$$\Rightarrow \quad \underline{Y}(s) = \underline{Y}_1(s) + \underline{Y}_2(s)$$

where:
$$\left| \begin{array}{l} \underline{Y}_1(s) = \underline{G}_1(s) \cdot U_1(s) \\ \underline{Y}_2(s) = \underline{G}_2(s) \cdot U_2(s) \end{array} \right|$$

are two SIMO – systems.

Each of these subsystems can be immediately realized, e.g. in controller-canonical form:

$$\underline{Y}_1(s) = \underline{G}_1(s) \cdot U_1(s) = \frac{1}{s^2+2s+1} \begin{bmatrix} 1 \\ (s+1) \end{bmatrix} \cdot U_1(s)$$

$$\Rightarrow \left| \begin{array}{l} \underline{\dot{x}}_1 = \begin{bmatrix} \phi & 1 \\ -1 & -2 \end{bmatrix} \underline{x}_1 + \begin{bmatrix} \phi \\ 1 \end{bmatrix} u_1 \\ \underline{y}_1 = \begin{bmatrix} 1 & \phi \\ 1 & 1 \end{bmatrix} \underline{x}_1 \end{array} \right|$$

and: $$\underline{Y}_2(s) = \underline{G}_2(s) \cdot U_2(s) = \frac{1}{s^3+4s^2+5s+2} \begin{bmatrix} (2s^2+4s+2 \\ 4 \end{bmatrix} \cdot U_2($$

$$\Rightarrow \left| \begin{array}{l} \underline{\dot{x}}_2 = \begin{bmatrix} \phi & 1 & \phi \\ \phi & \phi & 1 \\ -2 & -5 & -4 \end{bmatrix} \underline{x}_2 + \begin{bmatrix} \phi \\ \phi \\ 1 \end{bmatrix} u_2 \\ \underline{y}_2 = \begin{bmatrix} 2 & 4 & 2 \\ 4 & \phi & \phi \end{bmatrix} \underline{x}_2 \end{array} \right|$$

Concatenation:

$$\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_m \end{bmatrix} \; ; \quad \underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

$$\Rightarrow \left| \begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} A_1 & & \emptyset \\ & A_2 & \\ \emptyset & & \ddots \; A_m \end{bmatrix} \underline{x} + \begin{bmatrix} \underline{b}_1 & & \emptyset \\ & \underline{b}_2 & \\ \emptyset & & \ddots \; \underline{b}_m \end{bmatrix} \underline{u} \\ \underline{y} &= \begin{bmatrix} C_1 \, C_2 \cdots C_m \end{bmatrix} \underline{x} \end{aligned} \right|$$

$$\Rightarrow \left| \begin{aligned} \dot{\underline{x}} &= \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -5 & -4 \end{array}\right] \underline{x} + \left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right] \underline{u} \\ \underline{y} &= \left[\begin{array}{cc|ccc} 1 & 0 & 2 & 4 & 2 \\ 1 & 1 & 4 & 0 & 0 \end{array}\right] \underline{x} \end{aligned} \right|$$

- It turns out that also this representation is minimal (same system as before), but again, there is no guarantee for this to happen.

Notice: The A-matrix has now more than one nontrivial rows. The controller-canonical form is no longer unique.

(b) An observer-canonical realization:

Example:

$$\underline{\underline{G}}(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{2}{(s+2)} \\ \frac{1}{(s+1)} & \frac{4}{(s+1)^2(s+2)} \end{bmatrix}$$

$$\underline{Y}(s) = \underline{\underline{G}}(s) \cdot \underline{U}(s)$$

We let: $\underline{Y}(s) = \begin{bmatrix} Y_1(s) \\ \vdots \\ Y_p(s) \end{bmatrix}$

where: $Y_i(s) = \underline{G}_i'(s) \cdot \underline{U}(s)$

$\Rightarrow$ are p MISO-systems.

$$Y_1(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{2}{(s+2)} \end{bmatrix} \cdot \underline{U}(s)$$

$$Y_2(s) = \begin{bmatrix} \frac{1}{(s+1)} & \frac{4}{(s+1)^2(s+2)} \end{bmatrix} \cdot \underline{U}(s)$$

can immediately be realized, e.g. in observer-canonical form:

$$Y_1(s) = \frac{1}{s^3+4s^2+5s+2} \begin{bmatrix} (s+2) & (2s^2+4s+2) \end{bmatrix}$$

$$\left|\begin{aligned} \dot{\underline{x}}_1 &= \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -5 \\ 0 & 1 & -4 \end{bmatrix} \underline{x}_1 + \begin{bmatrix} 2 & 2 \\ 1 & 4 \\ 0 & 2 \end{bmatrix} \underline{u} \\ y_1 &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \underline{x}_1 \end{aligned}\right|$$

and: $\quad Y_2(s) = \dfrac{1}{s^3+4s^2+5s+2} \begin{bmatrix} (s^2+3s+2) & 4 \end{bmatrix}$

$$\Rightarrow \left|\begin{aligned} \dot{\underline{x}}_2 &= \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -5 \\ 0 & 1 & -4 \end{bmatrix} \underline{x}_2 + \begin{bmatrix} 2 & 4 \\ 3 & 0 \\ 1 & 0 \end{bmatrix} \underline{u} \\ y_2 &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \underline{x}_2 \end{aligned}\right|$$

Concatenation:

$$\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_p \end{bmatrix} \quad ; \quad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

$$\Rightarrow \left|\begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_p \end{bmatrix} \underline{x} + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_p \end{bmatrix} \underline{u} \\ \underline{y} &= \begin{bmatrix} \underline{c}_1' & & & 0 \\ & \underline{c}_2' & & \\ & & \ddots & \\ 0 & & & \underline{c}_p' \end{bmatrix} \underline{x} \end{aligned}\right|$$

$$\Rightarrow \quad \dot{\underline{x}} = \left[\begin{array}{ccc|ccc} 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & -5 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right] \underline{x} + \left[\begin{array}{cc} 2 & 2 \\ 1 & 4 \\ 0 & 2 \\ \hline 2 & 4 \\ 3 & 0 \\ 1 & 0 \end{array}\right] \underline{u}$$

$$\underline{y} = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \underline{x}$$

is an observer-canonical realization. It is <u>not</u> minimal (as there do exist 5th-order realizations). $\Longrightarrow$ There exists one uncontrollable mode (as the system is certainly observable).

<u>Verification</u>: $\text{Rank}(Q_c) = 5$

$\Longrightarrow$ One uncontrollable mode.

<u>Transformations into these Forms</u>:

<u>Problem</u>: The algorithms presented for the SISO-case no longer work. E.g. What would be the meaning of Q_c^{-1} or Q_o^{-1}? These matrices are not even square any longer !!!

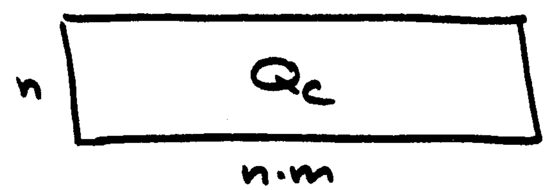
Algorithm to go into controller-canonical form:

Assume: $\text{Rank}(B) \equiv m$

(Otherwise, we have insignificant inputs.)

Assume: System is completely controllable

$\Longleftrightarrow Q_c = [B, AB, A^2B, \ldots, A^{n-1}B]$



$\text{Rank}(Q_c) \equiv n$

Recipe: We construct a matrix $\hat{L}$ out of the first n linearly independent columns of Q_c:

$\Rightarrow \hat{L}$ certainly contains all the $\underline{b}_i$ as they were assumed to be independent of each other.

• Now, we rearrange the columns of $\hat{L}$ such that:

$$L = \left[\underline{b}_1, A\underline{b}_1, \ldots A^{d_1-1} \cdot \underline{b}_1 \,\vdots\, \underline{b}_2, A\underline{b}_2, \ldots, A^{d_2-1} \right.$$
$$\left. \ldots \,\vdots\, \underline{b}_m, \ldots, A^{d_m-1} \underline{b}_m \right]$$

Def: • The m numbers: $d_1, \ldots, d_m$ are called the controllability indices of the system $\{A, B\}$.

• $\mu = \max_i \{d_i\}$; $i \in \{1, m\}$ is called the system controllability index.

Remarks:

(1) As B has full rank, all $\underline{b}_i$ are in $L \Rightarrow$ none of the d_i is zero.

(2) If $A^k \underline{b}_j$ is in $L \Rightarrow$ $A^{k-1} \cdot \underline{b}_j$ is also in L.

We let:

$$\begin{aligned}
\sigma_1 &= d_1 \\
\sigma_2 &= d_1 + d_2 \\
&\vdots \\
\sigma_k &= \sum_{i=1}^{k} d_i \\
&\vdots \\
\sigma_m &= \sum_{i=1}^{m} d_i \equiv n
\end{aligned}$$

- Now, we compute L^{-1}, and pick the following rows out of L^{-1}:

$$\underline{p}'_{\sigma_k} \quad ; \quad \forall k = \{1, \dots, m\}$$

- Now, we construct:

$$T = \left[\begin{array}{c}
\underline{p}_1' \\
\underline{p}_1' A \\
\vdots \\
\underline{p}_1' A^{d_1 - 1} \\
\hdashline
\underline{p}_2' \\
\vdots \\
\underline{p}_m' A^{d_m - 1}
\end{array}\right]$$

- We use T for a similarity transformation.

$\Rightarrow$

$$\hat{A} = \begin{array}{l} \left[\begin{array}{cccc}
\begin{bmatrix} \phi & 1 & & \phi \\ & & \ddots & \\ \phi & & & 1 \\ * & * & \cdots & * \end{bmatrix} & \begin{bmatrix} \phi \\ * \, * \, * \, * \, * \, * \end{bmatrix} & \cdots & \begin{bmatrix} \phi \\ * \, * \, * \, * \, * \end{bmatrix} \\
\begin{bmatrix} \phi \\ * \, * \, * \, * \, * \, * \end{bmatrix} & \begin{bmatrix} \phi & 1 & & \phi \\ & & \ddots & \\ \phi & & & 1 \\ * & * & \cdots & * \end{bmatrix} & \cdots & \begin{bmatrix} \phi \\ * \, * \, * \, * \, * \end{bmatrix} \\
\vdots & \vdots & & \vdots \\
\begin{bmatrix} \phi \\ * \, * \, * \, * \, * \, * \end{bmatrix} & \begin{bmatrix} \phi \\ * \, * \, * \, * \, * \, * \end{bmatrix} & \cdots & \begin{bmatrix} \phi & 1 & & \phi \\ & & \ddots & \\ \phi & & & 1 \\ * & * & \cdots & * \end{bmatrix}
\end{array}\right] \end{array}
\begin{array}{l} \leftarrow \sigma_1^{st} \\ \\ \leftarrow \sigma_2^{nd} \\ \\ \\ \leftarrow \sigma_m^{th} \text{ r} \end{array}$$

$$\hat{B} = \left[\begin{array}{c}
\begin{bmatrix} \phi \\ 1 \; * \, * \, * \, * \, * \, * \end{bmatrix} \\
\begin{bmatrix} \phi \\ \phi \; 1 \; * \, * \, * \, * \end{bmatrix} \\
\vdots \\
\begin{bmatrix} \phi \\ \phi \; \phi \cdots \phi \; 1 \end{bmatrix}
\end{array}\right]
\begin{array}{l} \leftarrow \sigma_1^{st} \text{ row} \\ \\ \leftarrow \sigma_2^{nd} \text{ row} \\ \\ \\ \leftarrow \sigma_m^{th} \text{ row} \end{array}$$

Example:

$$
\left|
\begin{aligned}
\dot{\underline{x}} &= \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \underline{u} \\
\underline{y} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 4 & 1 & -4 & 4 \end{bmatrix} \underline{x}
\end{aligned}
\right|
$$

cf. p. 34

$$
Q_c = \left[\begin{array}{cc:cc:cc:cc:cc}
0 & 0 & 1 & 0 & -2 & 0 & 3 & 0 & -4 & 0 \\
0 & 0 & 0 & 1 & 0 & -2 & 0 & 3 & 0 & -4 \\
1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & -2 & 0 & 4 & 0 & -8 & 0 & 16
\end{array}\right]
$$

[> RANK(qc(:, 1:3)) ⟶ 3 ✓

[> RANK(qc(:, 1:4)) ⟶ 4 ✓

[> RANK(qc(:, 1:5)) ⟶ 4 ⟹ leave out

[> RANK(qc(:, 1:6)) ⟶ $\underline{\underline{5}}$ ✓ okay.

[> LT= QC(:,[1:4, 6])

$$
\Rightarrow LT = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -2 & 4 \end{bmatrix}
$$

[> L = LT(:, [1,3,2,4,5])

$$\Rightarrow L = \left[\begin{array}{cc:ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 & 4 \end{array}\right]$$

$$\underbrace{\qquad}_{d_1=2}\quad\underbrace{\qquad\quad}_{d_2=3}$$

$$\Rightarrow L^{-1} = \begin{array}{c} \left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 3 & 0 & -2 & 2 \\ 0 & 1 & 0 & -1 & 1 \end{array}\right] \end{array} \begin{array}{l} \\ \leftarrow \sigma_1 \\ \\ \\ \leftarrow \sigma_2 \end{array}$$

$$\Rightarrow p_1' = [1 \; 0 \; 0 \; 0 \; 0]$$

$$p_2' = [0 \; 1 \; 0 \; -1 \; 1]$$

$$\Rightarrow T = \begin{bmatrix} p_1' \\ p_1'A \\ p_2' \\ p_2'A \\ p_2'A^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 2 & -2 \\ 0 & 1 & 0 & -3 & 4 \end{bmatrix}$$

$$\Rightarrow AH = T*A/T = \left[\begin{array}{cc:ccc} 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -5 & -4 \end{array}\right]$$

$$BH = T*B = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hdashline 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right] \; ; \quad CH = C/T = \left[\begin{array}{cc:ccc} 1 & 0 & 2 & 4 & 2 \\ 1 & 1 & 4 & 0 & 0 \end{array}\right]$$

Input-decoupling of uncontrollable modes

Assume: $\text{Rank}(B) = m$

$\text{Rank}(Q_c) = \bar{n} < n$

- We construct L as before. However, L has now only $\bar{n}$ columns. We extend thus L with anything that makes the rank full

$$\tilde{L} = [L, \text{extension}] .$$

From a numerical point of view, it may be best to extend with the nullspace:

$$\tilde{L} = [L, N(L)]$$

- Quite clearly, $\tilde{L}$ can be interpreted as the L-matrix of a different system:

$$\left| \begin{array}{l} \underline{\dot{x}} = A\underline{x} + \tilde{B}\underline{\tilde{u}} \\ \underline{y} = C\underline{x} + \tilde{D}\underline{\tilde{u}} \end{array} \right|$$

where: $$\tilde{B} = \left[B \, , \, N(L) \right]$$

Obviously: $\{A, \tilde{B}\}$ is controllable.

- Now, we can apply the same transformation as before and find:

$$\left| \begin{array}{l} \dot{\underline{\xi}} = \left[\begin{array}{c:c} \hat{A}_c & \hat{A}_{12} \\ \hdashline \phi & \hat{A}_{\bar{c}} \end{array} \right] \underline{\xi} + \left[\begin{array}{c} \hat{\tilde{B}} \\ \hdashline \phi \end{array} \right] \underline{\tilde{u}} \\ \\ \underline{y} = \left[\begin{array}{c:c} \hat{C}_c & C_2 \end{array} \right] \underline{\xi} \end{array} \right|$$

A minimal realization:

$$\left| \begin{array}{l} \dot{\underline{x}}_r = \hat{A}_c \, \underline{x}_r + \hat{\tilde{B}} \, \underline{\tilde{u}} \\ \underline{y} = \hat{C}_c \, \underline{x}_r \end{array} \right|$$

Now, we don't need those additional inputs any longer.

As :

$$\hat{\tilde{B}} = T \cdot \tilde{B}$$

$$\Longleftrightarrow \quad \left[\hat{B} \,\vdots\, *\right] = T \cdot \left[B \,\vdots\, *\right]$$

$$\Longrightarrow \quad \boxed{\hat{B} = T \cdot B}$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}}_r = \hat{A}_c \, \underline{x}_r + \hat{B} \, \underline{u} \\ \underline{y} = \hat{C}_c \, \underline{x}_r \end{array} \right|$$

is exactly what we want.

Example:

$$\left| \begin{array}{l} \dot{\underline{x}} = \begin{bmatrix} 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & -5 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix} \underline{x} + \begin{bmatrix} 2 & 2 \\ 1 & 4 \\ 0 & 2 \\ 2 & 4 \\ 3 & 0 \\ 1 & 0 \end{bmatrix} \underline{u} \\ \underline{y} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underline{x} \end{array} \right|$$

has one uncontrollable mode at -2

(cf. p. 353).

$$Q_c = \left[\begin{array}{cc:cc:cc:c} 2 & 2 & 0 & -4 & -2 & 8 & \\ 1 & 4 & 2 & -8 & -5 & 16 & \\ 0 & 2 & 1 & -4 & -2 & 8 & \cdots\cdots \\ 2 & 4 & -2 & 0 & 2 & 0 & \\ 3 & 0 & -3 & 4 & 3 & 0 & \\ 1 & 0 & -1 & 0 & 1 & 4 & \end{array}\right] ; \quad \mathrm{Rank}(Q_c) = 5$$

leave out

$$\Rightarrow \quad \hat{L} = \begin{bmatrix} 2 & 2 & 0 & -4 & 8 \\ 1 & 4 & 2 & -8 & 16 \\ 0 & 2 & 1 & -4 & 8 \\ 2 & 4 & -2 & 0 & 0 \\ 3 & 0 & -3 & 4 & 0 \\ 1 & 0 & -1 & 0 & 4 \end{bmatrix} ; \quad \mathrm{Rank}(\hat{L}) = 5$$

$$\Rightarrow \quad L = \underbrace{\left[\begin{array}{cc:ccc} 2 & 0 & 2 & -4 & 8 \\ 1 & 2 & 4 & -8 & 16 \\ 0 & 1 & 2 & -4 & 8 \\ 2 & -2 & 4 & 0 & 0 \\ 3 & -3 & 0 & 4 & 0 \\ 1 & -1 & 0 & 0 & 4 \end{array}\right]}_{d_1 \;\; | \;\; d_2}$$

```
[> L = [ L , ZROW(6,1) ] ;
[> [Q,R] = QR(L') ;
[> L(:,6) = Q(:,6) ;
```

$$\Rightarrow \quad \tilde{L} = \underbrace{\left[\begin{array}{cc:ccc:c} 2 & 0 & 2 & -4 & 8 & 0 \\ 1 & 2 & 4 & -8 & 16 & 0 \\ 0 & 1 & 2 & -4 & 8 & 0 \\ 2 & -2 & 4 & 0 & 0 & 0 \\ 3 & -3 & 0 & 4 & 0 & 0 \\ 1 & -1 & 0 & 0 & 4 & 1 \end{array}\right]}_{d_1 \;\; | \;\; d_2 \;\; | \;\; d_3}$$

$$\Rightarrow \tilde{L}^{-1} = \begin{array}{c}\left[\begin{array}{cccccc} 0 & 1 & -2 & 0 & 0 & 0 \\ -1 & 2 & -3 & 0 & 0 & 0 \\ -0.5 & 0.5 & -0.5 & 0.25 & 0 & 0 \\ -0.75 & 0.75 & -0.75 & 0 & 0.25 & 0 \\ -0.125 & 0 & 0.25 & -0.0625 & 0.125 & 0 \\ -0.5 & 1 & -2 & 0.25 & -0.5 & 1 \end{array}\right]\end{array}\begin{array}{l} \\ \leftarrow \underline{p}_1' \\ \\ \\ \leftarrow \underline{p}_2' \\ \leftarrow \underline{p}_3' \end{array}$$

$$\Rightarrow T = \begin{bmatrix} \underline{p}_1' \\ \underline{p}_1' A \\ \underline{p}_2' \\ \underline{p}_2' A \\ \underline{p}_2' A^2 \\ \underline{p}_3' \end{bmatrix} = \begin{bmatrix} -1 & 2 & -3 & 0 & 0 & 0 \\ 2 & -3 & 4 & 0 & 0 & 0 \\ -0.125 & 0 & 0.25 & -0.0625 & 0.125 & 0 \\ 0 & 0.25 & -0.75 & 0.125 & 0 & -0.5 \\ 0.25 & -0.75 & 1.75 & 0 & -0.5 & 1.75 \\ -0.5 & 1 & -2 & 0.25 & -0.5 & 1 \end{bmatrix}$$

$$\Rightarrow \hat{A} = T \cdot A / T = \left[\begin{array}{cc:ccc:c} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & -5 & -4 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & \boxed{-2} \end{array}\right]; \quad \hat{B} = T \cdot B = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hdashline 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ \hdashline 0 & 0 \end{array}\right]$$

$$\hat{C} = C / T = \left[\begin{array}{cc:ccc:c} 1 & 0 & 2 & 4 & 2 & -1.5 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$\Rightarrow$ Minimal realization:

$$\left|\begin{array}{l} \dot{\underline{x}}_r = \left[\begin{array}{cc:ccc} 0 & 1 & & 0 & \\ -1 & -2 & & & \\ \hdashline & & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \\ & & -2 & -5 & -4 \end{array}\right] \underline{x}_r + \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hdashline 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right] \underline{u} \\ \\ \underline{y} = \left[\begin{array}{cc:ccc} 1 & 0 & 2 & 4 & 2 \\ 1 & 1 & 4 & 0 & 0 \end{array}\right] \underline{x}_r \end{array}\right|$$

as desired.

Transformation into observer-canonical form:

All these algorithms have their equivalent observer form. However, we really do not need those as we can apply the <u>duality principle</u>.

Nevertheless, let us do it at least once:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -5 & -4 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & 0 & 2 & 4 & 2 \\ 1 & 1 & 4 & 0 & 0 \end{bmatrix} \underline{x}$$

Let us get an observer-canonical form for this:

$$\Rightarrow Q_o = \begin{bmatrix} 1 & 0 & 2 & 4 & 2 \\ 1 & 1 & 4 & 0 & 0 \\ \hdashline 0 & 1 & -4 & -8 & -4 \\ -1 & -1 & 0 & 4 & 0 \\ \hdashline -1 & -2 & 8 & 16 & 8 \\ 1 & 1 & 0 & 0 & 4 \\ \hdashline 2 & 3 & -16 & -32 & -16 \\ -1 & -1 & -8 & -20 & -16 \\ \hdashline -3 & -4 & 32 & 64 & 32 \\ 1 & 1 & 32 & 72 & 44 \end{bmatrix}$$

Rank is already $= 5$

$$\Rightarrow \hat{L} = \begin{bmatrix} 1 & 0 & 2 & 4 & 2 \\ 1 & 1 & 4 & 0 & 0 \\ 0 & 1 & -4 & -8 & -4 \\ -1 & -1 & 0 & 4 & 0 \\ -1 & -2 & 8 & 16 & 8 \end{bmatrix}$$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 2 & 4 & 2 \\ 0 & 1 & -4 & -8 & -4 \\ -1 & -2 & 8 & 16 & 8 \\ \hdashline 1 & 1 & 4 & 0 & 0 \\ -1 & -1 & 0 & 4 & 0 \end{bmatrix} \begin{matrix} \updownarrow e_1 = 3 \\ \\ \updownarrow e_2 = 2 \end{matrix}$$

e_i = observability indices

$\nu = 3$ = system observability index

$$\Rightarrow L^{-1} = \begin{array}{c} \begin{bmatrix} 0 & -2 & -1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 0 \\ -0.5 & -0.75 & -0.25 & 0.25 & 0 \\ 0.5 & 0.75 & 0.25 & 0 & 0.25 \\ 0 & 0.25 & 0.25 & -0.25 & -0.5 \end{bmatrix} \\ \begin{matrix} & & \uparrow \tau_1 & & \uparrow \tau_2 \end{matrix} \end{array} ; \quad \tau_1 = \begin{bmatrix} -1 \\ 2 \\ -0.25 \\ 0.25 \\ 0.25 \end{bmatrix}, \tau_2 = \Bigg[\cdots$$

$$\Rightarrow T^{-1} = \left[\underline{p}_1 ,\ A\underline{p}_1 ,\ A^2\underline{p}_1 ,\ \underline{p}_2 ,\ A\underline{p}_2 \right]$$

$$\Rightarrow T^{-1} = \begin{bmatrix} -1 & 2 & -3 & 0 & 0 \\ 2 & -3 & 4 & 0 & 0 \\ -0.25 & 0.25 & 0.25 & 0 & 0.25 \\ 0.25 & 0.25 & -1.75 & 0.25 & -0.5 \\ 0.25 & -1.75 & 5.25 & -0.5 & 0.75 \end{bmatrix}$$

$$\Rightarrow T = \begin{bmatrix} 4 & 2 & 2 & 4 & 2 \\ 4 & 1 & 4 & 8 & 4 \\ 1 & 0 & 2 & 4 & 2 \\ -3 & -1 & 8 & 4 & 0 \\ -1 & 1 & 0 & -8 & -4 \end{bmatrix}$$

$$\Rightarrow \hat{A} = T \cdot A / T = \left[\begin{array}{ccc|cc} 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & -5 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] ; \quad \hat{B} = T \cdot B = \left[\begin{array}{cc} 2 & 2 \\ 1 & 4 \\ 0 & 2 \\ \hline -1 & 0 \\ 1 & -4 \end{array}\right]$$

$$\hat{C} = C / T = \left[\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \quad \dot{\underline{x}} = \left[\begin{array}{ccc|cc} 0 & 0 & -2 & & \\ 1 & 0 & -5 & & 0 \\ 0 & 1 & -4 & & \\ \hline & 0 & & 0 & -1 \\ & & & 1 & -2 \end{array}\right] \underline{x} + \left[\begin{array}{cc} 2 & 2 \\ 1 & 4 \\ 0 & 2 \\ \hline -1 & 0 \\ 1 & -4 \end{array}\right] \underline{u}$$

$$\underline{y} = \left[\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 \end{array}\right] \underline{x}$$

is a legitimate observer-canonical form.

Notice: The resulting observer-canonical form is no longer the dual form of the controller-canonical form.

Reason:

$$
\begin{array}{ccc}
S_1 & \xleftrightarrow{\text{duality}} & \overline{S}_1 \\
\downarrow & & \downarrow \\
\underline{\underline{G}}(s) & \neq & \underline{\underline{G}}'(s)
\end{array}
$$

MIMO-system

but:

$$
\begin{array}{ccc}
S_{1_{CCF}} & \xrightarrow{\text{Trafo.}} & S_{1_{OCF}} \\
\downarrow & & \downarrow \\
\underline{\underline{G}}(s) & & \underline{\underline{G}}(s)
\end{array}
$$

the same system.

Decoupling Insignificant Inputs:

Given:
$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} + D\underline{u} \end{array} \right|$$

$$\underline{u} \in \mathbb{R}^{m} \implies B \in \mathbb{R}^{n \times m}$$

but: $\text{Rank}(B) = \overline{m} < m$

$\iff$ We could achieve exactly the same effect on our system with fewer inputs

$\iff$ There are insignificant inputs.

Idea: Find a regular transformation on the input vector:

$$\boxed{\underline{v} = T \cdot \underline{u}} \iff \boxed{\underline{u} = T^{-1} \underline{v}}$$

which pushes the non-zero elements of B to the left.

Notice: This is not a similarity transformation. The transformation is performed on the inputs and not on the states.

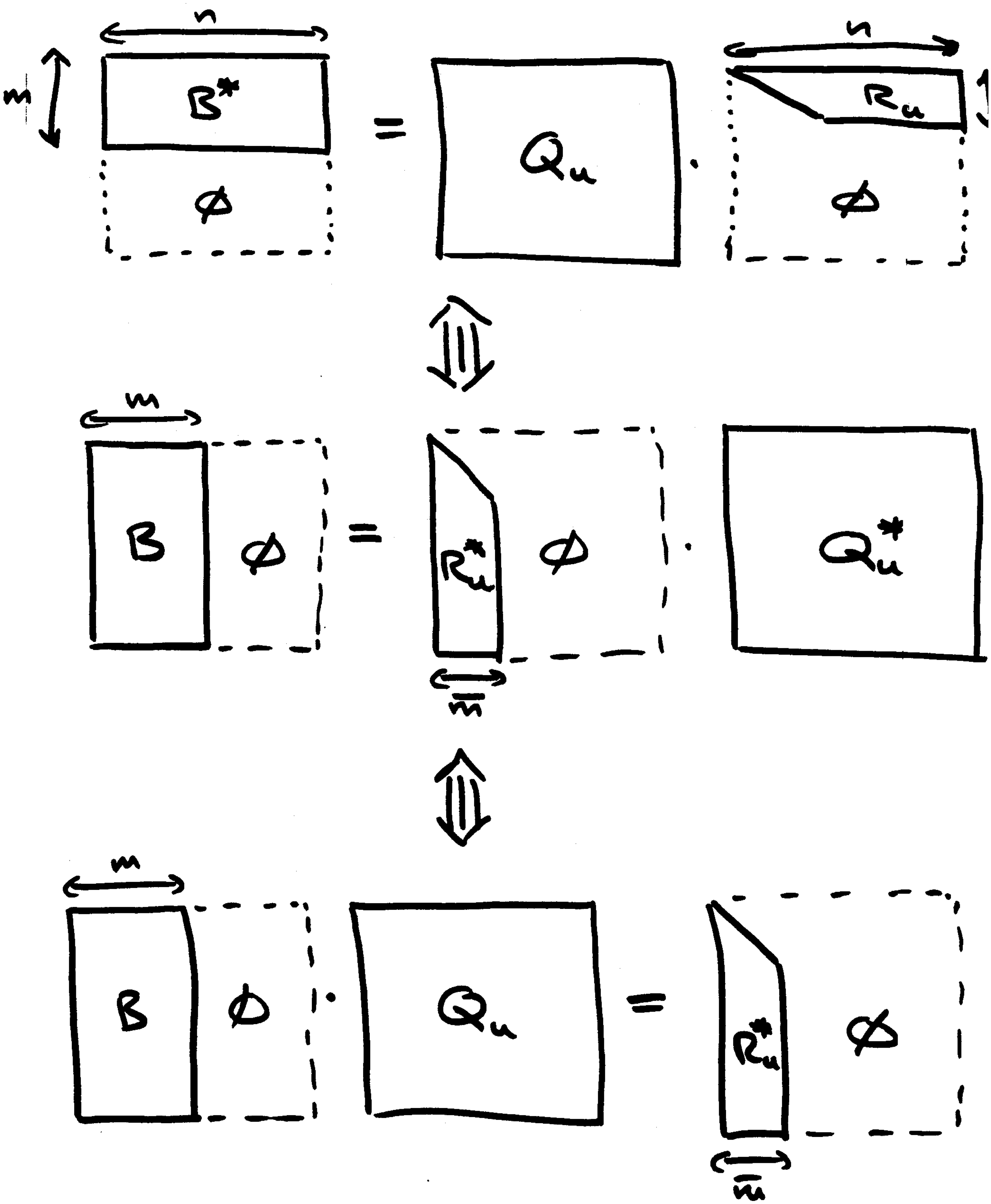
$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underbrace{BT^{-1}}_{\hat{B}}\underline{v} \\ \underline{y} = C\underline{x} + \underbrace{DT^{-1}}_{\hat{D}}\underline{v} \end{array} \right|$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \hat{B}\underline{v} \\ \underline{y} = C\underline{x} + \hat{D}\underline{v} \end{array} \right|$$

where: $\left| \begin{array}{l} \hat{B} = BT^{-1} \\ \hat{D} = DT^{-1} \end{array} \right|$

is the new representation.

Recipe: Take the hermitian transpose of B, concatenate from below with zero, and perform a QR-decomposition.



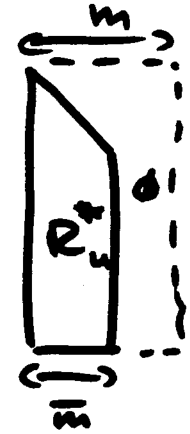
$\Rightarrow B \cdot Q_u = R_u^*$

in the desired form.

$$\Rightarrow \quad \boxed{T = Q_u^*}$$

applied to $\underline{u}$ will have the desired effect.

$$\hat{B} = B \cdot Q_u =$$



Obviously, the new input vector $\underline{v}$ (still of length m) can be cut in an upper portion of length $\bar{m}$ ($\underline{v}_1$) and a lower portion ($\underline{v}_2$) of length $m - \bar{m}$:

$$\underline{v} = \left.\begin{bmatrix} \left.\underline{v}_1\right\} \bar{m} \\ \hdashline \underline{v}_2 \end{bmatrix}\right\} m$$

$$\Rightarrow \hat{B}\underline{v} = \begin{bmatrix} R_u^* & \emptyset \end{bmatrix} \cdot \begin{bmatrix} \underline{v} \end{bmatrix} = \begin{bmatrix} \emptyset \\ R_u^* \end{bmatrix} \cdot \begin{bmatrix} \underline{v}_1 \end{bmatrix} + \emptyset \cdot \underline{v}_2$$

$\Rightarrow$ $\underline{v}_1$ are the significant inputs
$\underline{v}_2$ are the insignificant inputs.

Assume, we have removed the direct input/output coupling:

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \hat{B}\underline{v} \\ \underline{y} = C\underline{x} \end{array} \right|$$

can also be written as:

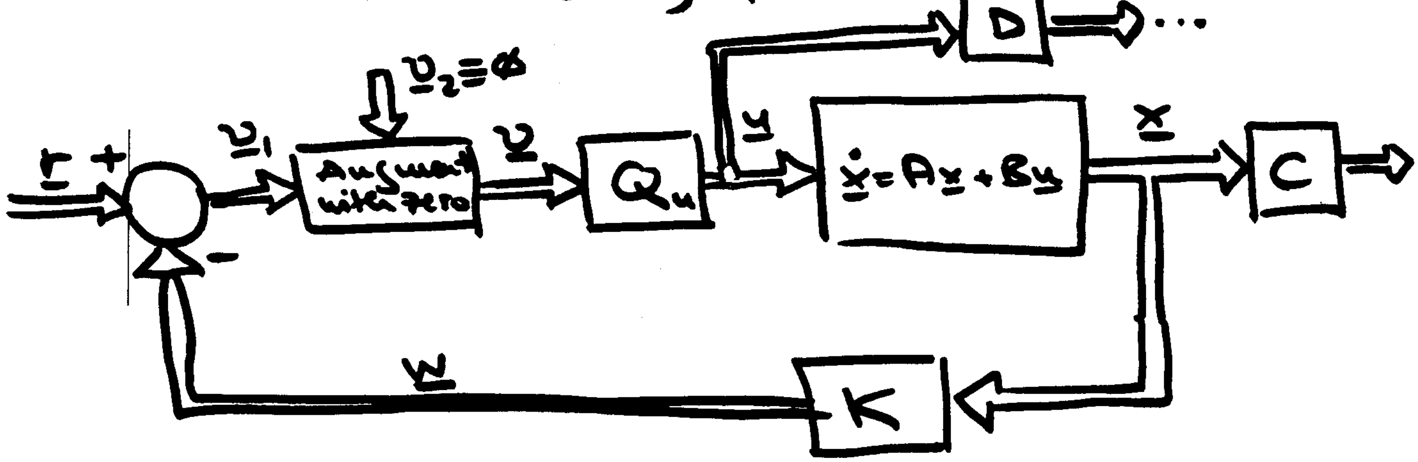
$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \hat{\hat{B}}\underline{v}_1 \\ \underline{y} = C\underline{x} \end{array} \right|$$

with: $\hat{\hat{B}} = \begin{bmatrix} \emptyset \\ R_u^* \end{bmatrix} \in \mathbb{R}^{n \times \bar{m}}$

and $\text{Rank}(\hat{\hat{B}}) = \bar{m}$

In this new representation, our previous MIMO-algorithms will work just fine.

E.g., state feedback (not yet discussed in detail for the MIMO-case):



$\underline{x} \in \mathbb{R}^n$

$\underline{u} \in \mathbb{R}^m$; $\underline{v} \in \mathbb{R}^m$

$\underline{v}_1 \in \mathbb{R}^{\bar{m}}$; $\underline{w} \in \mathbb{R}^{\bar{m}}$; $\underline{r} \in \mathbb{R}^{\bar{m}}$

$Q \in \mathbb{R}^{m \times m}$

$K \in \mathbb{R}^{\bar{m} \times n}$

(*Notice*: In our block diagram, only the Q_u-matrix appears, but *not* the R_u-matrix.)

Decoupling of insignificant Outputs:

(Dual problem)

$$\boxed{\underline{\eta} = T \cdot \underline{y}} \iff \boxed{\underline{y} = T^{-1} \cdot \underline{\eta}}$$

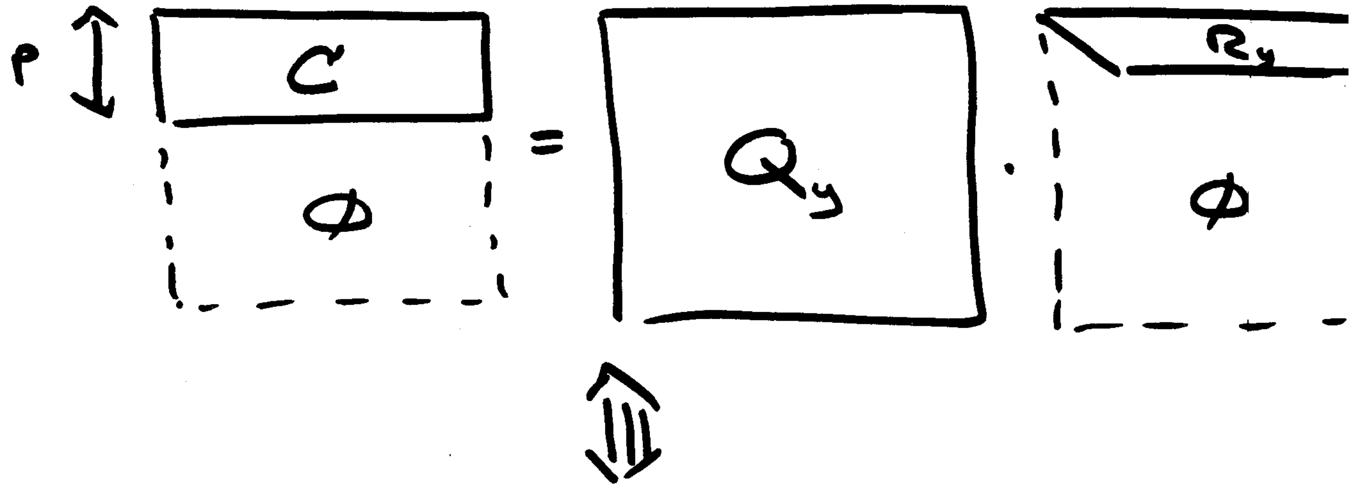
- Push the non-zero elements of C' up.

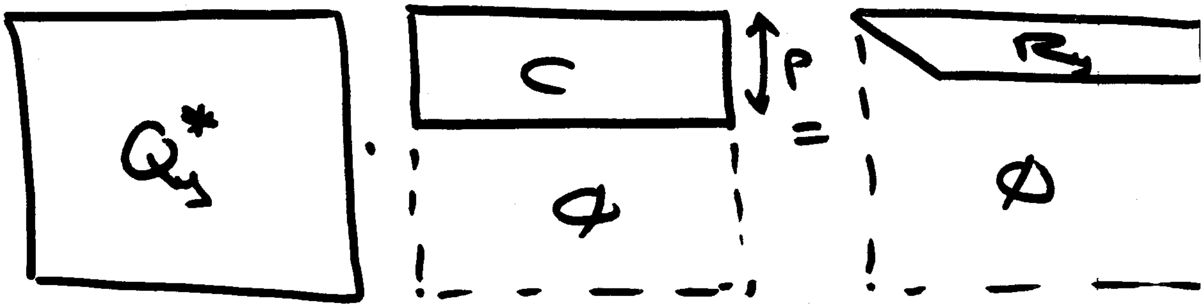
$$\left| \begin{array}{l} \underline{\dot{x}} = A\underline{x} + B\underline{u} \\ \underline{\eta} = \underbrace{T \cdot C}_{\hat{C}}\underline{x} + \underbrace{T \cdot D}_{\hat{D}}\underline{u} \end{array} \right|$$

$$\Rightarrow \left| \begin{array}{l} \underline{\dot{x}} = A\underline{x} + B\underline{u} \\ \underline{\eta} = \hat{C}\underline{x} + \hat{D}\underline{u} \end{array} \right|$$

where: $\left| \begin{array}{l} \hat{C} = TC \\ \hat{D} = TD \end{array} \right|$

is the new representation.

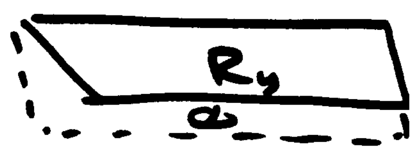




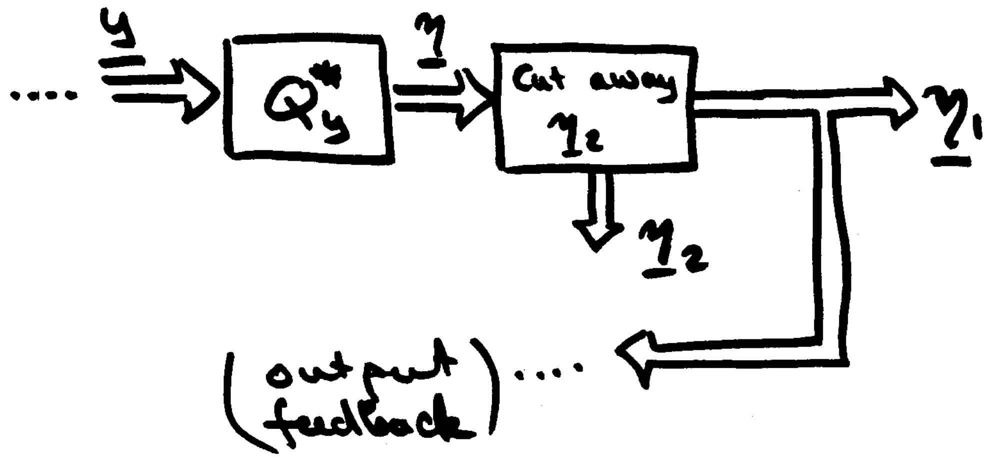
$\Rightarrow$ $\boxed{T = Q_y^*}$

applied to output $\underline{y}$ will have the desired effect:

$$\hat{C} = T \cdot C = Q_y^* \cdot C =$$



$\Longrightarrow$ We can use the same concept as before:



$\underline{y}, \underline{\eta} \in \mathbb{R}^p$

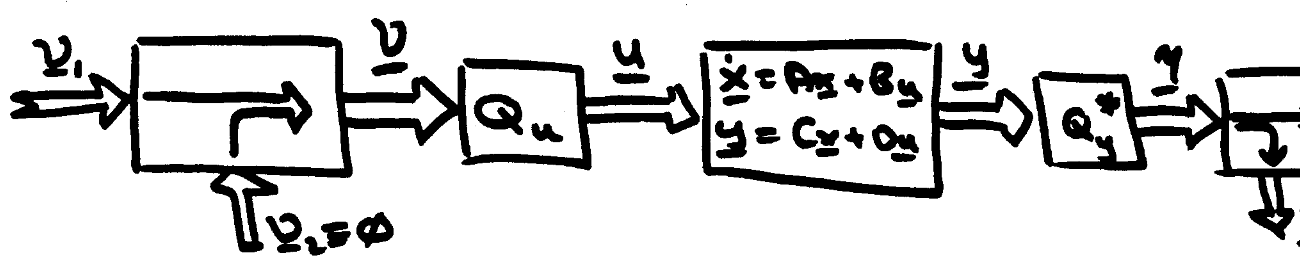
$\underline{\eta}_1 \in \mathbb{R}^{\bar{p}}$

Assume: $\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} + D\underline{u} \end{array} \right|$

$\underline{x} \in \mathbb{R}^n$

$\underline{u} \in \mathbb{R}^m$; Rank$(B) = \bar{m} < m$

$\underline{y} \in R^p$; Rank $(C) = \bar{p} < p$



Then, design controller around this system.