b}\underline{k}')\,\underline{x} + \underline{b}r$$

as desired.