Multi-Input / Multi-Output (MIMO) Systems

$$\left|\begin{array}{l} \underline{\dot{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} + D\underline{u} \end{array}\right| \qquad \begin{array}{l} \underline{x} \in \mathbb{R}^n \\ \underline{u} \in \mathbb{R}^m \\ \underline{y} \in \mathbb{R}^p \end{array}$$

$$\begin{array}{ccc} & \begin{array}{cc} \;n\; & \;m\; \end{array} & \\ S = \begin{array}{c} n \\ p \end{array} & \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] & \begin{array}{c} n \\ p \end{array} \\ & \begin{array}{cc} \;n\; & \;m\; \end{array} & \end{array}$$

n :. # of states
m :. # of inputs
p :. # of outputs

Transformation into the Frequency domain:

$$\left|\begin{array}{l} s\underline{X} = A\underline{X} + B\underline{U} \\ \underline{Y} = C\underline{X} + D\underline{U} \end{array}\right|$$

$$\Rightarrow (sI - A)\underline{X} = B\underline{U}$$

$$\Rightarrow \quad \underline{X}(s) = (sI-A)^{-1} B \underline{U}(s)$$

$$\Rightarrow \quad \underline{Y}(s) = \underbrace{\left[C(sI-A)^{-1} B + D \right]}_{\underline{\underline{G}}(s)} \underline{U}(s)$$

$$\boxed{\underline{\underline{G}}(s) = C(sI-A)^{-1} B + D}$$

is now a rational function matrix with p rows and m columns.

Example:

$$\left| \begin{array}{l} \underline{\dot{x}} = \begin{bmatrix} \phi & 1 \\ -2 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix} \underline{u} \\ \\ \underline{y} = \begin{bmatrix} 5 & 6 \\ 1 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} \phi & \phi \\ \phi & 1 \end{bmatrix} \underline{u} \end{array} \right|$$

$$(sI-A) = \begin{bmatrix} s & -1 \\ 2 & (s+3) \end{bmatrix}$$

$$\Rightarrow (sI-A)^{+} = \begin{bmatrix} (s+3) & 1 \\ -2 & s \end{bmatrix}$$

$$|sI-A| = s(s+3)+2 = s^2+3s+2$$
$$= (s+1)(s+2) \; ;$$

$$\Rightarrow (sI-A)^{-1} \quad \frac{\begin{bmatrix} (s+3) & 1 \\ -2 & s \end{bmatrix}}{(s+1)(s+2)}$$

$$\Rightarrow \quad (sI-A)^{-1}B = \frac{\begin{bmatrix} (s+3) & 1 \\ -2 & s \end{bmatrix}}{(s+1)(s+2)}$$

$$\Rightarrow \quad C(sI-A)^{-1}B = \frac{\begin{bmatrix} (5s+3) & (6s+5) \\ (s+7) & (-2s+1) \end{bmatrix}}{(s+1)(s+2)}$$

$$\Rightarrow \underline{\underline{\underset{\sim}{G}(s) = C(sI-A)^{-1}B+D}} = \underline{\underline{\frac{\begin{bmatrix} (5s+3) & (6s+5) \\ (s+7) & (s^2+s+3) \end{bmatrix}}{(s+1)(s+2)}}}$$

$\Rightarrow$ The smallest common multiple of all denominators can be taken out as a "global" denominator which is somewhat similar to the characteristic polynomial of the SISO-case.

<u>Warning</u>: The system order is no longer necessarily equivalent to the order of this global denominator polynomial.

<u>Warning:</u> Pole/zero-cancellations do no longer necessarily indicate problems with con=trollability and/or observability.

<u>Example:</u>

$$\dot{\underline{x}} = \left[\begin{array}{cc|c} -1 & \phi & \phi \\ \phi & -1 & \phi \\ \hline \phi & \phi & -2 \end{array}\right] \underline{x} + \left[\begin{array}{cc} 1 & \phi \\ \phi & 1 \\ \hline 1 & 1 \end{array}\right] \underline{u}$$

$$\underline{y} = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & \phi & \phi \end{array}\right] \underline{x}$$

$$Q_c = \left[B \;\; AB \;\; A^2B \right]$$

$$= \left[\begin{array}{cc|cc|cc} 1 & \phi & -1 & \phi & 1 & \phi \\ \phi & 1 & \phi & -1 & \phi & 1 \\ 1 & 1 & -2 & -2 & 4 & 4 \end{array}\right]$$

$$\Rightarrow Q_c \cdot Q_c^* = \begin{bmatrix} 3 & \phi & 7 \\ \phi & 3 & 7 \\ 7 & 7 & 42 \end{bmatrix}$$

$$\Rightarrow \det(Q_c \cdot Q_c^*) = 84 \neq \phi$$

$$\Rightarrow \text{Rank}(Q_c) = 3$$

$\Rightarrow$ System is controllable.

$$Q_o = \begin{bmatrix} 1 & 2 & 1 \\ -1 & \phi & \phi \\ \hdashline -1 & -2 & -2 \\ 1 & \phi & \phi \\ \hdashline 1 & 2 & 4 \\ -1 & \phi & \phi \end{bmatrix} \Rightarrow Q_o^* \cdot Q_o = \begin{bmatrix} 6 & 6 & 7 \\ 6 & 12 & 14 \\ 7 & 14 & 21 \end{bmatrix}$$

$$\Rightarrow \det(Q_o^* \cdot Q_o) = 168 \neq \phi$$

$$\Rightarrow \text{Rank}(Q_o) = 3$$

$\Rightarrow$ System is observable.

$\Rightarrow$ There seem to be two Jordan blocks associated with the pole at -1. Yet, there is neither an uncontrollable nor an unobservable mode in this system.

<u>Characteristic Polynomial</u>:

$$\underline{Q(\lambda)} = \det(\lambda I - A) = \underline{\underline{(\lambda+1)^2(\lambda+2)}}$$

- Let us find the transfer function matrix:

$$(sI-A) = \begin{bmatrix} (s+1) & \phi & \phi \\ \phi & (s+1) & \phi \\ \phi & \phi & (s+2) \end{bmatrix}$$

$$\Rightarrow (sI-A)^{-1} = \begin{bmatrix} \frac{1}{s+1} & \phi & \phi \\ \phi & \frac{1}{s+1} & \phi \\ \phi & \phi & \frac{1}{s+2} \end{bmatrix}$$

$$\Rightarrow (sI-A)^{-1}B = \begin{bmatrix} \frac{1}{s+1} & \phi \\ \phi & \frac{1}{s+1} \\ \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

$$\Rightarrow \underline{G}(s) = C(sI-A)^{-1}B = \begin{bmatrix} \left(\frac{1}{s+1} + \frac{1}{s+2}\right) & \left(\frac{2}{s+1} + \frac{1}{s+2}\right. \\ \frac{-1}{s+1} & \phi \end{bmatrix}$$

$$\Rightarrow \underline{\underline{G}}(s) = \begin{bmatrix} \frac{2s+3}{(s+1)(s+2)} & \frac{3s+5}{(s+1)(s+2)} \\ \frac{-1}{s+1} & \phi \end{bmatrix}$$

$$\Rightarrow \underline{\underline{G}}(s) = \frac{\begin{bmatrix} (2s+3) & (3s+5) \\ (-s-2) & \phi \end{bmatrix}}{(s+1)(s+2)}$$

⇒ • The transfer function matrix is of second order, yet this is a third order system.

• The "global" denominator ≠ characteristic polynomial:

$$\underline{\underline{Q(s) = (s+1)(s+2)}}$$

→ • A pole/zero cancellation has taken place without indicating a problem with controllability/observability.

In Matlab:

```
S = ss(A,B,C,D);
G = tf(S)
```

produces:

$$\underline{\underline{G}}(s) = \frac{\begin{bmatrix} (2s^2+5s+3) & (3s^2+8s+5) \\ (-s^2-3s-2) & \phi \end{bmatrix}}{s^3+4s^2+5s+2}$$

which is the correct transfer function matrix prior to pole/zero cancellation.

```
g11 = G(1,1)
```

produces:

$$g_{11}(s) = \frac{2s^2+5s+3}{s^3+4s^2+5s+2}$$

We can extract the numerator and denominator polynomial

Coefficients of $g_{11}(s)$ as vectors using:

[p11, q11] = tfdata(g11, 'v')

producing:

p11 = [∅ 2 5 3]
q11 = [1 4 5 2]

Then:

rp11 = roots(p11)
rq11 = roots(q11)

gives: $rp11 = \begin{bmatrix} -1.5 \\ -1 \end{bmatrix}$; $rq11 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$

There is a common pole/zero at -1.

r = [1 1] is the polynomial (s+1)

p11 = poly(rp11) gets the coefficient vector back without front zero.

p11 = deconv(p11, r) divides common pole out

Notice: This system has 3 poles @

$2 * (-1)$
$1 * (-2)$

The system has 6 zeroes @

$3 * (-1)$
$1 * (-2)$
$1 * (-1.5)$
$1 * (-1.6667)$

⇒ • MIMO systems can have more zeroes than poles, and yet be strictly proper.

• MIMO systems can have poles and zeroes at the same place which may or may not cancel away, and yet be perfectly controllable and observable.

Gilbert's Diagonal Realization:

Problem: Given a transfer function matrix, find a diagonal realization (corresponding to a Jordan-canonical form).

$\Rightarrow$ Use partial fraction expansion:

Example:

$$\underline{\underline{G}}(s) = \begin{bmatrix} \dfrac{2s+3}{s^2+3s+2} & \dfrac{3s+5}{s^2+3s+2} \\ \dfrac{-1}{s+1} & \emptyset \end{bmatrix}$$

$$\Rightarrow \underline{\underline{G}}(s) = \frac{R_1}{s+1} + \frac{R_2}{s+2}$$

$$R_1 = \lim_{s\to -1} (s+1)\underline{\underline{G}}(s) = \lim_{s\to -1} \begin{bmatrix} \dfrac{2s+3}{s+2} & \dfrac{3s+5}{s+2} \\ -1 & \emptyset \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & \emptyset \end{bmatrix}$$

$$R_2 = \lim_{s\to -2} (s+2)\underline{\underline{G}}(s) = \lim_{s\to -2} \begin{bmatrix} \dfrac{2s+3}{s+1} & \dfrac{3s+5}{s+1} \\ \dfrac{-(s+2)}{s+1} & \emptyset \end{bmatrix} = \begin{bmatrix} 1 & \\ \emptyset & \end{bmatrix}$$

$$\Rightarrow \underline{\underline{G}}(s) = \frac{\begin{bmatrix} 1 & 2 \\ -1 & \phi \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 1 & 1 \\ \phi & \phi \end{bmatrix}}{s+2}$$

$R_1 = C_1 B_1$

$\mathrm{Rank}(R_1) = 2 \Longrightarrow$ make e.g. $\begin{cases} B_1 = I^{(2)} \\ C_1 \equiv R_1 \end{cases}$

$$\underline{\underline{B_1 = \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix}}} \quad ; \quad \underline{\underline{C_1 = \begin{bmatrix} 1 & 2 \\ -1 & \phi \end{bmatrix}}}$$

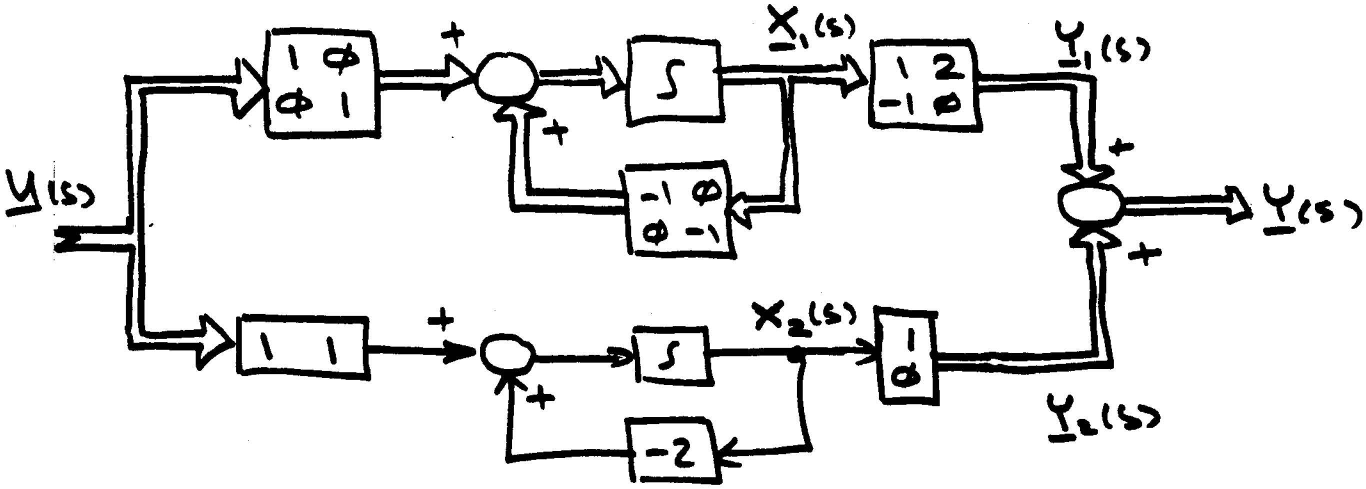
$R_2 = C_2 B_2$

$\mathrm{Rank}(R_2) = 1 \Longrightarrow$ e.g.

$$\begin{bmatrix} 1 & 1 \\ \phi & \phi \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ \phi \end{bmatrix}}_{C_2} \cdot \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{B_2}$$

$$\underline{Y}(s) = \underline{Y}_1(s) + \underline{Y}_2(s)$$

$$\underline{Y}_1(s) = \frac{\begin{bmatrix} 1 & 2 \\ -1 & \phi \end{bmatrix}}{s+1} \cdot \underline{U}(s) \; ; \quad \underline{Y}_2(s) = \frac{\begin{bmatrix} 1 & 1 \\ \phi & \phi \end{bmatrix}}{s+2} \cdot \underline{U}$$



$$\Rightarrow \underline{x} = \begin{bmatrix} \underline{x}_1 \\ x_2 \end{bmatrix}$$

$$\dot{\underline{x}} = \left[\begin{array}{cc|c} -1 & \phi & \phi \\ \phi & -1 & \phi \\ \hline \phi & \phi & -2 \end{array}\right] \underline{x} + \left[\begin{array}{cc} 1 & \phi \\ \phi & 1 \\ \hline 1 & 1 \end{array}\right] \underline{u}$$

$$\underline{y} = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & \phi & \phi \end{array}\right] \underline{x}$$

$$\dot{\underline{x}} = \begin{bmatrix} A_1 & & \phi \\ & A_2 & \\ \phi & & \ddots \; A_k \end{bmatrix} \underline{x} + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_k \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} C_1 & C_2 & \cdots & C_k \end{bmatrix} \underline{x}$$

Multiple poles of $\underline{\underline{G}}(s)$:

Example:

$$\underline{\underline{G}}(s) = \begin{bmatrix} \dfrac{1}{(s+1)^2} & \dfrac{2}{(s+2)} \\ \dfrac{1}{(s+1)} & \dfrac{4}{(s+1)^2(s+2)} \end{bmatrix}$$

$$\equiv \frac{R_{11}}{s+1} + \frac{R_{12}}{(s+1)^2} + \frac{R_2}{s+2}$$

$$R_2 = \lim_{s \to -2} (s+2)\underline{\underline{G}}(s) = \lim_{s \to -2} \begin{bmatrix} \dfrac{s+2}{(s+1)^2} & 2 \\ \dfrac{s+2}{s+1} & \dfrac{4}{(s+1)^2} \end{bmatrix} = \begin{bmatrix} \phi \\ \phi \end{bmatrix}$$

$\Rightarrow$ Rank$(R_2) = 1$

$$\Rightarrow \begin{bmatrix} \phi & 2 \\ \phi & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}_{C_2} \cdot \underbrace{[\phi \;\; 1]}_{B_2}$$

$$R_{12} = \lim_{s \to -1} (s+1)^2 \underline{\underline{G}}(s) = \lim_{s \to -1} \begin{bmatrix} 1 & \dfrac{2(s+1)^2}{s+2} \\ (s+1) & \dfrac{4}{s+2} \end{bmatrix} = \begin{bmatrix} 1 \\ \phi \end{bmatrix}$$

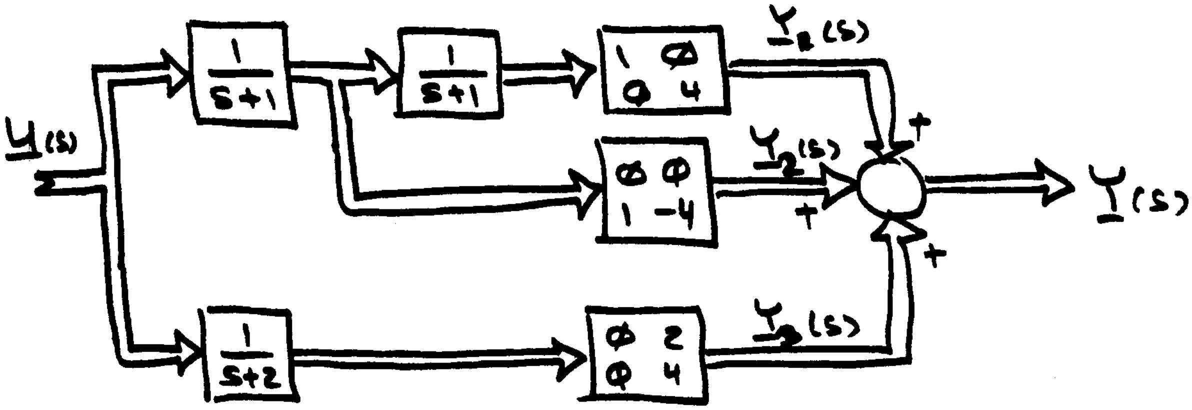
$\Rightarrow$ Rank$(R_{12}) = 2$

$$\Rightarrow B_2 = \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix} \;\; ; \;\; C_2 = \begin{bmatrix} 1 & \phi \\ \phi & 4 \end{bmatrix}$$

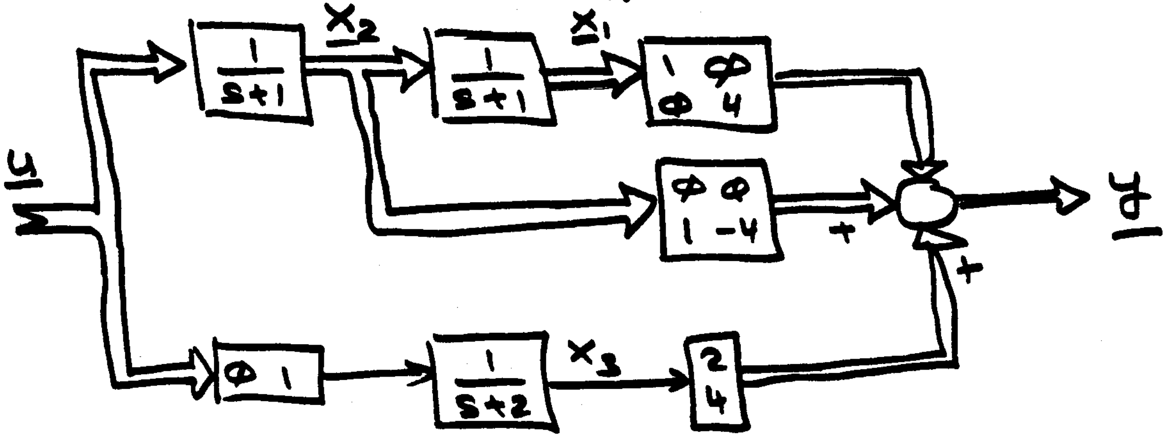
$$R_{11} = \lim_{s \to -1} \frac{d}{ds}\left\{(s+1)^2 \underline{\underline{G}}(s)\right\} = \lim_{s \to -1} \begin{bmatrix} 0 & \dfrac{4(s+1)(s+2) - 2(s+1)^2}{(s+2)^2} \\ 1 & \dfrac{-4}{(s+2)^2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & -4 \end{bmatrix}$$

$$\Rightarrow \underline{\underline{G}}(s) = \frac{\begin{bmatrix} 0 & 0 \\ 1 & -4 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}}{(s+1)^2} + \frac{\begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}}{s+2}$$



$$\equiv$$



$$\Rightarrow \left|\begin{aligned} \dot{\underline{x}}_1 &= -\underline{x}_1 + \underline{x}_2 \\ \dot{\underline{x}}_2 &= -\underline{x}_2 + \underline{u} \\ \dot{x}_3 &= -2x_3 + \begin{bmatrix} 0 & 1 \end{bmatrix}\underline{u} \\ \underline{y} &= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\underline{x}_1 + \begin{bmatrix} 0 & 0 \\ 1 & -4 \end{bmatrix}\underline{x}_2 + \begin{bmatrix} 2 \\ 4 \end{bmatrix}x_3 \end{aligned}\right|$$

with: $\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow \left|\begin{aligned} \dot{\underline{x}} &= \left[\begin{array}{cc:cc:c} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ \hdashline 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -2 \end{array}\right]\underline{x} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hdashline 1 & 0 \\ 0 & 1 \\ \hdashline 0 & 1 \end{array}\right]\underline{u} \\ \underline{y} &= \left[\begin{array}{cc:cc:c} 1 & 0 & 0 & 0 & 2 \\ 0 & 4 & 1 & -4 & 4 \end{array}\right]\underline{x} \end{aligned}\right|$$

Verification shows that
Rank$(Q_c) = 5 \Rightarrow$ system controllable
Rank$(Q_o) = 5 \Rightarrow$ system observable
$\Rightarrow$ This is a minimal realization.

- The superdiagonal 1-elements of the Jordan blocks are replaced by superdiagonal unity matrices.

- Even though $\text{Rank}(R_{11}) = 1$
  $\Longrightarrow$ we needed the full representation. Here, this worked just fine, but we can no longer <u>guarantee</u> that Gilbert's realization is minimal.

<u>Transformation to Jordan form:</u>

In our algorithm from the SISO-case, we did not make anywhere use of the fact that our system was of type SISO
$\Longrightarrow$ the algorithm must still work.

<u>Example:</u> We use the example that we just used above:

$$\dot{\underline{x}} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 4 & 1 & -4 & 4 \end{bmatrix} \underline{x}$$

$$\Rightarrow \quad \lambda_1 = -1 \quad ; \quad m_1 = 4$$
$$\lambda_2 = -2 \quad ; \quad m_2 = 1$$

$$\Rightarrow (A - \lambda_1 I) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} = A^{(1)}$$

$$\Rightarrow \rho_1 = 3 \quad ; \quad \nu_1 = 2$$

$$\Rightarrow (A - \lambda_1 I)^2 = A^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \rho_2 = 1 \quad ; \quad \underline{\underline{\nu_2 = 4}} \quad \Rightarrow \underline{\underline{p_k = 2}}$$

$\Rightarrow$ There are two Jordan-blocks each of size 2 associated with λ_1.

Find the generalized eigenvectors by QR-decomposition:

[> $[q_2, r_2] = QR(a2')$

$$q_2 = \begin{bmatrix} 1 & & & & \\ & 1 & & \phi & \\ & & 1 & & \\ & \phi & & 1 & \\ & & & & 1 \end{bmatrix} ; \quad r_2 = \begin{bmatrix} \phi & & & & \phi \\ & \phi & & & \\ & & \phi & & \\ & \phi & & \phi & \\ & & & & 1 \end{bmatrix}$$

$$\Rightarrow i_2 = \begin{bmatrix} \phi \\ \phi \\ \phi \\ \phi \\ 1 \end{bmatrix} ; \quad n_2 = \begin{bmatrix} 1 & \phi & \phi & \phi \\ \phi & 1 & \phi & \phi \\ \phi & \phi & 1 & \phi \\ \phi & \phi & \phi & 1 \\ \phi & \phi & \phi & \phi \end{bmatrix}$$

Warning: Not always can 6 push the ze rows to the e

[> $[q_1, r_1] = QR(a1')$

$$q_1 = \begin{bmatrix} \phi & \phi & -1 & \phi & \phi \\ \phi & \phi & \phi & -1 & \phi \\ -1 & \phi & \phi & \phi & \phi \\ \phi & -1 & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & 1 \end{bmatrix} ; \quad r_1 = \begin{bmatrix} -1 & \phi & \phi & \phi & \phi \\ \phi & -1 & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi & -1 \end{bmatrix}$$

$\Rightarrow$ [> $i_1 = q_1(:, [1,2,5]) = \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ -1 & \phi & \phi \\ \phi & -1 & \phi \\ \phi & \phi & 1 \end{bmatrix}$

[> $n_1 = q_1(:, [3,4]) = \begin{bmatrix} -1 & \phi \\ \phi & -1 \\ \phi & \phi \\ \phi & \phi \\ \phi & \phi \end{bmatrix}$

$\Rightarrow$ [> $x = [i_2, n_1] = \begin{bmatrix} \phi & -1 & \phi \\ \phi & \phi & -1 \\ \phi & \phi & \phi \\ \phi & \phi & \phi \\ 1 & \phi & \phi \end{bmatrix}$; $x = [x, \text{zeros}(5,2)]$

$\Rightarrow$ [> [qq, rr] = QR(x)

$$\Rightarrow qq = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} ; \quad rr = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \underline{v}_2 = qq(:,4) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \underline{v}_1 = a_1 * \underline{v}_2 ;$$

$$\underline{v}_4 = qq(:,5) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v}_3 = a_1 * \underline{v}_4 ;$$

$\Rightarrow$ [> a1 = a + 2*EYE(a);

[> [q, r] = QR(a1);

[> v_5 = q(:,5);

[> V = [v_1, v_2, v_3, v_4, v_5]

$$\Rightarrow V = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ [> T = INV(v)

$$\Rightarrow T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ [> AH = T*A/T

$$\Rightarrow AH = \left[\begin{array}{cc|cc|c} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

[> BH = T*B $\Rightarrow$ $BH = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$

[> CH = C/T $\Rightarrow$ $CH = \begin{bmatrix} 0 & 0 & -1 & 0 & 2 \\ 4 & -4 & 0 & -1 & 4 \end{bmatrix}$

$\Rightarrow$ Jordan-canonical form:

$$\dot{\underline{x}} = \left[\begin{array}{cc:cc:c} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -2 \end{array}\right] \underline{x} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ \hdashline 0 & 0 \\ -1 & 0 \\ \hdashline 0 & 1 \end{array}\right] \underline{u}$$

$$\underline{y} = \left[\begin{array}{cc:cc:c} 0 & 0 & -1 & 0 & 2 \\ 4 & -4 & 0 & -1 & 4 \end{array}\right] \underline{x}$$

looks as in the SISO-case.

Notice: In case of multiple poles in the transfer function matrix, Gilbert's "diagonal" realization and the Jordan-canonical form look slightly different

Notice: Multiple Jordan blocks do no longer necessarily indicate problems with controllability and/or observability.

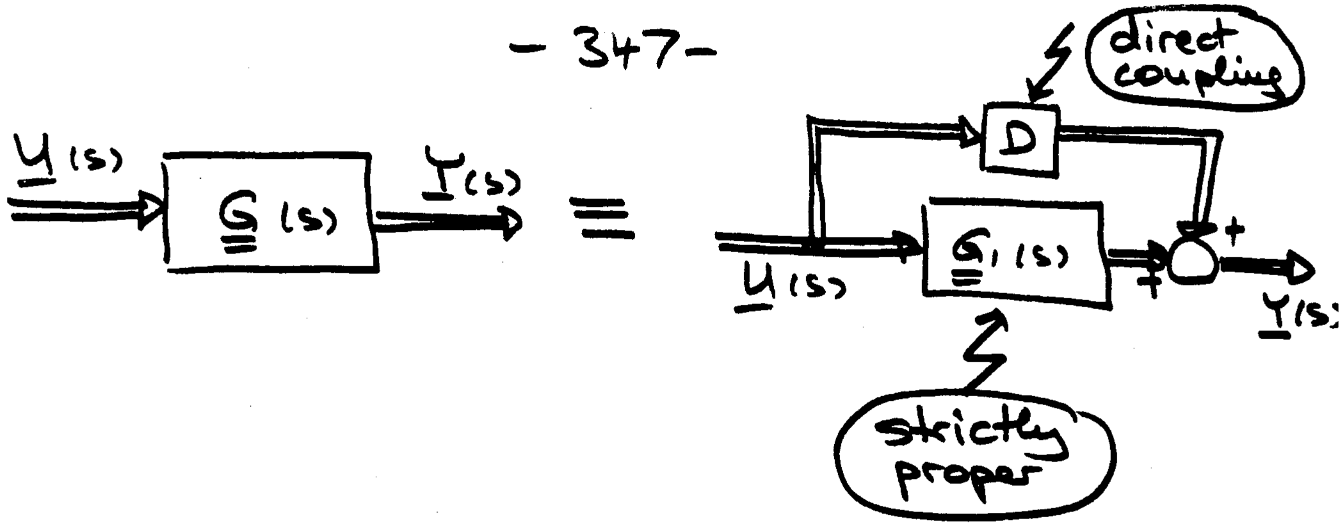
$D \neq \phi$:

Gilbert's technique works only for strictly proper transfer functio matrices. $\Rightarrow$ Decouple the direct coupling first.

Example: $\underline{\underline{G}}(s) = \begin{bmatrix} \frac{s+1}{s+2} & -5 \\ \frac{1}{s+1} & \frac{2(s+7)}{s+1} \end{bmatrix}$

$$\Rightarrow \underline{\underline{G}}(s) = \begin{bmatrix} \left(1 + \frac{-1}{s+2}\right) & -5 \\ \frac{1}{s+1} & \left(2 + \frac{12}{s+1}\right) \end{bmatrix}$$

$$\Rightarrow \underline{\underline{G}}(s) = \underbrace{\begin{bmatrix} \frac{-1}{s+2} & \phi \\ \frac{1}{s+1} & \frac{12}{s+1} \end{bmatrix}}_{\underline{\underline{G}}_1(s)} + \underbrace{\begin{bmatrix} 1 & -5 \\ \phi & 2 \end{bmatrix}}_{D}$$



- Now, the Gilbert algorithm can be applied to $\underline{\underline{G}}_1(s) \Longrightarrow \{A, B, C\}$, and D can finally be added again

Example (continued):

$$\underline{\underline{G}}_1(s) = \frac{\begin{bmatrix} 0 & 0 \\ 1 & 12 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}}{s+2}$$

$$R_1 = \begin{bmatrix} 0 & 0 \\ 1 & 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 12 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \end{bmatrix}$$

$$\Rightarrow \left| \begin{array}{l} \underline{\dot{x}} = \left[\begin{array}{c:c} -1 & 0 \\ \hdashline 0 & -2 \end{array}\right] \underline{x} + \begin{bmatrix} 1 & 12 \\ \hdashline -1 & 0 \end{bmatrix} \underline{u} \\ \underline{y} = \left[\begin{array}{c:c} 0 & 1 \\ 1 & 0 \end{array}\right] \underline{x} \end{array} \right|$$

⇒ The total representation of this system is:

$$\dot{\underline{x}} = \begin{bmatrix} -1 & \phi \\ \phi & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 12 \\ -1 & \phi \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} \phi & 1 \\ 1 & \phi \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & -5 \\ \phi & 2 \end{bmatrix} \underline{u}$$

Blockdiagonal Realizations:

(a) A controller-canonical realization:

Example:

$$\underline{\underline{G}}(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{2}{(s+2)} \\ \frac{1}{(s+1)} & \frac{4}{(s+1)^2(s+2)} \end{bmatrix}$$

$$\underline{Y}(s) = \underline{\underline{G}}(s) \cdot \underline{U}(s) \equiv \underline{G}_1(s) \cdot U_1(s) + \underline{G}_2(s) \cdot U_2(s)$$

$$\Rightarrow \underline{Y}(s) = \begin{bmatrix} \frac{1}{(s+1)^2} \\ \frac{1}{(s+1)} \end{bmatrix} \cdot U_1(s) + \begin{bmatrix} \frac{2}{(s+2)} \\ \frac{4}{(s+1)^2(s+2)} \end{bmatrix} \cdot U_2(s)$$