Polynomial Matrix Representation:

Beside from state-space representations and transferfunction matrices, there exists yet another representation which is commonly used. This is the polynomial matrix representation.

Problem:

In the frequency domain, we usually operate on rational functions. This is potentially harmful in a numerical sense, as there are poles in unpredictable places.

⟹ The sensitivity of parameters in any such representation (coefficients, roots, etc.) will be poorly balanced in the neighborhood of these poles.

Idea: Spare the division in each algorithm to the very last step, and operate strictly on polynomial matrices ever before. Thus:

$$\underline{Y}(s) = \underline{\underline{G}}(s) \cdot \underline{u}(s) = \frac{\underline{\underline{P}}(s)}{Q(s)} \cdot \underline{u}(s)$$

is now represented as:

$$\boxed{Q(s) \cdot \underline{Y}(s) = \underline{\underline{P}}(s) \cdot \underline{u}(s)}$$

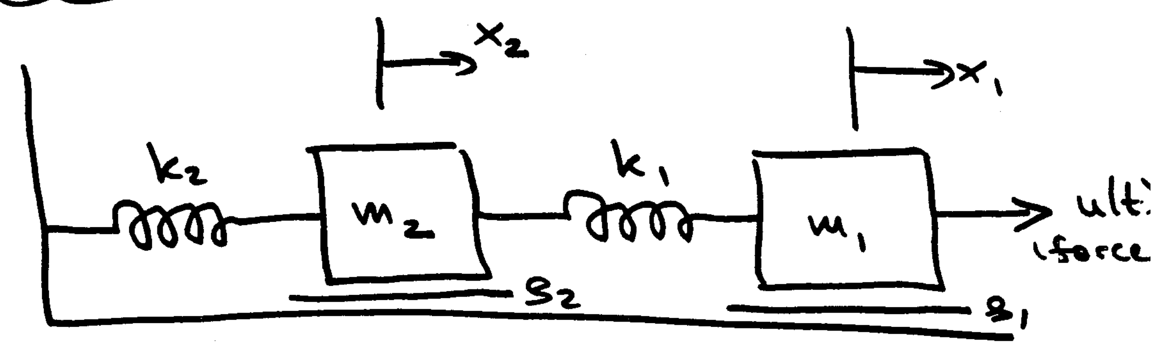
This is a possible polynomial matrix representation.

⇒ This idea was generalized e.g. by:

W.A. Wolovich: "Linear Multivariable Systems", Springer 1974.

- The basic idea works as follows: In the time domain, we always operated on sets of 1^{st} order ODE's. In the frequency domain, we operated on characteristic polynomials $\Longleftrightarrow$ one n^{th}-order ODE. However, any representation between these two extremes is also possible.

Example:



$$\Rightarrow \left| \begin{array}{l} m_1\ddot{x}_1 + s_1\dot{x}_1 + k_1(x_1 - x_2) = u(t) \\ m_2\ddot{x}_2 + s_2\dot{x}_2 - k_1(x_1 - x_2) + k_2 x_2 = \emptyset \\ y_1 = x_1 \\ y_2 = \dot{x}_1 \end{array} \right|$$

$$\Rightarrow \left| \begin{array}{l} \begin{bmatrix} (m_1 s^2 + s_1 s + k_1) & -k_1 \\ -k_1 & (m_2 s^2 + s_2 s + k_1 + k_2) \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \emptyset \end{bmatrix} u \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & \emptyset \\ s & \emptyset \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array} \right|$$

would be a possible polynomial matrix representation. While the state-space representations operate on 4 1^{st}-order ODE's, and the transfer function operates on 1 4^{th}-order ODE, this representation operates on 2 2^{nd}-order ODE's.

In general:

$$\left| \begin{aligned} \underline{\underline{P}}(s)\cdot \underline{x}(t) &= \underline{\underline{Q}}(s)\cdot \underline{u}(t) \\ \underline{y}(t) &= \underline{\underline{R}}(s)\cdot \underline{x}(t) + \underline{\underline{W}}(s)\cdot \underline{u}(t) \end{aligned} \right|$$

$\underline{u}(t) \in \mathbb{R}^{m\times 1}$ ∴ Input vector

$\underline{y}(t) \in \mathbb{R}^{p\times 1}$ ∴ Output vector

$\underline{x}(t) \in \mathbb{R}^{q\times 1}$ ∴ Partial state vector

$1 \le q \le n$

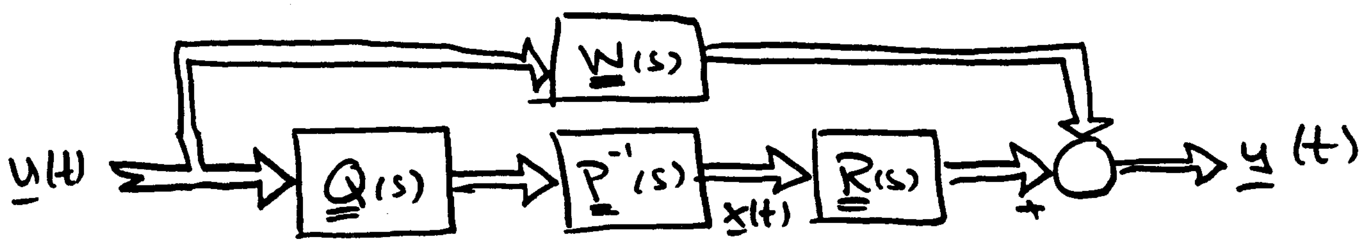
$\Rightarrow \underline{\underline{P}}(s) \in \mathbb{R}^{q\times q}$; $\underline{\underline{Q}}(s) \in \mathbb{R}^{q\times m}$;

$\underline{\underline{R}}(s) \in \mathbb{R}^{p\times q}$; $\underline{\underline{W}}(s) \in \mathbb{R}^{p\times m}$

- Notice the mixture between time domain ($\underline{x}(\underline{\underline{t}})$) and frequency domain ($\underline{P}(\underline{\underline{s}})$)

Graphical representation:

$$\underline{x}(t) = \underline{\underline{P}}^{-1}(s) \cdot \underline{\underline{Q}}(s) \cdot \underline{u}(t)$$



- $\Rightarrow$ $\underline{\underline{W}}(s)$ is the direct input/outpu coupling $\hat{=}$ D - matrix.

$\Leftrightarrow$ $\begin{cases} \text{System strictly proper} \Leftrightarrow \underline{\underline{W}}(s) \equiv \emptyset \\ \text{System proper} \Leftrightarrow \underline{\underline{W}}(s) \equiv \text{con} \\ \text{System not proper} \Leftrightarrow \underline{\underline{W}}(s) = f(s \end{cases}$

- $\Rightarrow$ We can and will make the transfer function matrix:

$$R(s) \cdot P^{-1}(s) \cdot Q(s)$$

<u>always</u> strictly proper.

$\Rightarrow$ We can and will always make $P(s)$ nonsingular

$$\Rightarrow \quad |P(s)| \neq \emptyset .$$

Notice: We can again define a "system"-matrix $\underline{\underline{S}}(s)$ as:

$$\underline{\underline{S}}(s) = \begin{array}{cc} \left[\begin{array}{c|c} \underline{\underline{P}}(s) & \underline{\underline{Q}}(s) \\ \hline \underline{\underline{R}}(s) & \underline{\underline{W}}(s) \end{array}\right] & \begin{array}{c} \updownarrow q \\ \updownarrow p \end{array} \\ \begin{array}{cc} \overset{\longleftrightarrow}{q} & \overset{\longleftrightarrow}{m} \end{array} & \end{array}$$

similar to the "S"-matrix in the time domain.

Notice: Our "s"-operator is really not the normal Laplacian operator, but simply:

$$\boxed{s \mathrel{\hat{=}} \frac{d}{dt}}$$

This is equivalent to the Laplacian iff all initial conditions are zero which we, however, do <u>not</u> request. For this reason, many authors prefer to use a different letter:

Wolovich : D
Zadeh : p

We shall stick to s, but we shall remember what we just did. In particular:

$$\boxed{s \cdot \frac{1}{s} \neq \frac{1}{s} \cdot s}$$

$\Longrightarrow$ The division must be properly defined.

<u>However</u>, this is not really a problem as long as we stick to polynomials only ($\frac{1}{s}$ does not even exist). <u>Moreover</u>, matrix multiplications are anyway <u>not</u> commutative:

$$\boxed{\underline{\underline{P}}(s) \cdot \underline{\underline{Q}}(s) \neq \underline{\underline{Q}}(s) \cdot \underline{\underline{P}}(s)}$$

$$\Rightarrow \boxed{\underline{\underline{R}}_1(s)} \Rightarrow \boxed{\underline{\underline{R}}_2(s)} \Rightarrow$$

$$\underline{\underline{R}}(s) \equiv \underline{\underline{R}}_2(s) \cdot \underline{\underline{R}}_1(s)$$

normal matrix multiplication

<u>**but**</u> $\underline{\underline{R}}_1(s) \cdot \underline{\underline{R}}_2(s) \neq \underline{\underline{R}}_2(s) \cdot \underline{\underline{R}}_1(s)$

CASCADING MEANS MULTIPLICATION IN REVERSED ORDER.

<u>Problem</u>: As we just flushed the division down the drain, we no longer operate on an <u>algebraic body</u>, but only on a <u>commutative ring</u> structure.

⇒ We must "relearn" the basic operations available for these data structures.

Idea: We remember that we know such a structure since 1st year grammar school. The integer numbers have the same properties:

$$\underbrace{5/3}_{\text{integers}} = \underbrace{1.666\ldots}_{\text{not integer}}$$

⇒ Algorithms that work on integers will work fine on polynomial matrices.

Solution: We define a set of elementary operations that are welldefined for polynomial matrices, and base all other operations on those.

Elementary Operations

(1) Row-Operations

There exist three elementary row operations. Each of them involves one <u>left-multiplication</u> with a special matrix.

(a) E^1-operator:

$$E_{ij}^{1(n)} = \begin{array}{cc}
\left[\begin{array}{c|c|c|c|c}
1 \ \ddots & \phi & \phi & \phi & \phi \\ \hline
\phi & \phi & \phi & 1 & \phi \\ \hline
\phi & \phi & 1 \ \ddots & \phi & \phi \\ \hline
\phi & 1 & \phi & \phi & \phi \\ \hline
\phi & \phi & \phi & \phi & \ddots \ 1
\end{array}\right] & \begin{array}{c} \\ i \\ \\ j \\ n \end{array} \\
\begin{array}{ccccc} \quad & i & \quad & j & n \end{array} &
\end{array}$$

<u>Example</u>:

$$E_{25}^{1(6)} = \begin{bmatrix}
1 & \phi & \phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi & 1 & \phi \\
\phi & \phi & 1 & \phi & \phi & \phi \\
\phi & \phi & \phi & 1 & \phi & \phi \\
\phi & 1 & \phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi & \phi & 1
\end{bmatrix}$$

$$\underline{\underline{P}}_2(s) = E_{ij}^{1(n)} \cdot \underline{\underline{P}}_1(s)$$

$$\boxed{\begin{array}{l}\Rightarrow \underline{\underline{P}}_1(s) \text{ and } \underline{\underline{P}}_2(s) \text{ are almost the same, except that the} \\ i^{th} \text{ and the } j^{th} \text{ row have been exchanged.}\end{array}}$$

$$\begin{bmatrix} 1 & \phi & \phi & \phi \\ \phi & \phi & 1 & \phi \\ \phi & 1 & \phi & \phi \\ \phi & \phi & \phi & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{1} \\ a_{31} & a_{32} & a_{3} \\ a_{21} & a_{22} & a_{2} \\ a_{41} & a_{42} & a_{4} \end{bmatrix}$$

(b) $\underline{E^2\text{- operator}}$:

$$E_i^{2(n)}(c) = \begin{array}{c|c} \begin{bmatrix} \begin{array}{cc|cc} 1 \ddots & \phi & & \phi \\ \phi & 1 & & \\ \hline & & c & \\ & \phi & & 1 \ddots \ \phi \\ & & & \phi \quad 1 \end{array} \end{bmatrix} & \begin{array}{c} \\ \\ i \\ \\ n \end{array} \\ \begin{array}{cc} \qquad\qquad i & \qquad n \end{array} & \end{array}$$

c is a $\underline{\underline{\text{scalar}}} \neq \phi$

Example:

$$E_3^{2(s)}(18.7) = \begin{bmatrix} 1 & \phi & \phi & \phi & \phi \\ \phi & 1 & \phi & \phi & \phi \\ \phi & \phi & 18.7 & \phi & \phi \\ \phi & \phi & \phi & 1 & \phi \\ \phi & \phi & \phi & \phi & 1 \end{bmatrix}$$

$$\underline{\underline{P}}_2(s) = E_i^{2(n)}(c) \cdot \underline{\underline{P}}_1(s)$$

$\Rightarrow$ $\boxed{\text{The } i^{th} \text{ row of } \underline{\underline{P}}_1(s) \text{ got multiplied by } c,\ c \neq \phi.}$

$$\begin{bmatrix} 1 & \phi & \phi & \phi \\ \phi & 1 & \phi & 0 \\ \phi & \phi & c & \phi \\ \phi & \phi & \phi & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1} \\ a_{21} & a_{22} & a_{23} & a_{2} \\ c \cdot a_{31} & c \cdot a_{32} & c \cdot a_{33} & c \\ a_{41} & a_{42} & a_{43} & a \end{bmatrix}$$

(c) $\underline{E^3\text{- operator}}$:

$$E_{ij}^{3(n)}(b(s)) = \begin{array}{c} \left[\begin{array}{c|c|c} \begin{matrix} 1 & \phi \\ \phi & \ddots \end{matrix} & \phi & \phi \\ \hline & 1 \cdots \cdots b(s) & \cdots \\ \phi & \begin{matrix} \ddots & \phi \\ \phi & \ddots \end{matrix} & 0 \\ \hline \phi & \phi & \begin{matrix} \ddots & \phi \\ \phi & 1 \end{matrix} \end{array}\right] \begin{array}{l} \\ i \\ \\ j \\ n \end{array} \\ \begin{array}{ccc} \qquad\quad i & \qquad j & \quad n \end{array} \end{array}$$

Example:

$$E_{52}^{3(6)}(3s+5) = \begin{bmatrix} 1 & \phi & \phi & \phi & \phi & \phi \\ \phi & 1 & \phi & \phi & \phi & \phi \\ \phi & \phi & 1 & \phi & \phi & \phi \\ \phi & \phi & \phi & 1 & \phi & \phi \\ \phi & (3s+5) & \phi & \phi & 1 & \phi \\ \phi & \phi & \phi & \phi & \phi & 1 \end{bmatrix}$$

$\boxed{\Rightarrow \text{The } i^{th} \text{ row of } \underline{\underline{P}}_1(s) \text{ gets replaced by:}}$

$$i^{th}\text{ row} \Leftarrow i^{th}\text{ row} + b(s) * j^{th}\text{ row}$$

$$\begin{bmatrix} 1 & \phi & \phi & \phi \\ \phi & 1 & \phi & 5 \\ \phi & \phi & 1 & \phi \\ \phi & \phi & \phi & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ (a_{21}+5a_{41}) & (a_{22}+5a_{42}) \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \cdots$$

(2) Column-operations:

The column-operations are the same except that now the multiplication is done from the right.

Notice:

- All elementary matrices are nonsingular:

$$\det(E^1) = \det(E^3) \equiv 1$$
$$\det(E^2) = c \neq \phi$$

- The determinant of all elementary matrices is constant $\neq f(s)$.
- The product of several elementary matrices is a polynomial matrix with constant determinant $\neq \phi$.

Def: Polynomial matrices with determinant $= c \neq f(s) \neq \phi$ are called unimodular.

Lemma: (without proof) All unimodular matrices can be written as products of elementary matrices.

Lemma: (without proof) All algorithm of linear algebra can be expressed through series of elementary operations.

Example: Given $P(s)$, find $P^{-1}(s)$

$\Rightarrow$ We use Gaussian elimination.

Algorithm:

(1) We add the product of the first row and $e_{i1}(s) = -\, p_{i1}(s)/p_{11}(s)$ to the i^{th} row:

$$K_1(s) = \begin{bmatrix} 1 & & & & \\ e_{21}(s) & 1 & & \emptyset & \\ e_{31}(s) & & 1 & & \\ \vdots & \emptyset & & \ddots & \\ e_{n1}(s) & & & & 1 \end{bmatrix}$$

$$K_1(s) = E_{21}^{3(n)}(e_{21}(s)) \cdot E_{31}^{3(n)}(e_{31}(s)) \cdot \ldots \cdot E_{n1}^{3(n)}(e_{n1}(s))$$

$\Rightarrow K_1(s)$ is a unimodular matrix.

$$\Rightarrow P_1(s) = K_1(s) \cdot P(s) = \begin{bmatrix} p_{11}(s) & p_{12}(s) & \cdots & p_{1n}(s) \\ \emptyset & p^1_{22}(s) & \cdots & p^1_{2n}(s) \\ \emptyset & p^1_{32}(s) & \cdots & p^1_{3n}(s) \\ \vdots & \vdots & & \vdots \\ \emptyset & p^1_{n2}(s) & \cdots & p^1_{nn}(s) \end{bmatrix}$$

where: $P^1_{ij}(s) = P_{ij}(s) + e_{i1}(s) \cdot P_{1j}(s)$

(2) We add the product of the 2nd row and $e_{i2}(s) = -P^1_{i2}(s)/P^1_{22}(s)$ to the i^{th} row.

$$K_2(s) = \begin{bmatrix} 1 & & & & \\ \phi & 1 & \phi & & \\ \phi & e_{32}(s) & 1 & & \\ \vdots & \vdots & \phi & \ddots & \\ \phi & e_{n2}(s) & & & 1 \end{bmatrix}$$

$$\Rightarrow P_2(s) = K_2(s) \cdot P_1(s) = K_2(s) \cdot K_1(s) \cdot P(s)$$

$$= \begin{bmatrix} P_{11}(s) & P_{12}(s) & \cdots & P_{1n}(s) \\ \phi & P^1_{22}(s) & \cdots & P^1_{2n}(s) \\ \phi & \phi & P^2_{33}(s) \cdots & P^2_{3n}(s) \\ \vdots & \vdots & \vdots & \vdots \\ \phi & \phi & P^2_{n3}(s) \cdots & P^2_{nn}(s) \end{bmatrix}$$

where: $P^2_{ij}(s) = P^1_{ij}(s) + e_{i2}(s) \cdot P^1_{2j}(s)$

etc.

$$\Rightarrow P_{n-1}(s) = \underbrace{K_{n-1}(s)\cdot K_{n-2}(s)\cdot \ldots \cdot K_1(s)}_{K(s)}\cdot P(s) =$$

$$\Rightarrow \boxed{P_{n-1}(s) = K(s)\cdot P(s)}$$

upper triangular

unimodular

$$\boxed{P(s) = K^{-1}(s)\cdot P_{n-1}(s)}$$

$$\Rightarrow \boxed{P^{-1}(s) = P_{n-1}^{-1}(s)\cdot K(s)}$$

⇒ The determination of $P^{-1}(s)$ has been reduced to the determination of an inverse of an upper triangular matrix. This can be solved by simply plugging in from the ~~bottom~~ to the top:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ \phi & & & \vdots \\ & & & a_{nn} \end{bmatrix}\cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

$$y_n = a_{nn} x_n \implies x_n = \frac{1}{a_{nn}} y_n$$

$$y_{n-1} = a_{n-1,n-1} \cdot x_{n-1} + a_{n-1,n} \cdot x_n$$

$$\implies x_{n-1} = \frac{1}{a_{n-1,n-1}} y_{n-1} - \frac{a_{n-1,n}}{a_{nn}} y_n$$

etc.

$\implies$ This works even on rational function matrices.

Example:

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{2}{(s+2)} & \frac{-(s+3)}{(s+2)} \\ \frac{1}{(s+1)} & \frac{4}{(s+1)^2(s+2)} & \phi \\ \frac{s}{(s+1)} & \frac{5}{(s+1)(s+2)} & \frac{-3}{(s+2)} \end{bmatrix}$$

$$\implies K_1(s) = \begin{bmatrix} 1 & \phi & \phi \\ -(s+1) & 1 & \phi \\ -s(s+1) & \phi & 1 \end{bmatrix}$$

$$G_1(s) = K_1(s)G(s) = \begin{bmatrix} \dfrac{1}{(s+1)^2} & \dfrac{2}{(s+2)} & \dfrac{-(s+3)}{(s+2)} \\ \phi & \dfrac{4-2(s+1)^3}{(s+1)^2(s+2)} & \dfrac{(s+1)(s+3)}{(s+2)} \\ \phi & \dfrac{5-2s(s+1)^2}{(s+1)(s+2)} & \dfrac{s(s+1)(s+3)-3}{(s+2)} \end{bmatrix}$$

$$\Rightarrow K_2(s) = \begin{bmatrix} 1 & \phi & \phi \\ \phi & 1 & \phi \\ \phi & \dfrac{2s(s+1)^3 - 5(s+1)}{4-2(s+1)^3} & 1 \end{bmatrix}$$

$$\Rightarrow G_2(s) = \begin{bmatrix} \dfrac{1}{(s+1)^2} & \dfrac{2}{(s+2)} & \dfrac{-(s+3)}{(s+2)} \\ \phi & \dfrac{4-2(s+1)^3}{(s+1)^2(s+2)} & \dfrac{(s+1)(s+3)}{(s+2)} \\ \phi & \phi & * \end{bmatrix}$$

$$* = \frac{\left[2s(s+1)^2-5\right](s+1)^2(s+3) + \left[s(s+1)(s+3)-3\right]\left[4-2(s+1)^3\right.}{(s+2)\left[4-2(s+1)^3\right]}$$

$$K(s) = K_2(s)K_1(s) = \begin{bmatrix} 1 & \phi & \phi \\ -(s+1) & 1 & \phi \\ \dfrac{(s+1)(s+5)}{4-2(s+1)^3} & \dfrac{2s(s+1)^3-5(s+1)}{4-2(s+1)^3} & 1 \end{bmatrix}$$

$$\det(K(s)) \equiv 1 \quad \Longrightarrow \quad \underline{\underline{\text{unimodular}}}$$

$\Rightarrow G(s)$ has been decomposed:

$$K(s) \cdot G(s) = G_2(s)$$

$$\Rightarrow G(s) = K^{-1}(s) \cdot G_2(s)$$

$$\Rightarrow G(s) = K^{+}(s) \cdot G_2(s)$$

$\Rightarrow$ LU-decomposition

- In similar ways, we can define the QR-decomposition as well.

- Notice: This gets soon pretty messy $\Rightarrow$ WE NEED A PROGRAM !!!

• Notice that the inverse of a unimodular matrix is a unimodular matrix.

Def: 2 polynomial matrices $P(s)$ and $Q(s)$ are:

→ ROW-EQUIVALENT, if $P(s) \equiv \underset{\uparrow \text{unimodular}}{U_L(s)} \cdot Q(s)$

→ COLUMN-EQUIVALENT, if $P(s) \equiv Q(s) \cdot \underset{\uparrow \text{unimodular}}{U_R(s)}$

→ EQUIVALENT; if $P(s) \equiv U_L(s) \cdot Q(s) \cdot U_R(s)$

Theorem:

Any polynomial matrix is row-equivalent to an upper-triangular matrix.

Proof: ⟹ Use Gaussian Elimination.

A small modification will ensure that we don't leave the polynomial operations. Let me explain at hand of an example:

Example: $$P(s) = \begin{bmatrix} s^2 & 2 \\ (s+1) & 1 \end{bmatrix}$$

(i) Move the polynomial of the lowest degree in column 1 to the position (1,1) by row exchange:

$$P^1(s) = \begin{bmatrix} \emptyset & 1 \\ 1 & \emptyset \end{bmatrix} \cdot P(s) = \begin{bmatrix} (s+1) & 1 \\ s^2 & 2 \end{bmatrix}$$

(ii) Divide any nonvanishing element in column 1 by the element (1,1). We obtain a quotient q^1_{i1} and a remainder r^1_{i1}:

$$P^1_{i1}(s) / P^1_{11}(s) = q^1_{i1}(s) + r^1_{i1}(s) / p^1_{11}(s)$$

$$\frac{s^2}{s+1} = \underbrace{(s-1)}_{q^1_{21}(s)} + \frac{\overbrace{1}^{r^1_{21}(s)}}{s+1}$$

The order of $r_{i1}^1(s)$ is always smaller than that of $p_{11}^1(s)$

(iii) Subtract the first row times the corresponding quotient $q_{i1}^1(s)$ from all other rows. In the first column we have thus now

$$\begin{bmatrix} p_{11}^1(s) \\ r_{21}^1(s) \\ \vdots \\ r_{p1}^1(s) \end{bmatrix}$$

$$P^2(s) = \begin{bmatrix} 1 & \phi \\ -(s-1) & 1 \end{bmatrix} \cdot P^1(s) = \begin{bmatrix} (s+1) & 1 \\ 1 & (-s+3) \end{bmatrix}$$

(iv) Repeat steps (i)...(iii) until all remainders have become zero.

$$P^3(s) = \begin{bmatrix} \phi & 1 \\ 1 & \phi \end{bmatrix} \cdot P^2(s) = \begin{bmatrix} 1 & (-s+3) \\ (s+1) & 1 \end{bmatrix}$$

$$P^4(s) = \begin{bmatrix} 1 & \phi \\ -(s+1) & 1 \end{bmatrix} P^3(s) = \begin{bmatrix} 1 & (-s+3) \\ \phi & (s^2-2s-2) \end{bmatrix}$$

(v) Repeat steps (i)...(iv) with all the elements below p_{22} until they all are zero

(vi) Compare p_{12} with p_{22}. If the order of $p_{12} \geq$ order of p_{22}, apply steps (ii) and (iii) again to make the order of $p_{12} <$ than the order of p_{22} since we want the highest orders always on the diagonal.

(vii) Repeat for columns 3...m

$\Longrightarrow$ Upper triangular form with the highest order polynomials on the diagonal.

$$U_L(s) = \begin{bmatrix} 1 & \phi \\ -(s+1) & 1 \end{bmatrix} \cdot \begin{bmatrix} \phi & 1 \\ 1 & \phi \end{bmatrix} \cdot \begin{bmatrix} 1 & \phi \\ -(s-1) & 1 \end{bmatrix} \cdot \begin{bmatrix} \phi & 1 \\ 1 & \phi \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (-s+1) \\ (-s-1) & s^2 \end{bmatrix} \text{ is } \underline{\underline{\text{unimodular}}}$$

$P^4(s) \equiv U_L(s) \cdot P(s)$; $P^4(s)$ is rowequivalent to $P(s)$.

Example:

$$P(s) = \begin{bmatrix} (s+1) & (s^2+2s+1) & 2s^3 \\ (2s-2) & (-2s^2+1) & 2s^2 \\ -s^3 & (5s-2) & s \end{bmatrix}$$

(i) Nothing to do.

(ii) $$\frac{2s-2}{s+1} = 2 + \frac{-4}{s+1}$$

$$\frac{-s^3}{s+1} = (-s^2+s-1) + \frac{1}{s+1}$$

$$P^1(s) = \begin{bmatrix} 1 & \phi & \phi \\ -2 & 1 & \phi \\ -(-s^2+s-1) & \phi & 1 \end{bmatrix} \cdot P(s)$$

$$= \begin{bmatrix} (s+1) & (s^2+2s+1) & 2s^3 \\ -4 & (-4s^2-4s-1) & (-4s^3+2s^2) \\ 1 & (s^4+s^3+6s-1) & (2s^5-2s^4+2s^3+s) \end{bmatrix}$$

(i) $$P^2(s) = \begin{bmatrix} \phi & \phi & 1 \\ \phi & 1 & \phi \\ 1 & \phi & \phi \end{bmatrix} \cdot P^1(s) = \begin{bmatrix} 1 & (s^4+s^3+6s-1) & (2s^5-2s^4+2s^3+s) \\ -4 & (-4s^2-4s-1) & (-4s^3+2s^2) \\ (s+1) & (s^2+2s+1) & 2s^3 \end{bmatrix}$$

(ii) $$\frac{-4}{1} = -4 \quad ; \quad \frac{(s+1)}{1} = (s+1)$$

$$P^3(s) = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -(s+1) & 0 & 1 \end{bmatrix} \cdot P^2(s) = \begin{bmatrix} 1 & (s^4+s^3+6s-1) & (2s^5-2s^4+2s^3+s \\ 0 & (4s^4+4s^3-4s^2+20s-5) & (8s^5-8s^4+4s^3+2s^2 \\ 0 & (-s^5-2s^4-s^3-5s^2-3s+2) & (-2s^6-2s^3-s^2-s \end{bmatrix}$$

(i) Get lowest degree of column 2 into (2,2)
$\Rightarrow$ already the case.

(ii) $$\frac{-s^5-2s^4-s^3-5s^2-3s+2}{4s^4+4s^3-4s^2+20s-5} = \left(-\tfrac{1}{4}s-\tfrac{1}{4}\right) + \frac{-s^3-s^2+\frac{3}{4}s}{4s^4+4s^3-4s^2+2}$$

$$P^4(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \left(\frac{1}{4}s+\frac{1}{4}\right) & 1 \end{bmatrix} \cdot P^3(s)$$

$$= \begin{bmatrix} 1 & (s^4+s^3+6s-1) & (2s^5-2s^4+2s^3+s) \\ 0 & (4s^4+4s^3-4s^2+20s-5) & (8s^5-8s^4+4s^3+2s^2+4s) \\ 0 & \left(-s^3-s^2+\frac{3}{4}s+\frac{3}{4}\right) & \left(-s^4-\frac{1}{2}s^3+\frac{1}{2}s^2\right) \end{bmatrix}$$

(i) $$P^5(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot P^4(s)$$

<u>etc.</u>

$\Rightarrow$ this goes on for some time. Each time the degree of P_{22} and P_{23} get reduced by one. We end up with:

$$P^{x}(s) = \begin{bmatrix} 1 & (s^4+s^3+6s-1) & (2s^5-2s^4+2s^3+s) \\ \phi & (as+b) & * \\ \phi & \phi & * \end{bmatrix}$$

(vi)

$$P^{x+1}(s) = \begin{bmatrix} \phi & 1 & \phi \\ 1 & \phi & \phi \\ \phi & \phi & 1 \end{bmatrix} P^{x}(s) = \begin{bmatrix} \phi & (as+b) & * \\ 1 & (s^4+s^3+6s-1) & * \\ \phi & \phi & * \end{bmatrix}$$

(ii) $$\frac{s^4+s^3+6s-1}{as+b} = (cs^3+ds^2+es+f) + \frac{g}{as+b}$$

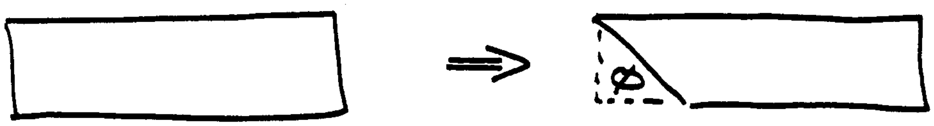
$$P^{x+2}(s) = \begin{bmatrix} 1 & -(cs^3+ds^2+es+f) & \phi \\ \phi & 1 & \phi \\ \phi & \phi & 1 \end{bmatrix} P^{x+1}(s)$$

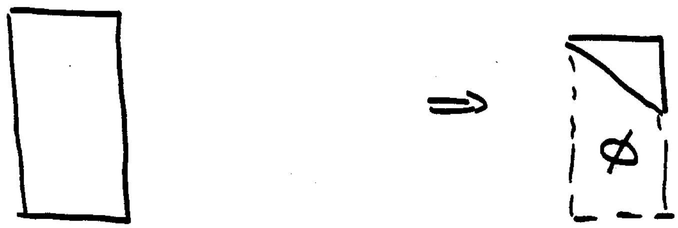
$$= \begin{bmatrix} \phi & (as+b) & * \\ 1 & g & * \\ \phi & \phi & * \end{bmatrix}$$

$$P^{x+3}(s) = \begin{bmatrix} \phi & 1 & \phi \\ 1 & \phi & \phi \\ \phi & \phi & 1 \end{bmatrix} P^{x+2}(s) = \begin{bmatrix} 1 & g & * \\ \phi & (as+b) & * \\ \phi & \phi & * \end{bmatrix}$$

<u>etc.</u>

With this algorithm, we can eventually reduce <u>any</u> polynomial matrix to the product of a unimodular matrix and an upper-triangular matrix where the highest order polynomial in each column is on the diagonal.

$P(s) =$ 

$P(s) =$ 

is always possible.

<u>Theorem</u>:

Any polynomial matrix is column-equivalent to a lower-triangular matrix.

<u>Algorithm</u>:

(1) Use the above algorithm with column operations in place of row operations. <u>or</u>:

(2) Use the duality principle.

$$\begin{tikzcd} P(s) \arrow[r, "\text{duality}"] \arrow[d, dashed] & P^*(s) \arrow[d] \\ T_L(s)\cdot U_R(s) & U_L(s)\cdot T_u(s) \arrow[l, "\text{duality}"'] \end{tikzcd}$$