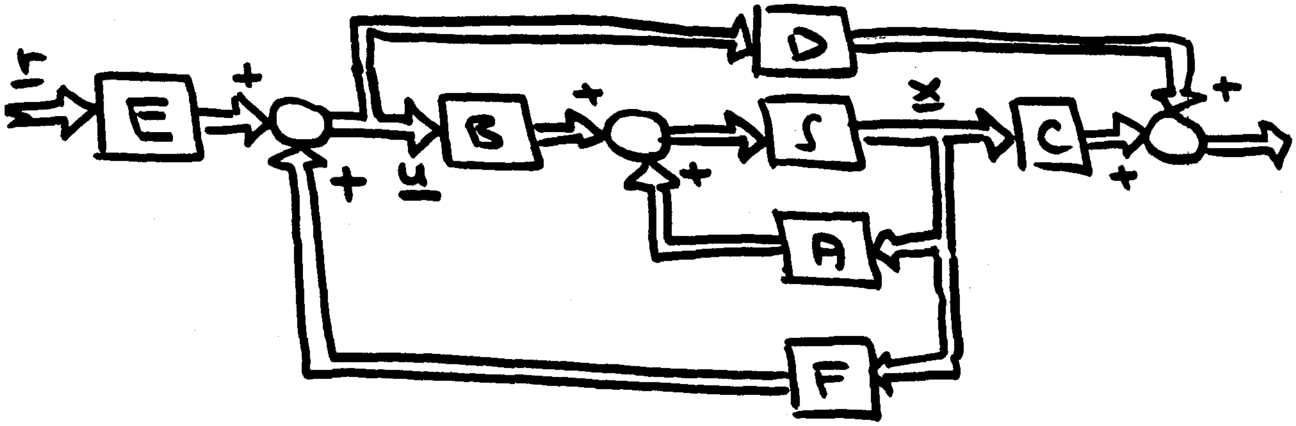
Poleplacement for MIMO Systems



$$\underline{u}(t) = E \cdot \underline{r}(t) + F \cdot \underline{x}(t)$$

Notice: Most textbooks use a positive instead of a negative feedback for MIMO-systems. (Does not matter, simply changes the sign of all elements of F.)

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = (A+BF)\,\underline{x} + BE\,\underline{r} \\ \underline{y} = (C+DF)\,\underline{x} + DE\,\underline{r} \end{array} \right|$$

is the closed-loop system.

If $\{A, B\}$ was in any controller-canonical form,

$\Rightarrow$ $\{(A+BF), BE\}$ is still in a controller-canonical form

Remarks:

(1) $(A+BF)$ has the same structure (same controllabilit indices) as A.

(2) The nontrivial σ_k-rows of $(A+BF)$ can be chosen freely by selecting F appropriately.

(3) (BE) has basically the same form as B. Its nontrivial σ_k-rows are determined by E.

$\Rightarrow$ We can make the

matrix

$$\hat{B}_m \in \mathbb{R}^{m\times m}$$

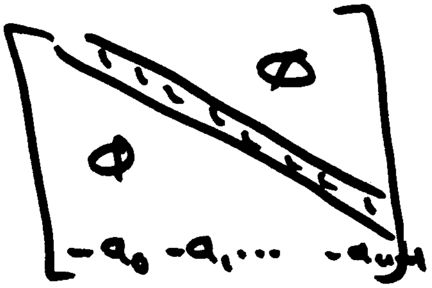
that consists of the nontrivial σ_k-rows only a nonsingular uppertriangular matrix.

(4) If $D \equiv 0 \Rightarrow C_{CL} = C_{OL}$.

(5) If $\{A, B\}$ is not fully controllable, the algorithm can be applied to the controllable subsystem.

Notice: The poleplacement problem is no longer unique.

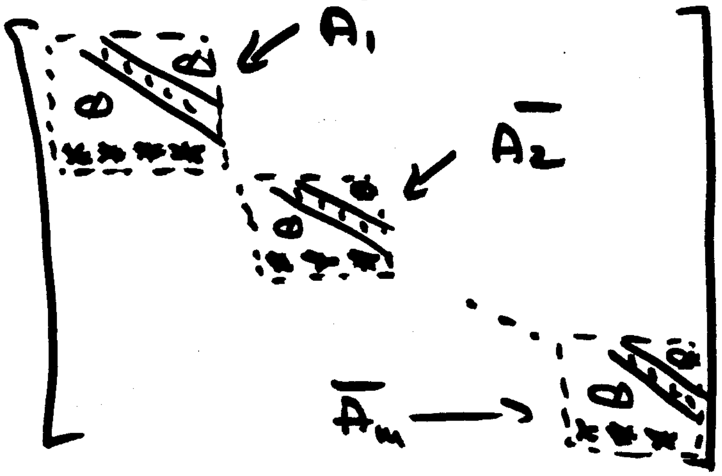
Proof: We first choose F such that $(A+BF)$ takes the form:



$$\Rightarrow \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

Alternately, we can choose:

$A+BF =$



$$\Rightarrow \det(\lambda I - A - BF) = \det(\lambda I - \bar{A}_1) \cdot \det(\lambda I - \bar{A}_2) \cdot \cdots \cdot \det(\lambda I - \bar{A}_m)$$

is another controller-canonical realization.

- In fact, we have $(\underset{\text{significant inputs}}{\bar{m}} \times \underset{\text{controllable states}}{\bar{n}})$ coefficients available to match $\bar{n}$ parameters.

Theorem: Only the controllable poles are influenced by F,

Proof:

$$A = \left[\begin{array}{c|c} A_c & A_{12} \\ \hline 0 & A_{\bar{c}} \end{array}\right] ; \quad B = \left[\begin{array}{c} B_c \\ \hline 0 \end{array}\right]$$

$$\Rightarrow \quad F = \left[\begin{array}{c|c} F_1 & F_2 \end{array}\right]$$

$$\Rightarrow (A+BF) = \left[\begin{array}{c|c} (A_c + B_c F_1) & \begin{array}{c} A_{12} \\ + \\ B_c F_2 \end{array} \\ \hline 0 & A_{\bar{c}} \end{array}\right]$$

$$\Rightarrow \det(\lambda I - A - BF)$$

$$= \underbrace{\det(\lambda I - A_c - B_c F_1)}_{\substack{\text{controllable} \\ \text{poles}}} \cdot \underbrace{\det(\lambda I - A}_{\substack{\text{uncontrolla} \\ \text{poles}}}$$

F_2 is arbitrary (e.g. 0). Has no influence on the poles.

Example:

Given:

$$\underline{\dot{x}} = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -5 & -4 \end{array}\right] \underline{x} + \left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right] \underline{u}$$

(cf. p. 35

Find a state feedback that places the poles at:

$$\{ -1 \pm j \ ; \ -2 \pm j \ ; \ -4 \}$$

One solution would be to make:

$$Q_1(s) = (s+1+j)(s+1-j) = s^2+2s+2$$

$$Q_2(s) = (s+2+j)(s+2-j)(s+4) = s^3+8s^2+2$$

$$+20$$

$$\Rightarrow A_{cl} = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -20 & -21 & -8 \end{array}\right]$$

has the desired eigenvalues.

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} \end{bmatrix}$$

$$\Rightarrow BF = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ F_{11} & F_{12} & F_{13} & F_{14} & F_{15} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} \end{bmatrix}$$

$$\Rightarrow A_{cl} = A + BF = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ (F_{11}-1) & (F_{12}-2) & F_{13} & F_{14} & F_{15} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ F_{21} & F_{22} & (F_{23}-2) & (F_{24}-5) & (F_{25}-4) \end{bmatrix}$$

Comparison of coefficients:

$$\underline{\underline{F = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -18 & -16 & -4 \end{bmatrix}}}$$

will place the poles at the desired positions.

- Alternately, we could have created our characteristic polynomial:

$$Q_{cl}(s) = s^5 + 10 s^4 + 39 s^3 + 78 s^2 + 82 s + 40$$

$$\Rightarrow A_{cl} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -40 & -82 & -78 & -39 & -10 \end{bmatrix}$$

has also the desired eigenvalues.

$\Rightarrow$ Comparison of coefficients:

$$F = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ -40 & -82 & -76 & -34 & -6 \end{bmatrix}$$

would also place the poles at the desired positions.

- We can make use of this freedom to optimize some other characteristics, e.g. obtain a strong decoupling of subsystems (as with our first solution), or minimize parameter sensitivity.

  $\longrightarrow$ cf. ECE 544 .

Observers for MIMO-systems

(1) Full-order Luenberger Observer:

Given:
$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} + Dy \end{array} \right|$$

We use the same general idea as in the SISO-case:

$$\hat{\dot{\underline{x}}} = M \cdot \hat{\underline{x}} + L \cdot \underline{u} + K \cdot \underline{y}$$

where: $\underline{e} = \underline{x} - \hat{\underline{x}}$

$$\Rightarrow \dot{\underline{e}} = \dot{\underline{x}} - \hat{\dot{\underline{x}}}$$

$$\Rightarrow \dot{\underline{e}} = A\underline{x} + B\underline{u} - M \cdot \hat{\underline{x}} - L \cdot \underline{u} - K \cdot \underline{y}$$
$$= A\underline{x} + B \cdot \underline{u} - M \cdot \hat{\underline{x}} - L \cdot \underline{u} - K \cdot C \cdot \underline{x} - K \cdot D \cdot \underline{u}$$

$$\Rightarrow \dot{\underline{e}} = M \cdot \underline{x} - M\hat{\underline{x}} - M\underline{x} + A\underline{x} - KC\underline{x} + B\underline{u} - L\underline{u} - KD\underline{u}$$

$$\Rightarrow \dot{\underline{e}} = M\underline{e} + (A - KC - M)\underline{x} + (B - L - KD)$$

We want:

$$\boxed{\underline{\dot{e}} = M\underline{e}}$$

$\uparrow$ stable

$\Rightarrow \quad M = A - KC$

$\Rightarrow$ pole-placement problem.

$L = B - KD$

$\Rightarrow$ simply plug in.

Example: (continued)

$$\underline{y} = \left[\begin{array}{cc|ccc} 1 & 0 & 2 & 4 & 2 \\ 1 & 1 & 4 & 0 & 0 \end{array}\right] \underline{x}$$

(cf. p.350)

We want to solve the output feedback problem.

$\Rightarrow$ We solve the pole placement problem for the (dual) system:

$$\left| \begin{array}{l} \underline{\dot{x}} = A'\underline{x} + C'\underline{u} \\ \underline{y} = B'\underline{x} + D'\underline{u} \end{array} \right|$$

- We start by converting this into controller-canonical form:

$$Q_c = \left[\begin{array}{cc|cc|cc|cc|cc} 1 & 1 & 0 & -1 & -1 & 1 & 2 & -1 & -3 & 1 \\ 0 & 1 & 1 & -1 & -2 & 1 & 3 & -1 & -4 & 1 \\ 2 & 4 & -4 & 0 & 8 & 0 & -16 & -8 & 32 & 32 \\ 4 & 0 & -8 & 4 & 16 & 0 & -32 & -20 & 64 & 72 \\ 2 & 0 & -4 & 0 & 8 & 4 & -16 & -16 & 32 & 44 \end{array}\right]$$

$$\Rightarrow \hat{L} = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 & -2 \\ 2 & 4 & -4 & 0 & 8 \\ 4 & 0 & -8 & 4 & 16 \\ 2 & 0 & -4 & 0 & 8 \end{bmatrix}$$

has full rank

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & -2 & 1 & -1 \\ 2 & -4 & 8 & 4 & 0 \\ 4 & -8 & 16 & 0 & 4 \\ 2 & -4 & 8 & 0 & 0 \end{bmatrix} \Rightarrow L^{-1} = \begin{bmatrix} 0 & 2 & -.5 & .5 & \\ -2 & 5 & -.75 & .75 & . \\ -1 & 2 & -.25 & .25 & . \\ 0 & 0 & .25 & 0 & -. \\ 0 & 0 & 0 & .25 & -. \end{bmatrix}$$

$$\underbrace{\longleftrightarrow}_{d_1} \quad \underbrace{\longleftrightarrow}_{d_2}$$

$$\Rightarrow \underline{g}_1' = [-1 \quad 2 \quad -.25 \quad .25 \quad .25]$$

$$\underline{g}_2' = [0 \quad 0 \quad 0 \quad .25 \quad -.5]$$

$$\Rightarrow T = \begin{bmatrix} \underline{g}_1' \\ \underline{g}_1' A' \\ \underline{g}_1' A'^2 \\ \underline{g}_2' \\ \underline{g}_2' A' \end{bmatrix} = \begin{bmatrix} -1 & 2 & -.25 & .25 & .25 \\ 2 & -3 & .25 & .25 & -1.75 \\ -3 & 4 & .25 & -1.75 & 5.25 \\ 0 & 0 & 0 & .25 & -.5 \\ 0 & 0 & .25 & -.5 & .75 \end{bmatrix}$$

$$\Rightarrow \quad \hat{A} = T \cdot A' / T = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2 & -5 & -4 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$$

$$\hat{B} = T \cdot C' = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right]$$

$$\hat{C} = B' / T = \left[\begin{array}{ccc|cc} 2 & 1 & 0 & -1 & 1 \\ 2 & 4 & 2 & 0 & -4 \end{array}\right]$$

is in the desired form.

- We choose the observer poles

$$\underline{p}_o = 2 \cdot \underline{p}_c$$

$$\Rightarrow Q_1(s) = s^3 + 16 s^2 + 84 s + 160$$
$$Q_2(s) = s^2 + 4s + 8$$

$$\Rightarrow A_{CL} = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -160 & -84 & -16 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -8 & -4 \end{array}\right]$$

has the desired eigenvalues.

Comparison of coefficients:

$$\hat{F} = \begin{bmatrix} -158 & -79 & -12 & 14 & 4 \\ 0 & 0 & 0 & -7 & -2 \end{bmatrix}$$

solves the poleplacement problem in controller-canonical coordinates.

$$\underline{u} = E \cdot \underline{r} + \hat{F} \cdot \underline{\xi}$$

$$\underline{\xi} = T_{CCF} \cdot \underline{x}$$

$$\Rightarrow \quad \underline{u} = E \cdot \underline{r} + (\underbrace{\hat{F} \cdot T_{CCF}}_{F}) \cdot \underline{x}$$

Backtransformation:

$$\boxed{F = \hat{F} \cdot T_{CCF}}$$

$$\Rightarrow F = \begin{bmatrix} 36 & -127 & 17.75 & -36.75 & 31.75 \\ 0 & 0 & -0.5 & -0.75 & 2 \end{bmatrix}$$

is the feedback in the original coordinates.

$$\Rightarrow \quad \boxed{K = -F'}$$

$$K = \begin{bmatrix} -36 & 0 \\ 127 & 0 \\ -17.75 & 0.5 \\ 36.75 & 0.75 \\ -31.75 & -2 \end{bmatrix}$$

is the required observer-gain matrix.

$$\Rightarrow L = B - KD \equiv B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M = A - KC = \begin{bmatrix} 36 & 1 & 72 & 144 & 72 \\ -128 & -2 & -254 & -508 & -254 \\ 17.25 & -0.5 & 33.5 & 72 & 35.5 \\ -37.5 & -0.75 & -76.5 & -147 & -72.5 \\ 33.75 & 2 & 69.5 & 122 & 59.5 \end{bmatrix}$$

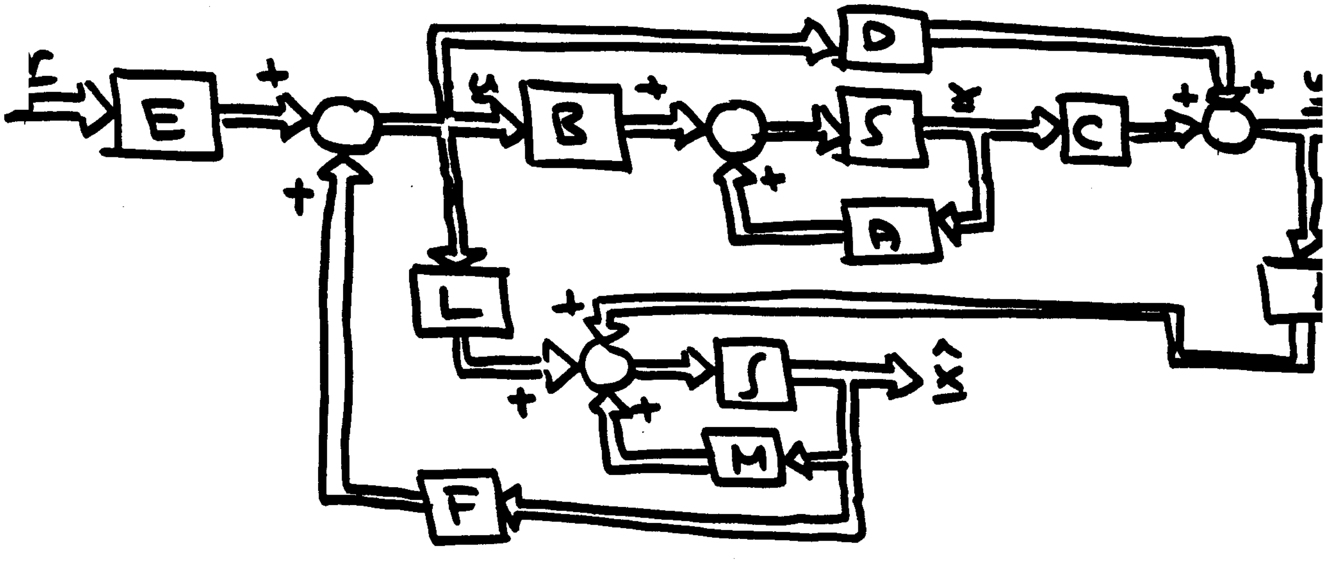
- Eig(M) are the observer poles as requested.

- This completes the observer design.
- We have not yet chosen E of the controller. We can still pick a desired E, e.g. to obtain desired DC-gains or whatever.
- Let $\underline{\underline{E = I^{(2)}}}$ ✓ .

$$\Rightarrow \text{System:} \left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} + D\underline{u} \end{array} \right|$$

$$\text{Observer:} \left| \dot{\hat{\underline{x}}} = M\hat{\underline{x}} + L\underline{u} + K\underline{y} \right|$$

$$\text{Controller:} \left| \underline{u} = E\underline{r} + F\hat{\underline{x}} \right|$$



Example: (continued)

$$\underline{x}_{cl} = \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix}$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + BE\underline{r} + BF\hat{\underline{x}} \\ \dot{\hat{\underline{x}}} = M\hat{\underline{x}} + LE\underline{r} + LF\hat{\underline{x}} + KC\underline{x} + KDE\underline{r} \\ \qquad\qquad\qquad\qquad\qquad\qquad + KDF\hat{\underline{x}} \\ \underline{y} = C\underline{x} + DE\underline{r} + DF\hat{\underline{x}} \end{array} \right|$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + BF\hat{\underline{x}} + BE\underline{r} \\ \dot{\hat{\underline{x}}} = KC\underline{x} + (M+LF+KDF)\hat{\underline{x}} + (LE+KDE) \\ \underline{y} = C\underline{x} + DF\hat{\underline{x}} + DE\underline{r} \end{array} \right.$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}}_{cl} = \left[\begin{array}{c|c} A & BF \\ \hline KC & \begin{array}{c} M+LF \\ +KDF \end{array} \end{array} \right] \underline{x}_{cl} + \left[\begin{array}{c} BE \\ \hdashline LE+KDE \end{array} \right] \underline{r} \\ \underline{y} = \left[\begin{array}{c:c} C & DF \end{array} \right] \underline{x}_{cl} + \left[DE \right] \underline{r} \end{array} \right|$$

is the closed-loop system.

- We choose e.g. F from p. 384:

$$F = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -18 & -16 & -4 \end{bmatrix}$$

$$\Rightarrow A_{cl} = \left[\begin{array}{ccccc|ccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -5 & -4 & 0 & 0 & -18 & -16 & -4 \\ \hline -36 & 0 & -72 & -144 & -72 & 36 & 1 & 72 & 144 & 72 \\ 127 & 0 & 254 & 508 & 254 & -129 & -2 & -254 & -508 & -254 \\ -16.25 & 0.5 & -31.5 & -67 & -33.5 & 16.25 & -0.5 & 31.5 & 68 & 33.5 \\ 39 & 0.75 & 79.5 & 153 & 76.5 & -39 & -0.75 & -79.5 & -153 & -75.5 \\ -37.75 & -2 & -79.5 & -143 & -71.5 & 37.75 & 2 & 59.5 & 122 & 63.5 \end{array}\right]$$

$$\mathrm{Eig}(A_{cl}) = \{-1 \pm j\ ;\ -2 \pm j\ ;\ -2 \pm 2j\ ;\ -4\ ;\ -4 \pm 2j\ ;\ -$$

as expected.

$$B_{cl} = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right] \Rightarrow \underline{\underline{\mathrm{Rank}(Q_c) = 5}}$$

$\Rightarrow$ The observer poles are uncontrollable as expected.

(2) Minimum-order Luenberger Observer:

Idea: If C were a non-singular matrix (same number of outputs as states), i.e.

$$\underline{y} = C\underline{x}$$

$$\Rightarrow \boxed{\hat{\underline{x}} = C^{-1}\underline{y}}$$

can be directly computed $\Rightarrow$ no "observer" is necessary (or at least no observer poles are necessary).

General case: If p outputs are measured, and if

$$\text{Rank}(C) \equiv p$$

(which we require always)

$\Rightarrow$ It should be possible to design an observer with (n-p) poles only, and use the measurements directly.

Algorithm:

$$\underline{\dot{x}} = \left[\begin{array}{c} \underline{\dot{x}}_1 \\ \hline \underline{\dot{x}}_2 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \cdot \left[\begin{array}{c} \underline{x}_1 \\ \hline \underline{x}_2 \end{array}\right] + \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] \cdot \underline{u}$$

$$\underline{y} = \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right] \cdot \left[\begin{array}{c} \underline{x}_1 \\ \hline \underline{x}_2 \end{array}\right]$$

(after eliminating the direct input/output coupling for the time being).

$\underline{x} \in \mathbb{R}^n$; $\underline{u} \in \mathbb{R}^m$; $\underline{y} \in \mathbb{R}^p$

$\underline{x}_1 \in \mathbb{R}^{n-p}$; $\underline{x}_2 \in \mathbb{R}^p$

- We can assume C_2 to be regular. Otherwise, we simply shift p linearly independent columns of C to the right by renumbering the states.

- We apply the following similarity transformation:

$$\underline{\xi} = T\,\underline{x} \;\; ; \;\; T = \left[\begin{array}{c:c} I^{(n-p)} & 0 \\ \hdashline C_1 & C_2 \end{array}\right]$$

$$\Rightarrow \underline{\xi} = \left[\begin{array}{c:c} I^{(n-p)} & 0 \\ \hdashline C_1 & C_2 \end{array}\right] \cdot \left[\begin{array}{c} \underline{x}_1 \\ \hdashline \underline{x}_2 \end{array}\right] = \left[\begin{array}{c} \underline{x}_1 \\ \hdashline \underline{y} \end{array}\right]$$

$\Rightarrow$ The outputs have become state variables in this new representation.

$$\Rightarrow T^{-1} = \left[\begin{array}{c|c} I^{(n-p)} & \phi \\ \hline -C_2^{-1}C_1 & C_2^{-1} \end{array}\right]$$

(With this choice, there is no need to invert an (n×n) matrix, only a (p×p) matrix.)

$$\Rightarrow \dot{\underline{\xi}} = \underbrace{TAT^{-1}}_{\hat{A}}\, \underline{\xi} + \underbrace{TB}_{\hat{B}}\, \underline{u}$$

$$\Rightarrow \hat{A} = \left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right] \quad ; \quad \hat{B} = \left[\begin{array}{c} B_1 \\ \hline \hat{B}_2 \end{array}\right]$$

where:

$$\left|\begin{aligned} &P = A_{11} - A_{12}C_2^{-1}C_1 \\ &Q = A_{12}\,C_2^{-1} \\ &R = C_1A_{11} + C_2A_{21} - (C_1A_{12} + C_2A_{21}) \cdot C_2^{-1} \cdot C_1 \\ &S = (C_1A_{12} + C_2A_{22})\,C_2^{-1} \\ &\hat{B}_2 = C_1B_1 + C_2B_2 \end{aligned}\right.$$

or cheaper:

$$\left|\begin{array}{l} Q = A_{12} C_2^{-1} \\ S = (C_1 A_{12} + C_2 A_{22}) C_2^{-1} \\ P = A_{11} - Q C_1 \\ R = C_1 A_{11} + C_2 A_{21} - S C_1 \\ \hat{B}_2 = C_1 B_1 + C_2 B_2 \end{array}\right|$$

$$\underline{y} = \underbrace{C T^{-1}}_{\hat{C}} \cdot \underline{\xi}$$

where: $\hat{C} = [\,0 \;\vdots\; I^{(p)}\,]$

Example: (continued)

$$\underline{y} = \left[\begin{array}{ccc:cc} 1 & 0 & 2 & 4 & 2 \\ 1 & 1 & 4 & 0 & 0 \end{array}\right] \underline{x}$$

$$\underbrace{}_{C_1} \quad \underbrace{}_{C_2}$$

$\mathrm{Rank}(C_2) = 1$

$\Rightarrow$ renumber the states, e.g. exchange $x_1 \leftrightarrow x_5$

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \dot{\underline{z}} = \left[\begin{array}{ccc:cc} -4 & 0 & -2 & -5 & 0 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \underline{z} + \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ \hdashline 0 & 0 \\ 0 & 0 \end{array}\right] \underline{u}$$

$$\underline{y} = \left[\begin{array}{ccc:cc} 2 & 0 & 2 & 4 & 1 \\ 0 & 1 & 4 & 0 & 1 \end{array}\right] \underline{z}$$

is of course no longer in controller-canonical form.

- Now, we choose:

$$T = \left[\begin{array}{ccc:cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hdashline 2 & 0 & 2 & 4 & 1 \\ 0 & 1 & 4 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \quad \dot{\underline{\xi}} = \left[\begin{array}{ccc:cc} -1.5 & -1.25 & -4.5 & -1.25 & 1.25 \\ 0 & -1 & 4 & 0 & -1 \\ 0.5 & 0.25 & 0.5 & 0.25 & -0.25 \\ \hdashline 0 & -1 & -8 & -2 & 2 \\ -2 & 1 & 6 & 1 & -2 \end{array}\right] \underline{\xi} + \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ \hdashline 0 & 2 \\ 1 & 0 \end{array}\right] \underline{u}$$

$$\underline{y} = \left[\begin{array}{ccc:cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \underline{\xi}$$

The idea is to create an observer for $\underline{x}_1$ alone, and use $\underline{x}_2 \equiv \underline{y}$ directly.

- We try the following approach:

$$\dot{\hat{\underline{x}}}_1 = M\hat{\underline{x}}_1 + L\underline{u} + K\underline{y}$$

(as before).

$$\Rightarrow \underline{e}_1 = \underline{x}_1 - \hat{\underline{x}}_1$$

$$\Rightarrow \dot{\underline{e}}_1 = -M\hat{\underline{x}}_1 - L\underline{u} - K\underline{y} + P\underline{x}_1 + Q\underline{y} + B_1\underline{u}$$

$$\Rightarrow \dot{\underline{e}}_1 = M\underline{x}_1 - M\hat{\underline{x}}_1 + P\underline{x}_1 - M\underline{x}_1 + Q\underline{y} - K\underline{y} + B_1\underline{u} - L$$

$$\Rightarrow \dot{\underline{e}}_1 = M\underline{e}_1 + \underbrace{(P-M)}_{0}\underline{x}_1 + \underbrace{(Q-K)}_{0}\underline{y} + \underbrace{(B_1-L)}_{0}\underline{u}$$

$\Longrightarrow$ Does not work. Not enough freedom.

- We try the following modified approach:

$$\left|\begin{aligned} \dot{\underline{w}} &= M\underline{w} + L\underline{u} + K\underline{y} \\ \hat{\underline{x}}_1 &= \underline{w} + N\underline{y} \end{aligned}\right|$$

$$\Rightarrow \dot{\underline{e}}_1 = \dot{\underline{x}}_1 - \dot{\hat{\underline{x}}}_1 = P\underline{x}_1 + Q\underline{y} + B_1\underline{u} - \dot{\underline{w}} - N\dot{\underline{y}}$$

$$= P\underline{x}_1 + Q\underline{y} + B_1\underline{u} - M\underline{w} - L\underline{u} - K\underline{y}$$

$$- NR\underline{x}_1 - NS\underline{y} - N\hat{B}_2\underline{u}$$

$$= P\underline{x}_1 + Q\underline{y} + B_1\underline{u} - M\hat{\underline{x}}_1 + MN\underline{y} - L\underline{u} - K\underline{y}$$

$$- NR\underline{x}_1 - NS\underline{y} - N\hat{B}_2\underline{u}$$

$$\Rightarrow \dot{\underline{e}}_1 = M\underline{x}_1 - M\hat{\underline{x}}_1 + P\underline{x}_1 - NR\underline{x}_1 - M\underline{x}_1$$

$$+ Q\underline{y} + MN\underline{y} - K\underline{y} - NS\underline{y} + B_1\underline{u} - L\underline{u} - N\hat{B}_2$$

$$\Rightarrow \dot{\underline{e}}_1 = M\underline{e}_1 + (P - NR - M)\underline{x}_1 + (Q + MN - K - NS)$$

$$+ (B_1 - L - N\hat{B}_2)\underline{u}$$

We want: $\boxed{\dot{\underline{e}}_1 = M\underline{e}_1}$

$$\Rightarrow \quad M = P - NR$$

$\leftarrow$ { Pole placement problem

Then: $K = Q + MN - NS$

$L = B_1 - N\hat{B}_2$

Finally: $\hat{\underline{x}}_2 = C_2^{-1}\underline{y} - C_2^{-1}C_1\hat{\underline{x}}_1$

Example: (continued)

We solve the following poleplacement problem:

$$\dot{\underline{y}} = P'\underline{y} + R'\underline{u}$$

$$\Rightarrow \quad \dot{\underline{y}} = \begin{bmatrix} -1.5 & 0 & -0.5 \\ -1.25 & -1 & 0.25 \\ -4.5 & 4 & 0.5 \end{bmatrix} \underline{y} + \begin{bmatrix} 0 & -2 \\ -1 & 1 \\ -8 & 6 \end{bmatrix} \underline{u}$$

We transform into controller-canonical form:

$$Q_c = \left[\begin{array}{cc:cc:cc} 0 & -2 & 4 & 0 & -2 & -8 \\ -1 & 1 & -1 & 3 & -6 & 1 \\ -8 & 6 & -8 & 16 & -26 & 20 \end{array}\right]$$

$$\Rightarrow \hat{L} = \begin{bmatrix} 0 & -2 & 4 \\ -1 & 1 & -1 \\ -8 & 6 & -8 \end{bmatrix} \Rightarrow L = \begin{array}{c} \begin{bmatrix} 0 & 4 & -2 \\ -1 & -1 & 1 \\ -8 & -8 & 6 \end{bmatrix} \\ \quad \underset{d_1}{\longleftrightarrow} \ \underset{d_2}{\leftrightarrow} \end{array}$$

$$\Rightarrow L^{-1} = \begin{bmatrix} -0.25 & 1 & -0.25 \\ 0.25 & 2 & -0.25 \\ 0 & 4 & -0.5 \end{bmatrix}$$

$$\Rightarrow \underline{q}_1' = [\,0.25 \quad 2 \quad -0.25\,]$$

$$\underline{q}_2' = [\,0 \quad 4 \quad -0.5\,]$$

$$\Rightarrow T = \begin{bmatrix} \underline{q}_1' \\ \underline{q}_1'P' \\ \underline{q}_2' \end{bmatrix} = \begin{bmatrix} 0.25 & 2 & -0.25 \\ -1.75 & -3 & 0.25 \\ 0 & 4 & -0.5 \end{bmatrix}$$

$$\Rightarrow \dot{\underline{\xi}} = \begin{bmatrix} 0 & 1 & 0 \\ -21 & -6 & 7 \\ -11 & 0 & 4 \end{bmatrix} \underline{\xi} + \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \underline{u}$$

We place the observer poles at:

$$\underline{P}_o = \{ -2 \pm 2j \ ; \ -4 \}$$

$$\Rightarrow Q_1(s) = s^2 + 4s + 8$$

$$Q_2(s) = s + 4$$

$$\Rightarrow A_{cl} = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ -8 & -4 & 0 \\ \hdashline 0 & 0 & -4 \end{array}\right]$$

$$\hat{B} \cdot \hat{F} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ (F_{11} + 2F_{21}) & (F_{12} + 2F_{22}) & (F_{13} + 2F_{23}) \\ F_{21} & F_{22} & F_{23} \end{bmatrix}$$

$$\Rightarrow \hat{A} + \hat{B} \cdot \hat{F} = \begin{bmatrix} 0 & 1 & 0 \\ (F_{11} + 2F_{21} - 21) & (F_{12} + 2F_{22} - 6) & (F_{13} + 2F_{23} + 7) \\ (F_{21} - 11) & F_{22} & (F_{23} + 4) \end{bmatrix}$$

$$\equiv A_{cl}$$

$$\Rightarrow \quad \hat{F} = \begin{bmatrix} -9 & 2 & 9 \\ 11 & 0 & -8 \end{bmatrix}$$

$$\Rightarrow F = \hat{F} \cdot T_{ccF} = \begin{bmatrix} -5.75 & 12 & -1.75 \\ 2.75 & -10 & 1.25 \end{bmatrix}$$

$$\Rightarrow \quad N = -F' = \begin{bmatrix} 5.75 & -2.75 \\ -12 & 10 \\ 1.75 & -1.25 \end{bmatrix}$$

$$\Rightarrow \quad M = P - NR = \begin{bmatrix} -7 & 7.25 & 58 \\ 20 & -23 & -152 \\ -3 & 3.25 & 22 \end{bmatrix}$$

Eig(M) are as requested.

$$\Rightarrow K = Q + MN - NS = \begin{bmatrix} -12.75 & 3.5 \\ 91 & -52 \\ -12.75 & 7 \end{bmatrix}$$

$$\Rightarrow L = B_1 - N\hat{B}_2 = \begin{bmatrix} 2.75 & -10.5 \\ -9 & 24 \\ 1.25 & -3.5 \end{bmatrix}$$

$\Rightarrow$ **System:** $\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} \end{array} \right|$

Observer: $\left| \begin{array}{l} \dot{\underline{w}} = M\underline{w} + L\underline{u} + K\underline{y} \\ \hat{\underline{x}}_1 = \underline{w} + N\underline{y} \\ \hat{\underline{x}}_2 = C_2^{-1}\underline{y} - C_2^{-1}C_1\hat{\underline{x}}_1 \end{array} \right|$

Controller: $\left| \underline{u} = E\underline{r} + F_1\hat{\underline{x}}_1 + F_2\hat{\underline{x}}_2 \right|$

$$\Rightarrow \dot{\underline{x}} = A\underline{x} + BE\underline{r} + BF_1\hat{\underline{x}}_1 + BF_2\hat{\underline{x}}_2$$

$$= A\underline{x} + BE\underline{r} + BF_1\hat{\underline{x}}_1 + BF_2C_2^{-1}\underline{y} - BF_2C_2^{-1}C_1$$

$$\Rightarrow \dot{\underline{x}} = \left[A + BF_2C_2^{-1}C\right]\underline{x} + \left[BF_1 - BF_2C_2^{-1}C_1\right]\hat{\underline{x}}_1$$

$$+ \left[BE\right]\underline{r}$$

$$\dot{\hat{\underline{x}}}_1 = \dot{\underline{w}} + N\dot{\underline{y}} = M\underline{w} + L\underline{u} + K\underline{y} + NC\dot{\underline{x}}$$

$$= M(\hat{\underline{x}}_1 - N\underline{y}) + L\underline{u} + K\underline{y} + NC(A\underline{x} + B\underline{u})$$

$$= M\hat{\underline{x}}_1 - MNC\underline{x} + KC\underline{x} + NCA\underline{x} + L\underline{u}$$

$$+ NCB\underline{u}$$

$$= M\hat{\underline{x}}_1 - MNC\underline{x} + KC\underline{x} + NCA\underline{x} + LE\underline{r} + LF_1\hat{\underline{x}}_1 + LF_2\hat{\underline{x}}_2 + NCBE\underline{r} + NCBF_1\hat{\underline{x}}_1 + NCBF_2\hat{\underline{x}}_2$$

$$= M\hat{\underline{x}}_1 - MNC\underline{x} + KC\underline{x} + NCA\underline{x} + LF_1\hat{\underline{x}}_1 + NCBF_1\hat{\underline{x}}_1 + LE\underline{r} + NCBE\underline{r} + LF_2C_2^{-1}C\underline{x} - LF_2C_2^{-1}C_1\hat{\underline{x}}_1 + NCBF_2C_2^{-1}C\underline{x} - NCBF_2C_2^{-1}C_1\hat{\underline{x}}_1$$

$$\Rightarrow \dot{\hat{\underline{x}}}_1 = [KC + NCA - MNC + LF_2C_2^{-1}C + NCBF_2C_2^{-} \cdot \underline{x} + [M + LF_1 + NCBF_1 - LF_2C_2^{-1}C_1 - NCBF_2C_2^{-1}C_1] + [LE + NCBE]\underline{r}$$

$$\underline{y} = C\underline{x}$$

Example: (continued)

$$\underline{x}_{cl} = \begin{bmatrix} \underline{x} \\ \hat{\underline{x}}_1 \end{bmatrix}$$

We start after renumbering the state variables. Thus:

$$A = \begin{bmatrix} -4 & 0 & -2 & -5 & 0 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \underbrace{\begin{matrix} 2 & 0 & 2 \\ 0 & 1 & 4 \end{matrix}}_{C_1} & \underbrace{\begin{matrix} 4 & 1 \\ 0 & 1 \end{matrix}}_{C_2} \end{bmatrix}$$

$$F = \begin{bmatrix} \underbrace{\begin{matrix} 0 & 0 & 0 \\ -4 & 0 & -18 \end{matrix}}_{F_1} & \underbrace{\begin{matrix} 0 & -1 \\ -16 & 0 \end{matrix}}_{F_2} \end{bmatrix}$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}}_{cl} = A_{cl}\, \underline{x}_{cl} + B_{cl} \cdot \underline{r} \\ \underline{y} = C_{cl}\, \underline{x}_{cl} \end{array} \right|$$

$$A_{cl} = \left[\begin{array}{ccccc|ccc} -12 & 4 & 6 & -21 & 0 & 4 & -4 & -26 \\ 0 & -3 & -4 & 0 & -2 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -5 & -3.25 & -52 & -21 & 0 & -3 & 3.25 & 32 \\ -20 & 20 & 148 & 0 & -2 & 20 & -22 & -148 \\ 3 & -3.25 & -22 & 1 & 0 & -3 & 3.25 & 22 \end{array} \right]$$

$$\mathrm{Eig}(A_{cL}) = \{ -1 \pm j \; ; -2 \pm j \; ; -2 \pm 2j \; ; -4 \; ; -4$$

as expected.

$$B_{cL} = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{array}\right] \Rightarrow \underline{\underline{\mathrm{Rank}(Q_c) = 5}}$$

Again, the observer poles are uncontrollable.

$$C_{cL} = \left[\begin{array}{ccccc|ccc} 2 & 0 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$