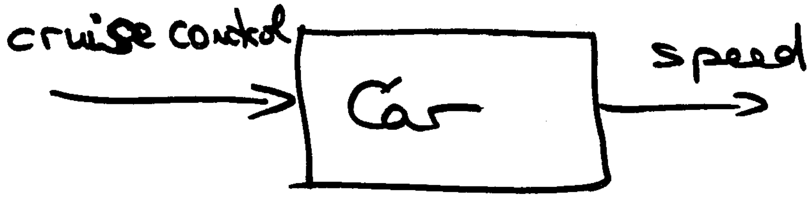
Steady-state Accuracy:

Problem: In a SISO-control circuit, we usually wish to scale our input in units of our output. E.g., in a cruise-control circuit of a fancy car, I may wish to scale the angular position of a knob in m/h.

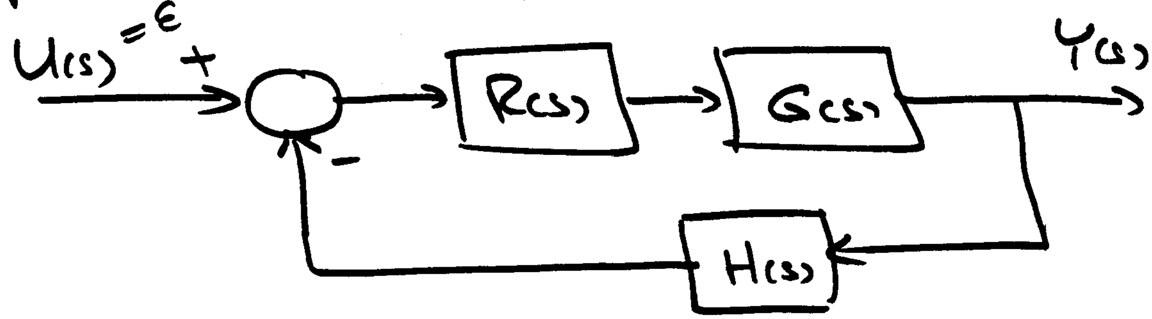


I want the ratio $\frac{y_{ss}}{u_{ss}}$ to be insensitive to parameter variations, i.e.

$$\boxed{G_{CL}(s) = -\underline{c}_{CL} A_{CL}^{-1} \underline{b}_{CL} + d_{CL}}$$

should be made insensitive to parameter changes such as gain factors in my controller, etc.

In ECE 441, we had found a solution to this problem for control circuits of the type:



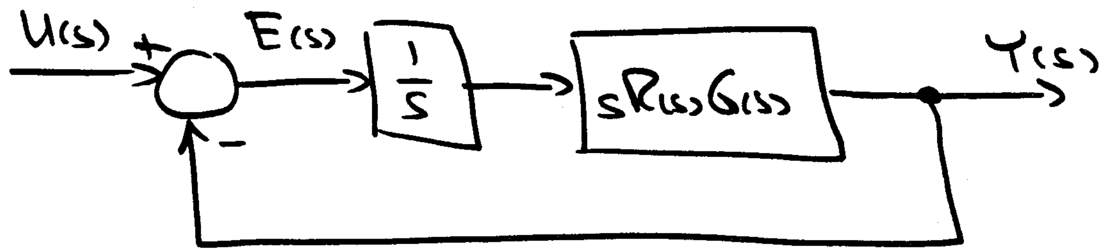
We had found that if the circuit is stable and if

$$H(\emptyset) = 1 \Longrightarrow \text{if Type}\{R(s)\cdot G(s)\}$$

$$\Longrightarrow \; y_{ss} = y(\infty) \equiv u_{ss} = u(\infty)$$

$$\Longleftrightarrow \; e_{ss} = u_{ss} - y_{ss} = \emptyset \; .$$

This can be illustrated easily.



In the steady-state, all derivatives are zero $\Longrightarrow$

$$e_{ss} = \emptyset \quad \Longrightarrow \quad y_{ss} = u_{ss}$$

q.e.d.

Question: Can we find a similar design for state-feedback control?

- Let me illustrate the problem at hand of an example.

$$\left| \begin{array}{l} \dot{\underline{x}} = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} 3 \end{bmatrix} u \end{array} \right|$$

We want to use pole placement to move the three poles to

$$\underline{p} = [\,-2\ ;\ -1+j\ ;\ -1-j\,]$$

We find:

$$\underline{k}' = [15.7222 \quad -19.3333 \quad 22.6111]$$

The closed-loop system looks as follows:

$$\dot{\underline{x}} = \begin{bmatrix} -14.7222 & 17.3333 & -19.6111 \\ -19.7222 & 24.3333 & -28.6111 \\ -8.7222 & 11.3333 & -13.6111 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = [-46.1667 \quad 59 \quad -66.8333]\,\underline{x} + [3]\,u$$

$$\Rightarrow \underline{\underline{G(0)}} = -\underline{c}'_{CL} \cdot A_{CL}^{-1}\,\underline{b}_{CL} + d_{CL} = \underline{\underline{-12}}$$

Thus, we could e.g. divide the input by -12, and have thereby normalized our DC-gain to 1.

. Now, I modify my $\underline{k}'$-vector by only ≈ 1% :

[> KR = K + Ø.Ø1 * NORM(K) * RAND(K);

$\Rightarrow \underline{k}_r' = [16.\text{Ø}195 \quad -19.1138 \quad 22.7146]$

$\Rightarrow$ The closed-loop system takes now the form:

$$\dot{\underline{x}} = \begin{bmatrix} -15.\text{Ø}195 & 17.1138 & -19.7146 \\ -2\text{Ø}.\text{Ø}195 & 24.1138 & -28.7146 \\ -9.\text{Ø}195 & 11.1138 & -13.7146 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = [-47.\text{Ø}584 \quad 58.3413 \quad -67.1439] \underline{x} + [3] u$$

which looks acceptably close.

However, the new DC-gain is:

$$\underline{\underline{G_r(\text{Ø})}} = \underline{\underline{+7.8966}}$$

Even the sign is different.

$\Rightarrow$ The DC-gain is extremely sensitive to variations in the controller gain.

By the way:

$$\mathrm{Eig}(A_{CL_r}) = \begin{bmatrix} -\emptyset.9428 \\ +1.2962 \\ -4.9737 \end{bmatrix}$$

This system is even unstable !!!

$$\boxed{\text{WE GOT A PROBLEM !}}$$

Idea: We add one more differential equation:

$$\boxed{\dot{q} = y - r}$$

and feed back also q as an additional state variable. If we make this system stable

$$\Longrightarrow \quad \dot{q}_{ss} \equiv \emptyset \quad \Longrightarrow \quad \boxed{y = r}$$

independent of the feedback gains.

Algorithm:

(1) We start with:

$$\dot{q} = y = \underline{c}'\underline{x} + du$$

With: $\underline{\xi} = \begin{bmatrix} \underline{x} \\ q \end{bmatrix}$

$$\Rightarrow \left| \begin{aligned} \dot{\underline{\xi}} &= \left[\begin{array}{c:c} A & \phi \\ \hdashline \underline{c}' & \phi \end{array}\right]\underline{\xi} + \left[\begin{array}{c} \underline{b} \\ \hdashline d \end{array}\right]u \\ y &= \left[\begin{array}{c:c} \underline{c}' & \phi \end{array}\right]\underline{\xi} + [d]u \end{aligned} \right|$$

is the augmented open-loop system of order $(n+1)$.

(2) We design a state-feedback (or output feedback) using whatever technique we like for this system:

$$u = r - \underline{k}'\underline{\xi}$$

With: $\underline{k}' = \left[\begin{array}{c:c} \underline{k}_1' & k_2 \end{array}\right]$

$$\Rightarrow \quad u = r - \underline{k}_1'\underline{x} - k_2 q$$

(3) Plugging in the state-feedback, we find for the closed-loop system:

$$\left|\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}r - \underline{b}\underline{k}_1'\underline{x} - \underline{b}k_2 q \\ \dot{q} &= \underline{c}'\underline{x} + dr - d\underline{k}_1'\underline{x} - dk_2 q \\ y &= \underline{c}'\underline{x} + dr - d\underline{k}_1'\underline{x} - dk_2 q \end{aligned}\right|$$

$$\Rightarrow \left| \begin{aligned} A_{CL} &= \left[\begin{array}{c|c} (A - \underline{b}\underline{k}_1') & -\underline{b}k_2 \\ \hline (\underline{c}' - d\underline{k}_1') & -dk_2 \end{array}\right] ; \quad \underline{b}_{CL} = \left[\begin{array}{c} \underline{b} \\ \hline d \end{array}\right] \\ \underline{c}_{CL}' &= \left[\begin{array}{c|c} (\underline{c}' - d\underline{k}_1') & -dk_2 \end{array}\right] ; \quad d_{CL} = [d] \end{aligned} \right|$$

which has obviously a DC-gain of zero, since:

$$\dot{q}_{ss} = 0 \implies \underline{\underline{y_{ss} = 0}}$$

(4) We now modify our design just a little by making

$$\dot{q} = y - r$$

but we use the feedback vector found before.

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underline{b}u \\ y = \underline{c}'\underline{x} + du \\ \dot{q} = y - r \\ u = r - \underline{k}_1'\underline{x} - k_2 q \end{array} \right|$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underline{b}r - \underline{b}\underline{k}_1'\underline{x} - \underline{b}k_2 q \\ y = \underline{c}'\underline{x} + dr - d\underline{k}_1'\underline{x} - dk_2 q \\ \dot{q} = \underline{c}'\underline{x} + dr - d\underline{k}_1'\underline{x} - dk_2 q - r \end{array} \right|$$

$\Rightarrow$ Modified closed-loop system:

$$\tilde{A}_{CL} = \left[\begin{array}{c|c} (A - \underline{b}\underline{k}_1') & -\underline{b}k_2 \\ \hline (\underline{c} - d\underline{k}_1') & -dk_2 \end{array} \right] \equiv A_{CL}$$

$\Rightarrow$ eigenvalues are unchanged

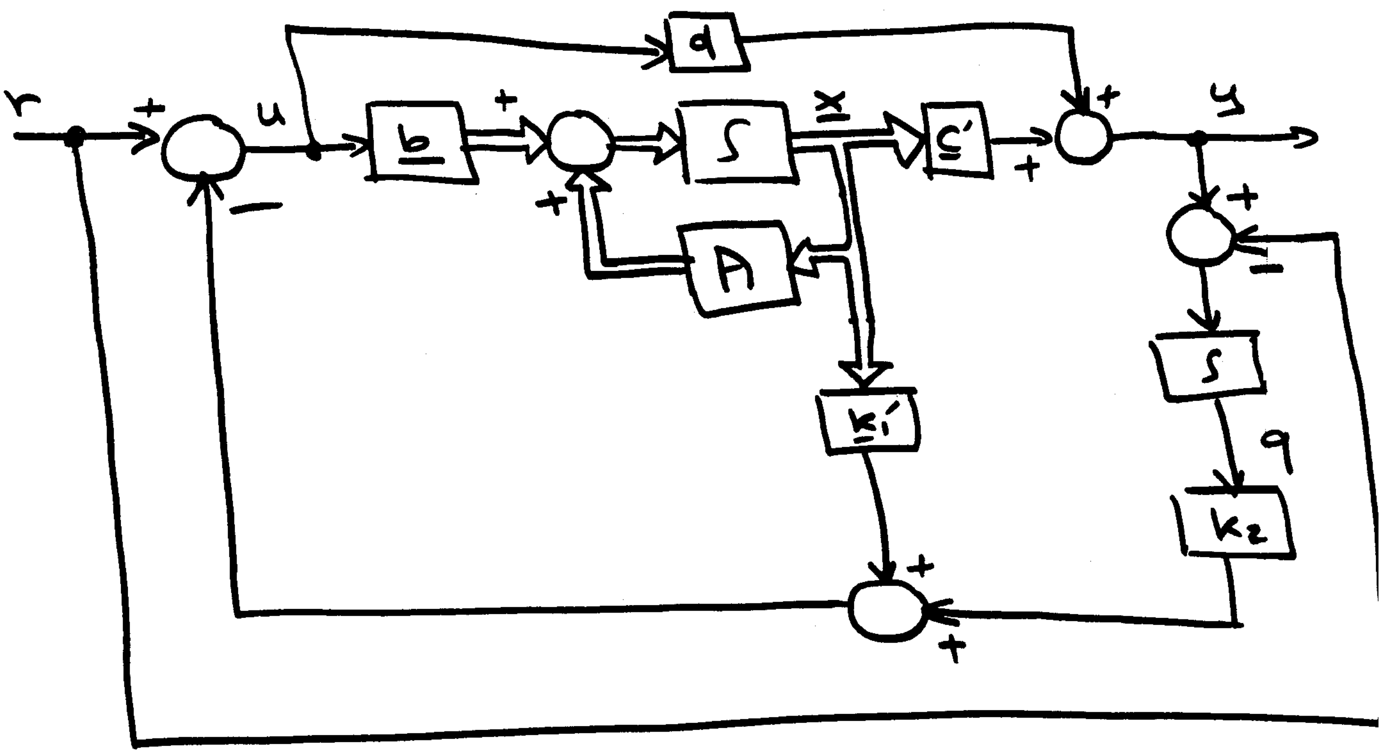
$$\tilde{\underline{b}}_{CL} = \left[\begin{array}{c} \underline{b} \\ \hdashline (d-1) \end{array} \right] \Leftarrow \text{modified}$$

$$\tilde{\underline{c}}'_{CL} \equiv \underline{c}'_{CL} \quad ; \quad \tilde{d}_{CL} \equiv d_{CL}$$

$$\Rightarrow \quad \dot{q}_{ss} = \emptyset \quad \Rightarrow \quad y_{ss} \equiv r_{ss}$$

$$\Rightarrow \boxed{G_{CL}(\emptyset) \equiv 1}$$

independent of $\underline{k}'$



This design will guarantee a DC-gain of 1 which is completely insensitive to changes in system parameters (as long as the system remain stable.)

Example: (continued)

$$\dot{\underline{\xi}} = \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -4 & 5 & -6 & 0 \\ 7 & -8 & 9 & 0 \\ \hline 1 & 1 & 1 & 0 \end{array}\right] \underline{\xi} + \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ \hline 3 \end{array}\right] u$$

$$y = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \end{array}\right] \underline{\xi} + [3] u$$

is the augmented open-loop system.

We choose:

$$\underline{p} = [-2;\ -1+j;\ -1-j;\ -4]$$

$$\Rightarrow \underline{k}' = [19.0648 \quad -23.7222 \quad 28.6574 \quad -0.3333]$$

$$A_{CL} = \begin{bmatrix} -18.0648 & 21.7222 & -25.6574 & 0.3333 \\ -23.0648 & 28.7222 & -34.6574 & 0.3333 \\ -12.0648 & 15.7222 & -19.6574 & 0.3333 \\ -56.1944 & 72.1667 & -84.9722 & 1 \end{bmatrix}$$

has the desired pole locations.

$$\underline{b}_{CL} = \underline{b}_{OL} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix};\quad d_{CL} = d_{OL} = [3]$$

$$\underline{c}'_{CL} = [-56.1944 \quad 72.1667 \quad -84.9722 \quad 1]$$

$$\Rightarrow \quad \underline{\underline{G_{CL}(\phi)}} = -\underline{c}'_{CL} A_{CL}^{-1} \underline{b}_{CL} + d_{CL} \underline{\underline{\equiv \phi}}$$

(as expected)

Modification: $\tilde{\underline{b}}_{CL} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \Leftarrow$ modified

$$\Rightarrow \underline{\underline{\tilde{G}_{CL}(\phi)}} = -\underline{c}'_{CL} A_{CL}^{-1} \tilde{\underline{b}}_{CL} + d_{CL} \underline{\underline{\equiv 1}} \quad \checkmark$$

- Now, we do the same type of variation:

$$\underline{k}'_r = [19.2923 \quad -23.6252 \quad 28.7541 \quad -\phi.2428$$

$$\Rightarrow A_{CL_r} = \begin{bmatrix} -18.2923 & 21.6252 & -25.7541 & \phi.2428 \\ -23.2923 & 28.6252 & -34.7541 & \phi.2428 \\ -12.2923 & 15.6252 & -19.7541 & \phi.2428 \\ -56.877\phi & 71.8756 & -85.2622 & \phi.7285 \end{bmatrix}$$

$$\underline{c}'_{CL_r} = [-56.877\phi \quad 71.8756 \quad -85.2622 \quad \phi.7285]$$

$$\Rightarrow \underline{\underline{\tilde{G}_{CL_r}(\phi) \equiv 1}} \quad \checkmark$$

Of course, the eigenvalues have changed again:

$$\mathrm{Eig}(A_{cl_r}) = \begin{bmatrix} -0.1349 \pm 1.1258\,j \\ -1.267 \\ -7.1559 \end{bmatrix}$$

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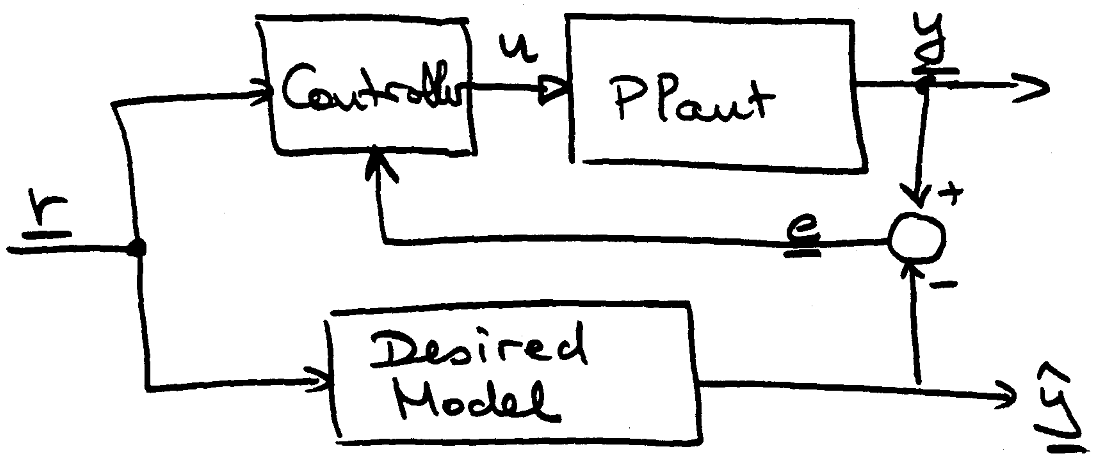
will have a considerable overshoot, but at least, the system remained stable (no guarantee !!!).

THIS TECHNIQUE DOES <u>NOT</u> HELP WITH DESENSITIZING THE DYNAMIC PROPERTIES, ONLY STEADY-STATE BEHAVIOR.

<u>Solution</u>: $\Longrightarrow$ $\boxed{\text{Model-Reference Adaptive Control}}$

(too advanced for now $\Rightarrow$ ECE 548)

General Idea :

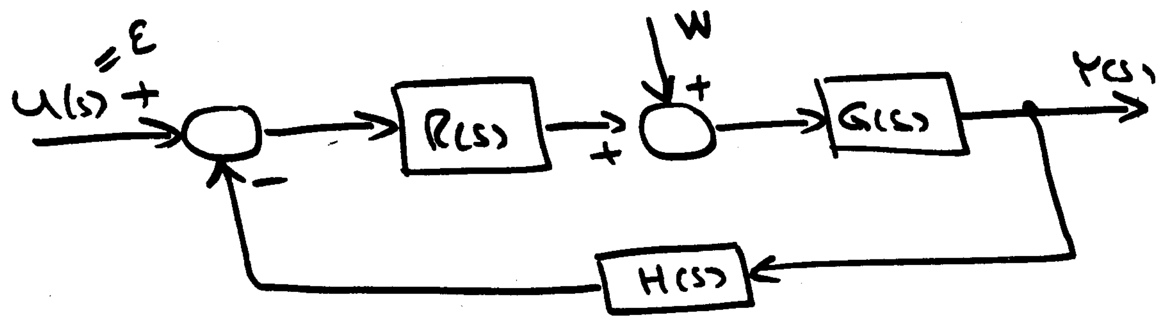


We modify a controller such that the plant + controller behaves like a desired model ($\underline{e} \to \emptyset$). As long as the desired model does not change, any changes in the plant or controller will be corrected at once. Unfortunately, the controller ends up nonlinear $\Rightarrow$ too complicated for now !!

Suppression of Disturbances:

Problem: Let us assume we have an additive constant but unknown and uninfluenceable disturbance on our system (e.g. the headwind of an aircraft). How can we guarantee that the output of the system remains biasfree?

- Remember from ECE 441:



→ Disturbances are ideally suppressed in the steady state if:

$$H(\emptyset) \equiv 1$$

and $\text{Type}\{R(s)\} > \text{Type}\{G(s)\}$.

[For reasons of stability:

$$\text{Type}\{G(s)\} = \emptyset$$
$$\text{Type}\{R(s)\} = 1 \quad .]$$

<u>Question</u>: Can we find a similar design using state-feedback?

- We notice that we can treat the discontinuity as <u>another input</u> to the system:

$$\underline{\dot{x}} = A\underline{x} + \underline{b}u + \underline{g} \cdot w$$

$\uparrow$ disturb

$$|||$$

$$\underline{\dot{x}} = A\underline{x} + B\,\underline{u}$$

where: $\underline{u} = \begin{bmatrix} u \\ w \end{bmatrix}$

and: $B = [\underline{b}, \underline{\gamma}]$

In the frequency domain:

$$\underline{G}(s) = \underline{c}'(sI-A)^{-1}B + \underline{d}'$$

$$\Rightarrow Y(s) = \boxed{G_u(s) \,\vdots\, G_w(s)} \cdot \begin{bmatrix} U(s) \\ \hline W(s) \end{bmatrix}$$

Due to the superposition principle, we can deal with each input separately:

$\Rightarrow$ Steady-state accuracy: $G_u(\phi) = 1$

Disturbance suppression: $G_w(\phi) = \phi$

- Let us look at the closed-loop system with state-feedback:

$$\left| \begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}u + \underline{\gamma}\cdot w \\ y &= \underline{c}'\underline{x} + du \\ u &= r - \underline{k}'\underline{x} \end{aligned} \right|$$

$$\Rightarrow \left| \begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}r - \underline{b}\underline{k}'\underline{x} + \underline{\gamma}w \\ y &= \underline{c}'\underline{x} + dr - d\underline{k}'\underline{x} \end{aligned} \right|$$

$$\Rightarrow \left| \begin{aligned} \dot{\underline{x}} &= \left[(A - \underline{b}\underline{k}')\right]\underline{x} + \left[\underline{b}, \underline{\gamma}\right]\cdot\begin{bmatrix} r \\ w \end{bmatrix} \\ y &= \left[(\underline{c}' - d\underline{k}')\right]\underline{x} + \left[d, 0\right]\cdot\begin{bmatrix} r \\ w \end{bmatrix} \end{aligned} \right|$$

$$Q_u(s) = \det(sI - A + \underline{b}\underline{k}')$$

$$\equiv Q_W(s)$$

$\Rightarrow$ The poles of $G_u(s)$ and $G_W(s)$ are the same.

Only the zeroes are different.

<u>Solution</u>: Let us try the same <u>Integral Feedback Design</u> as in the previous section:

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underline{b}u + \underline{\gamma}w \\ y = \underline{c}'\underline{x} + du \\ \dot{q} = y - r \\ u = r - \underline{k}_1'\underline{x} - k_2 q \end{array} \right|$$

For analysis of the disturbance suppression, we can assume the other input (r) to be zero.

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underline{b}u + \underline{\gamma}w \\ y_w = \underline{c}'\underline{x} + du \\ \dot{q} = y_w \\ u = -\underline{k}_1'\underline{x} - k_2 q \end{array} \right|$$

We know already that we can make the system stable by pole placement.

$$\Rightarrow \dot{q}_{ss} = \phi \Rightarrow \underline{\underline{y_{W_{ss}} \equiv \phi}}$$

$\Rightarrow$ The disturbances are ideally suppressed at steady-state.

- We notice again that this technique will filter out disturbances only in the steady-state, and <u>not</u> during the transient period.
- We realize also that

$$\boxed{w = \text{constant}}$$

was assumed from the beginning.

• Very often, we have a different problem to solve:

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underline{b}\,u + \underline{\gamma}\,w \\ y = \underline{c}'\underline{x} + du + v \end{array} \right|$$

where: $\underline{\gamma}\,w$ is the plant error. w is assumed to be a Gaussian white noise, and $\underline{\gamma}$ is the "coloring"-vector. v is the measurement error which is also assumed to be Gaussian white noise.

Idea: We build an observer that optimally reconstructs $\underline{x}$ out of measurements of

u and y.

$$\Longrightarrow \left| \begin{array}{l} \dot{\underline{\hat{x}}} = A\underline{\hat{x}} + \underline{b}u + \underline{h}(y-\hat{y}) \\ \hat{y} = \underline{c}'\underline{x} + du \end{array} \right|$$

is such an observer.

$$\underline{\hat{x}} \text{ observes } \underline{x}$$

$$\hat{v} = y - \hat{y} \text{ observes } v$$

and: $(\widehat{\underline{g}w}) = \dot{\underline{e}} = \dot{\underline{x}} - \dot{\underline{\hat{x}}}$ observes $(\underline{g}w)$

- In this context, the "observer" is usually referred to as a filter, and the variables $\underline{\hat{x}}$, $\hat{v}$, and $(\widehat{\underline{g}w})$ are referred to as estimators.

An "optimal" $\underline{h}$ can be found by solving a Riccati equation, where:

$$Q = \mathrm{Var}\{w\} \; ; \; R = \mathrm{Var}\{v\}$$

and $N = \mathrm{Cov}\{w, v\}$

(without proof)

This design is referred to as a <u>Kalman / Bucy – filter</u>.

[> $\underline{h} = \mathrm{LQE}(A, \underline{g}, \underline{c}', Q, R, N)$

will find the optimal filter-feedback vector.

$\Longrightarrow$ Problems of <u>stochastic control</u> (no class currently available!)