Theorem: Any polynomial matrix is equivalent to a diagonal matrix.

$$P(s) = U_L(s) \cdot \diagdown(s) \cdot U_R(s)$$

$\Rightarrow$ <u>Smith Decomposition</u>.

$$P(s) = U_L(s) \cdot S(s) \cdot U_R(s)$$

$$\Longleftrightarrow S(s) = U_L^{-1}(s) \cdot P(s) \cdot U_R^{-1}(s)$$

$$S(s) = \begin{bmatrix} \mathrm{diag}\{f_1(s), f_2(s), \dots, f_r(s)\} & \emptyset \\ \emptyset & \emptyset \end{bmatrix}$$

$$r = \mathrm{Rank}(S(s)) \equiv \mathrm{Rank}(P(s))$$

Def: The row (column) rank of a matrix $P(s)$ is equal to the number of linearly independent rows (columns) of the polynomial matrix, where the α_i-elements (weighting factors) can be from the rational functions in s.

$$\mathrm{Rank}(P(s)) = \mathrm{Min}(\mathrm{row Rank}(P(s)), \mathrm{col Rank}(P(s$$

- $f_i(s)$ divides $f_{i+1}(s)$

$\Longleftrightarrow$ $f_r(s)$ has the highest order. $f_{r-1}(s)$ contains a subset of the roots of $f_r(s)$, etc.

- $f_i(s)$ are called the invariant dividers of $P(s)$.

Algorithm: Similar to before, but toggling between row- and column-operations.

Example:

$$P(s) = \begin{bmatrix} 1 & 1 \\ s & (2s+1) \end{bmatrix}$$

(i) Reduce the first column with row operations.

$$P^1(s) = \begin{bmatrix} 1 & \phi \\ -s & 1 \end{bmatrix} P(s) = \begin{bmatrix} 1 & 1 \\ \phi & (s+1) \end{bmatrix}$$

(ii) Reduce the first row with column operations.

$$P^2(s) = P^1(s) \cdot \begin{bmatrix} 1 & -1 \\ \phi & 1 \end{bmatrix} = \begin{bmatrix} 1 & \phi \\ \phi & (s+1) \end{bmatrix}$$

$$\Rightarrow P(s) = U_L(s) \cdot S(s) \cdot U_R(s)$$

$$= \begin{bmatrix} 1 & \phi \\ -s & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & \phi \\ \phi & (s+1) \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ \phi & 1 \end{bmatrix}^{-1}$$

$$= \underbrace{\begin{bmatrix} 1 & \phi \\ s & 1 \end{bmatrix}}_{U_L(s)} \cdot \underbrace{\begin{bmatrix} 1 & \phi \\ \phi & (s+1) \end{bmatrix}}_{S(s)} \cdot \underbrace{\begin{bmatrix} 1 & 1 \\ \phi & 1 \end{bmatrix}}_{U_R(s)}$$

$$f_1(s) = 1$$
$$f_2(s) = s+1$$

(iii) It can happen that new non-zero elements get introduced into the first column while reducing the first row. However, these will always be of lower order $\Longrightarrow$ repeat (i) and (ii) until convergence

$$\Rightarrow P^*(s) = \left[\begin{array}{c|ccc} f_1(s) & \phi & \cdots & \phi \\ \hline \phi & & & \\ \vdots & & \hat{P}(s) & \\ \phi & & & \end{array}\right]$$

(iv) Repeat on $P^*(s)$.

Theorem:

Any transfer function matrix can be reduced to a diagonal form:

$$G(s) = U_L(s) \cdot M(s) \cdot U_R(s)$$

where: $U_L(s)$, $U_R(s)$ are unimodular polynomial matrices, and

$$M(s) = \begin{bmatrix} \mathrm{diag}\left(\frac{\varepsilon_i(s)}{\psi_i(s)}\right) & \phi \\ \phi & \phi \end{bmatrix}$$

- $\varepsilon_i(s)$ and $\psi_i(s)$ are prime (no common roots) $i \in 1, \dots, r$
- $r = \mathrm{Rank}(G(s))$
- $\varepsilon_i(s)$ divides $\varepsilon_{i+1}(s)$
  $\psi_{i+1}(s)$ divides $\psi_i(s)$

Proof:

$$G(s) = \frac{N(s)}{d(s)} = \frac{U_L(s) \cdot S(s) \cdot U_R(s)}{d(s)}$$

$$= U_L(s) \cdot \underbrace{\frac{S(s)}{d(s)}}_{M(s)} \cdot U_R(s)$$

- $\psi_1(s) \equiv d(s)$

$\Rightarrow$ This is called the Smith-McMillan form //.

Def:

$\partial[P(s)] :=$ degree of $P(s)$

$\equiv$ highest power in s.

$\partial_{c_i}[P(s)] :=$ degree of i-th column

$\partial_{r_j}[P(s)] :=$ degree of j-th row

$\Gamma_c[P(s)] :=$ scalar matrix whose elements in the i-th column consist of the coefficient of $s^{\partial_{c_i}[P(s)]}$

$\Gamma_r[P(s)]$:= defined accordingly for rows.

Example:

$$P(s) = \begin{bmatrix} (s^2-3) & 1 & (2s) \\ (4s+2) & 2 & \phi \\ (-s^2) & (s+3) & (-3s+2) \end{bmatrix}$$

$$\Rightarrow \partial[P(s)] = 2$$

$$\partial_{c_1}[P(s)] = 2 \, ; \quad \partial_{c_2}[P(s)] = 1 \, ; \quad \partial_{c_3}[P(s)] = 1$$

$$\partial_{r_1}[P(s)] = 2 \, ; \quad \partial_{r_2}[P(s)] = 1 \, ; \quad \partial_{r_3}[P(s)] = 2$$

$$\Gamma_c[P(s)] = \begin{bmatrix} 1 & \phi & 2 \\ \phi & \phi & \phi \\ -1 & 1 & -3 \end{bmatrix}$$

$$\Gamma_r[P(s)] = \begin{bmatrix} 1 & \phi & \phi \\ 4 & \phi & \phi \\ -1 & \phi & \phi \end{bmatrix}$$

We also define:

$$P^c(s) = \Gamma_c[P(s)] \cdot \begin{bmatrix} s^{\partial c_1[P(s)]} & & \phi \\ & s^{\partial c_2[P(s)]} & \\ \phi & \ddots & s^{\partial c_q[P(s)]} \end{bmatrix}$$

$$\Rightarrow P^c_{(s)} = \begin{bmatrix} 1 & \phi & 2 \\ \phi & \phi & \phi \\ -1 & 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} s^2 & \phi & \phi \\ \phi & s & \phi \\ \phi & \phi & s \end{bmatrix} = \begin{bmatrix} s^2 & \phi & 2 \\ \phi & \phi & \\ -s^2 & s & - \end{bmatrix}$$

It is easy to verify that:

$$|P^c_{(s)}| = \gamma_c \cdot s^p \quad \text{where} \quad \begin{cases} \gamma_c = |\Gamma_c| \\ p = \sum_{i=1}^{q} \partial_{ci} \end{cases}$$

<u>**Def**</u>: A polynomial matrix $P_{(s)} \in \mathbb{R}^{q_1 \times q_2}$ is called column- (row-) "proper", iff $\Gamma_c[P_{(s)}]$ $(\Gamma_r[P_{(s)}])$ has the full rank $= \min\{q_1, q_2\}$.

$\Rightarrow$ A $q \times q$ polynomial matrix $P_{(s)}$ is column- (row-) proper iff

$$\gamma_c = |\Gamma_c| \neq \phi \quad (\gamma_r = |\Gamma_r| \neq \phi)$$

Accordingly, our upper-triangular matrix is <u>column proper</u> iff $P_{(s)}$ is nonsingular. The lower-triangular matrix is <u>row proper</u> iff $P_{(s)}$ is nonsingular.

<u>Warning</u>: The "proper" property is <u>not</u> invariant to equivalence transformations.

<u>Notice</u>: By triangularization, you can get <u>any</u> non-singular $P(s)$ into proper form. However, this algorithm is expensive. There exist cheaper algorithms if this is all that you wish to achieve. One such algorithm is demonstrated at hand of an example.

<u>Example</u>:

$$P(s) = \begin{bmatrix} (s^2-3) & 1 & (2s) \\ (4s+2) & 2 & \emptyset \\ (-s^2) & (s+3) & (-3s+2) \end{bmatrix}$$

$$|P(s)| = 6s^3 + 44s^2 + 28s - 16 \neq \emptyset$$

$\Rightarrow P(s)$ is non-singular.

$$\Gamma_c[P(s)] = \begin{bmatrix} 1 & \emptyset & 2 \\ \emptyset & \emptyset & \emptyset \\ -1 & 1 & -3 \end{bmatrix} ; \; |\Gamma_c[P(s)]| = \emptyset$$

$\Rightarrow P(s)$ is not column proper.

Algorithm:

(i) $\Gamma_c = [\underline{\gamma}_1, \underline{\gamma}_2, \underline{\gamma}_3]$

Express the column with the highest degree $\partial c_i [P(s)]$ through the others.

$$\partial c_1 [P(s)] = 2 \; ; \; \partial c_2 [P(s)] = 1 \; ; \; \partial c_3 [P(s)] = 1$$

$$\Rightarrow \underline{\gamma}_1 = \alpha_1 \underline{\gamma}_2 + \alpha_2 \underline{\gamma}_3 \quad \Longrightarrow \quad \alpha_1 = \alpha_2 = 0.5$$

(ii) Subtract from the same column in $P(s) = [\underline{p}_1(s), \underline{p}_2(s), \underline{p}_3(s)]$ the expression

$$\alpha_1 \cdot s^{(\partial c_1 - \partial c_2)} \cdot \underline{p}_2(s) + \alpha_2 \cdot s^{(\partial c_1 - \partial c_3)} \cdot \underline{p}_3(s)$$

$$= 0.5 \cdot s \cdot \begin{bmatrix} 1 \\ 2 \\ (s+3) \end{bmatrix} + 0.5 \cdot s \cdot \begin{bmatrix} (2s) \\ 0 \\ (-3s+2) \end{bmatrix}$$

$$= \begin{bmatrix} (0.5\,s + s^2) \\ s \\ (-s^2 + 2.5s) \end{bmatrix}$$

In terms of elementary column operations:

$$P'(s) = P(s) \cdot \begin{bmatrix} 1 & 0 & 0 \\ -0.5s & 1 & 0 \\ -0.5s & 0 & 1 \end{bmatrix}$$

$$\Rightarrow P'(s) = \begin{bmatrix} (-0.5s-3) & 1 & (2s) \\ (3s+2) & 2 & 0 \\ (-2.5s) & (s+3) & (-3s+2) \end{bmatrix}$$

$$\partial_{c_1}[P'(s)] = \underline{1}$$

reduced

$$\Rightarrow \Gamma_c[P'(s)] = \begin{bmatrix} -0.5 & 0 & 2 \\ 3 & 0 & 0 \\ -2.5 & 1 & -3 \end{bmatrix}$$

$$|\Gamma_c[P'(s)]| = 6 \neq 0$$

$\Rightarrow$ $P'(s)$ is column-proper.

Otherwise, continue in the same manner.

- We use row-operations (or the duality principle) to get $P(s)$ into row-proper form.

Theorem: Any non-singular $P(s)$ can be brought into "proper" form by an equivalence transformation:

$$\hat{P}(s) = U_L(s) \cdot P(s) \cdot U_R(s)$$

where $\hat{P}(s)$ is proper.

⟹ Proof given through the previous algorithms.

Dividers of Polynomial Matrices:

Def: If $P(s) = H(s) \cdot G(s)$

$H(s)$:= left divider of $P(s)$
$G(s)$:= right divider of $P(s)$
$P(s)$:= left multiple of $G(s)$
$P(s)$:= right multiple of $H(s)$

Def: The largest-common-right-divider (LCRD) of two polynomial matrices $P(s)$ and $R(s)$ is a common right-divider which is a left-multiple of any common right-divider of $P(s)$ and $R(s)$.

Def: The <u>largest-common-left-divider</u> (LCLD) of two polynomial matrices $P(s)$ and $Q(s)$ is a common left-divider which is a right-multiple of any common left-divider of $P(s)$ and $Q(s)$

$$P(s) = H(s) \cdot G(s)$$

$$\underbrace{\underset{p_1 \times q_1}{P(s)}}_{\substack{\text{right multiple of } H(s) \\ \text{left multiple of } G(s)}} = \underbrace{\underset{p_1 \times r}{H(s)}}_{\substack{\text{left divider} \\ \text{of } P(s)}} \cdot \underbrace{\underset{r \times q_1}{G(s)}}_{\substack{\text{right divider} \\ \text{of } P(s)}}$$

$$\underset{p_1 \times q_2}{Q(s)} = \underbrace{\underset{p_1 \times r}{H(s)}}_{\substack{\text{Common left} \\ \text{divider of } P(s) \text{ and } Q(s)}} \cdot \underset{r \times q_2}{F(s)}$$

$$\underset{p_2 \times q_1}{R(s)} = \underset{p_2 \times r}{E(s)} \cdot \underbrace{\underset{r \times q_1}{G(s)}}_{\substack{\text{common right} \\ \text{divider of } P(s) \text{ and } R(s)}}$$

<u>Theorem</u>: If the polynomial matrix $[P(s) \,;\, R(s)]$ is reduced to $[T_u(s) \,;\, \phi]$ (upper triangular form) $\Rightarrow T_u(s)$ is the LCRD of $P(s)$ and $R(s)$.

<u>Theorem</u>: If the polynomial matrix $[P(s), Q(s)]$ is reduced to $[T_L(s), \phi]$ (lower triangular form) $\Rightarrow T_L(s)$ is the LCLD of $P(s)$ and $Q(s)$.

<u>Proof</u>:
$$U_L(s) \cdot \begin{bmatrix} P(s) \\ R(s) \end{bmatrix} = \begin{bmatrix} T_u(s) \\ \phi \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} P(s) \\ R(s) \end{bmatrix} = U_L^{-1}(s) \cdot \begin{bmatrix} T_u(s) \\ \phi \end{bmatrix} = \left[\begin{array}{c|c} L_1(s) & L_2(s) \\ \hline L_3(s) & L_4(s) \end{array}\right] \cdot \begin{bmatrix} T_u(s) \\ \phi \end{bmatrix}$$

$$= \begin{bmatrix} L_1(s) \cdot T_u(s) \\ L_3(s) \cdot T_u(s) \end{bmatrix}$$

$\Rightarrow$ $P(s) = L_1(s) \cdot T_u(s)$

$R(s) = L_3(s) \cdot T_u(s)$

$\Longrightarrow$ $T_u(s)$ is a CRD of $P(s)$ and $R(s)$.

<u>But</u>:

$$U_L(s) \cdot \begin{bmatrix} P(s) \\ R(s) \end{bmatrix} = \left[\begin{array}{c|c} L_5(s) & L_6(s) \\ \hline L_7(s) & L_8(s) \end{array}\right] \cdot \begin{bmatrix} P(s) \\ R(s) \end{bmatrix} = \begin{bmatrix} T_u(s) \\ \emptyset \end{bmatrix}$$

$\Rightarrow$ $L_5(s) \cdot P(s) + L_6(s) \cdot R(s) = T_u(s)$

$\Longrightarrow$ Any CRD of $P(s)$ and $R(s)$ is also a CRD of $T_u(s)$.

$\Rightarrow$ $T_u(s)$ is a left-multiple of any CRD of $P(s)$ and $R(s)$.

$\Longrightarrow$ $T_u(s) = \text{LCRD}$. <u>q.e.d.</u>

<u>Remarks</u>:

(1) If $P(s)$ is non-singular $\Longrightarrow$ $T_u(s)$ is non-singular.

(2) If $T_u(s)$ is LCRD of $P(s)$ and $R(s)$ and $P(s)$ is non-singular
$\Rightarrow$ any row-equivalent polynomial matrix to $T_u(s)$ is also LCRD

(3) If $T_u(s) = I^{(q)} \Rightarrow P(s)$ and $R(s)$ are called <u>relative prime</u> to each other.

<u>Lemma</u>: If $T_u(s)$ is unimodular,
$\Rightarrow P(s)$ and $R(s)$ are <u>relative right prime</u> to each other.

<u>Lemma</u>: If $T_L(s)$ is unimodular,
$\Rightarrow P(s)$ and $Q(s)$ are <u>relative left prime</u> to each other.

<u>Example</u>: $P(s) = \begin{bmatrix} s^2 & -1 \\ -s & s^2 \end{bmatrix}$; $R(s) = \begin{bmatrix} s & - \\ \phi & \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} P(s) \\ R(s) \end{bmatrix} = \begin{bmatrix} s^2 & -1 \\ -s & s^2 \\ s & -s \\ \phi & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s & -1 & 1 & \phi \\ -1 & s^2 & \phi & 1 \\ 1 & -s & \phi & \phi \\ \phi & 1 & \phi & \phi \end{bmatrix}}_{\text{unimodular}} \cdot \left.\begin{bmatrix} s & \phi \\ \phi & 1 \\ \phi & \phi \\ \phi & \phi \end{bmatrix}\right\} \text{upper triangu}$$

$$\Rightarrow T_u(s) = \begin{bmatrix} s & \phi \\ \phi & 1 \end{bmatrix} = \text{LCRD}$$

$$|T_u(s)| = s \quad \Rightarrow P(s) \text{ and } R(s) \text{ are not relative right prime.}$$

$$Q(s) = \begin{bmatrix} s & -s \\ \phi & 1 \end{bmatrix}$$

$$\begin{bmatrix} P(s) & Q(s) \end{bmatrix} = \begin{bmatrix} s^2 & -1 & s & -s \\ -s & s^2 & \phi & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & \phi & \phi & \phi \\ \phi & 1 & \phi & \phi \end{bmatrix}}_{\text{lower triangular}} \cdot \underbrace{\begin{bmatrix} s^2 & -1 & s & -s \\ -s & s^2 & \phi & 1 \\ 1 & \phi & \phi & \phi \\ s & \phi & 1 & -1 \end{bmatrix}}_{\text{unimodular}}$$

$$\Rightarrow T_L(s) = \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix} \quad \Rightarrow P(s) \text{ and } Q(s) \text{ are relative left prime.}$$

Let us now look at a set of differential equations of the form:

$$P(s) \cdot \underline{x}(t) = Q(s) \cdot \underline{u}(t)$$

- The application of elementary row operations on $P(s)$ means here:

  E^1: Exchanging two dif. eq.

  E^2: Multiplying a diff. eq. by a constan

  E^3: adding $b(s) \cdot$ (j-th diff. eq.) to the i-th diff. eq.

Obviously, none of these operations changes the solution at all.

- The application of elementary column operations on $P(s)$ means:

  E^1: substituting $\xi_i = x_j$; $\xi_j = x_i$

  E^2: substituting $\xi_i = x_i / c$

  E^3: substituting $\xi_i = x_i + b(s) \cdot x_j$

$\Rightarrow$ These are all linear transformations.

In particular, we can get $P(s)$ into upper-triangular form:

$$\left|\begin{array}{l} P_{11}(s)\cdot x_1(t) + P_{12}(s)\cdot x_2(t) + \ldots + P_{1q}(s)\cdot x_q(t) = \\ \qquad\qquad P_{22}(s)\cdot x_2(t) + \cdots + P_{2q}(s)\cdot x_q(t) = \\ \qquad\qquad\qquad \ddots \\ \qquad\qquad\qquad\qquad P_{qq}(s)\cdot x_q(t) = \cdot \end{array}\right.$$

where the polynomials on the diagonal are of the highest degrees.

$\Longrightarrow$ We can solve this system from the last to the first. Each time we solve only <u>one</u> diff. eq. in <u>one</u> variable.

$\Longrightarrow$ The number of required initial conditions equals the degree of the polynomials along the diagonal.

$\Longrightarrow$ n (the system order) can be computed as:

$$n = \partial\left[\prod_{i=1}^{q} p_{ii}(s)\right] \equiv \partial\left[|P(s)|\right]$$

Since elementary operations do not change the degree of the determinant, this is even always true (also if $P(s)$ is not in triangular form).

⇒ Obviously, the system is only solvable, if none of the $p_{ii}(s)$ vanishes.

⇒ $P(s)$ must be regular.

⇒ If any $p_{ii}(s)$ is a constant, the corresponding equation is algebraic in the others. Such a variable is called unimportant.

Example:

$$\left| \begin{array}{l} \ddot{x}_1(t) + 2x_2(t) = u(t) \\ \dot{x}_1(t) + x_1(t) + x_2(t) = \emptyset \end{array} \right|$$

$$\Rightarrow \begin{bmatrix} s^2 & 2 \\ (s+1) & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \emptyset \end{bmatrix} u$$

triangularization:

$$\begin{bmatrix} 1 & (-s+3) \\ \emptyset & (s^2-2s-2) \end{bmatrix} = \begin{bmatrix} 1 & (-s+1) \\ (-s-1) & s^2 \end{bmatrix} \cdot \begin{bmatrix} s^2 & \vdots \\ (s+1) & \end{bmatrix}$$

$\Rightarrow$ an equivalent set of diff.eq. is:

$$\left| \begin{array}{l} x_1 - \dot{x}_2 + 3x_2 = u \\ \ddot{x}_2 - 2\dot{x}_2 - 2x_2 = -\dot{u} - u \end{array} \right|$$

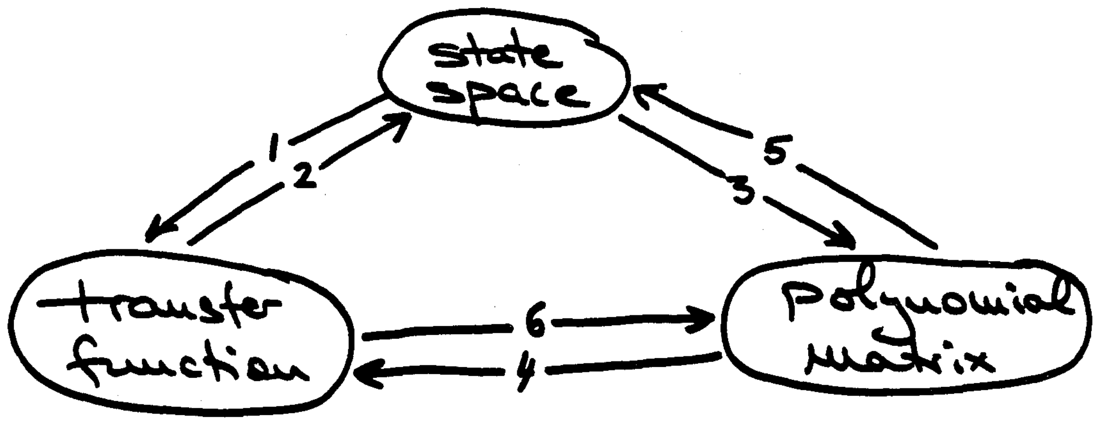
$\Rightarrow$ solve the secon diff.eq.

$\Rightarrow x_2(t)$

$\Rightarrow$ plug into the first

$\Rightarrow x_1(t)$ is unimportant.

Transfer- and Equivalence Relations



① $G(s) = C\cdot(sI-A)^{-1}B + D$

② Realization problem ⇒ discussed before for SISO systems and also a first look for MIMO systems (Gilbert, controller-canonical, observer-canonical). The most general case will be discussed later.

③ Trivial, since the state-space is just a special case of the polynomial matrix representation:

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}$$

$$\Rightarrow s\underline{x} = A\underline{x} + B\underline{u}$$

$$\Rightarrow (sI-A)\underline{x} = B\underline{u}$$

$$\underline{y} = C\underline{x} + D\underline{u}$$

Comparison:

$$\{P(s), Q(s), R(s), W(s)\} = \{(sI-A), B, C, D\}$$

(4) $$\left| \begin{array}{l} P(s)\,\underline{x}(t) = Q(s)\cdot\underline{u}(t) \\ \underline{y}(t) = R(s)\cdot\underline{x}(t) + W(s)\cdot\underline{u}(t) \end{array} \right|$$

Assume that all initial conditions vanish

$$\Rightarrow \quad \left| \begin{array}{l} P(s)\cdot\underline{x}(s) = Q(s)\cdot\underline{u}(s) \\ \underline{y}(s) = R(s)\cdot\underline{x}(s) + W(s)\cdot\underline{u}(s) \end{array} \right|$$

If $P(s)$ is non-singular:

$$\underline{y}(s) = [\underbrace{R(s)\cdot P^{-1}(s)\cdot Q(s) + W(s)}_{G(s)}]\cdot\underline{u}(s)$$

(5) For the time being use (4) followed by (2). A more general and direct approach will follow.

(6) Needs more work. (Use (2)+(3)).

Equivalence Transformations:

Def: Two systems:

(I')

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} + D\underline{u} \end{array} \right|$$

(II')

$$\left| \begin{array}{l} P(s) \cdot \underline{z}(t) = Q(s) \cdot \underline{u}(t) \\ \underline{y}(t) = R(s) \cdot \underline{z}(t) + W(s) \cdot \underline{u} \end{array} \right.$$

are equivalent, <u>if</u>

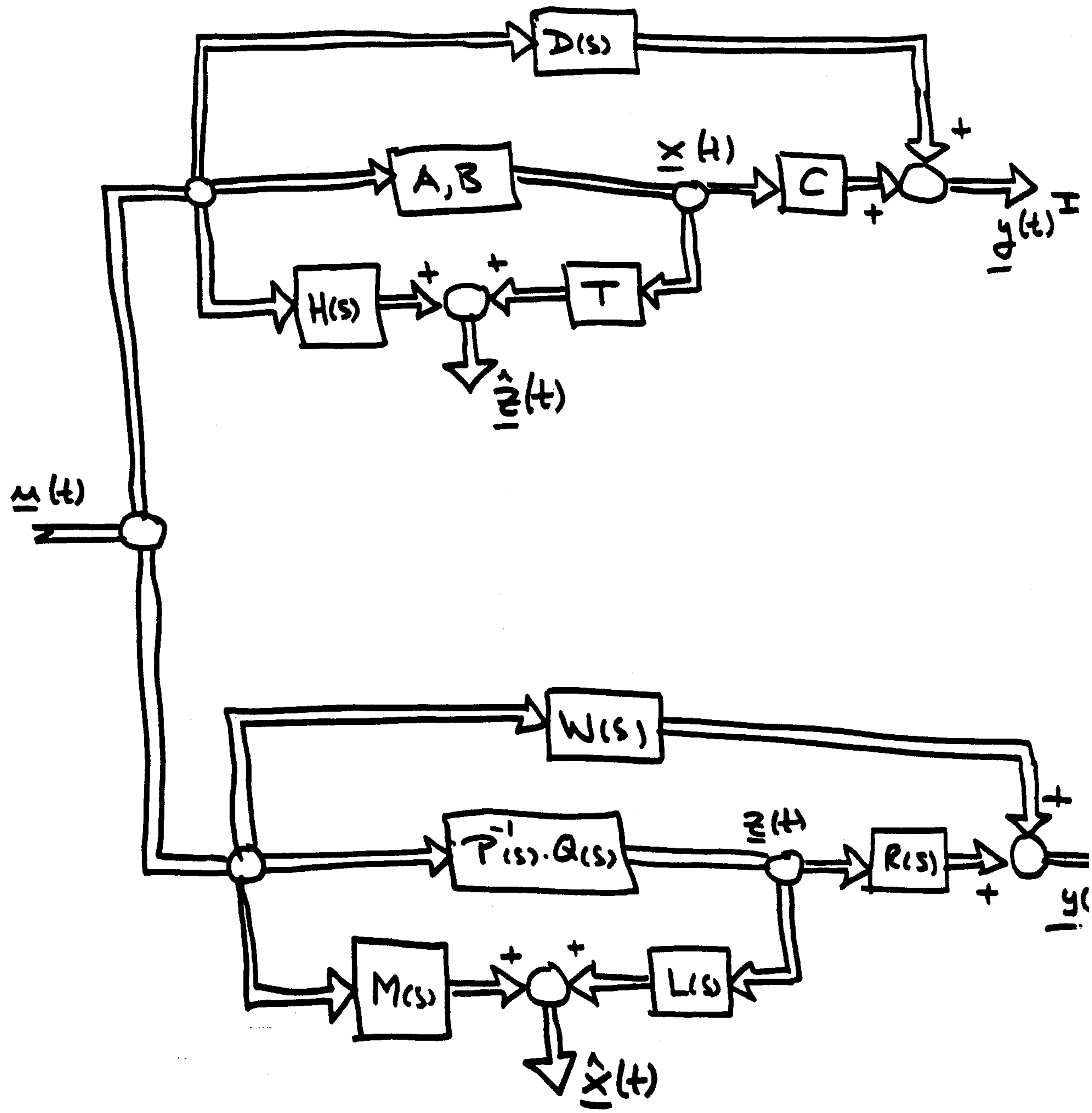
(1) for any $\underline{u}(t)$ and $\underline{x}(t_0)$, there exists exactly <u>one</u> $\underline{z}(t_0)$ such that:

$$\underline{z}(t) = T \cdot \underline{x}(t) + H(s) \cdot \underline{u}(t)$$

<u>where</u>: T := constant matrix
$H(s)$:= polynomial matrix

(2) $\underline{y}(t)^{I} \equiv \underline{y}(t)^{II}$

Let me explain at hand of a graph:



equivalent iff: $\hat{\underline{z}}(t) \equiv \underline{z}(t)$

$\underline{y}(t)^{II} \equiv \underline{y}(t)^{I}$

$\Rightarrow \hat{\underline{x}}(t) \equiv \underline{x}(t)$

- In order to be equivalent, it is not sufficient that the two systems have the same I/O-behavior. They must also internally be equivalent. In particular:

$$n \equiv \dim(\underline{x}) \equiv \partial\left[|P(s)|\right].$$

- It follows directly from ($\hat{\underline{z}}(t) = \underline{z}(t)$):

$$T(sI-A)^{-1}B + H(s) \equiv P^{-1}(s) \cdot Q(s)$$

$$\Rightarrow \quad \frac{T(sI-A)^{+}B + H(s)}{|sI-A|} \equiv \frac{P^{+}(s) \cdot Q(s)}{|P(s)|}$$

$$\Rightarrow \quad |sI-A| \equiv \alpha |P(s)| \quad ; \quad \alpha \neq \phi$$

and from ($\underline{y}(t)^{II} = \underline{y}(t)^{I}$):

$$C(sI-A)^{-1}B + D(s) \equiv R(s)P^{-1}(s)Q(s) + W(s)$$

- The pair $\{A, T\}$ is <u>observable</u>. This follows from the uniqueness condition between $\underline{x}(t_0)$ and $\underline{z}(t_0)$. Otherwise, we could choose the unobservable part of $\underline{x}(t_0)$ freely.

- $\hat{\underline{x}}(t) \equiv L(s)\cdot \underline{z}(t) + M(s)\cdot \underline{u}(t)$

Proof: $\hat{\underline{z}}(t) = T\cdot \underline{x}(t) + H(s)\cdot \underline{u}(t) = \underline{z}(t)$

$$\Rightarrow s\underline{z}(t) = T\cdot s\underline{x}(t) + sH(s)\cdot \underline{u}(t)$$

$$= TA\underline{x}(t) + (TB + sH(s))\cdot \underline{u}(t)$$

$$\vdots$$

$$s^{n-1}\underline{z}(t) = TA^{n-1}\underline{x}(t) + \left(TA^{n-2}B + \cdots + s^{n-1}H(s)\right)\cdot \underline{u}$$

$$\Rightarrow \underbrace{\begin{bmatrix} T \\ TA \\ \vdots \\ TA^{n-1} \end{bmatrix}}_{B_0} \underline{x}(t) = \underbrace{\begin{bmatrix} I^{(q)} \\ s\cdot I^{(q)} \\ \vdots \\ s^{n-1}\cdot I^{(q)} \end{bmatrix}}_{I(s)} \underline{z}(t) - \underbrace{\begin{bmatrix} & H(s) \\ & TB + sH(s) \\ \cdot^{\cdot^{\cdot}} & \vdots \\ TA^{n-2}B + \cdots + s^{n-1}H(s) & \end{bmatrix}}_{J(s)} \cdot \underline{u}$$

$$\Rightarrow B_0\underline{x}(t) = I(s)\cdot \underline{z}(t) - J(s)\underline{u}(t)$$

$$\Rightarrow (B_0^* B_0)\underline{x}(t) = (B_0^* I(s))\underline{z}(t) - (B_0^* J(s))\cdot \underline{u}(t)$$

$$\Rightarrow \underline{x}(t) = \underbrace{\underbrace{(B_0^* B_0)^{-1} B_0^*}_{\text{pseudoinverse of } B_0} I(s)}_{L(s)} \cdot \underline{z}(t) - \underbrace{(B_0^* B_0)^{-1} B_0^* J(s)}_{M(s)} \cdots$$

$$\Rightarrow \underline{x}(t) = L(s)\cdot \underline{z}(t) + M(s)\cdot \underline{u}(t)$$

q.e.d.

($L(s)$ and $M(s)$ are not unique).

- Let us look at the special case where:

$$\{P(s), Q(s), R(s), W(s)\} = \{(sI-\hat{A}), \hat{B}, \hat{C}, \hat{D}\}$$

$$P(s) = sI-\hat{A}$$
$$Q(s) = \hat{B}$$

$$\Rightarrow \quad (sI-\hat{A})\,\underline{z}(t) = \hat{B}\,\underline{u}(t)$$

$$\Rightarrow \quad \dot{\underline{z}}(t) = \hat{A}\,\underline{z}(t) + \hat{B}\,\underline{u}(t)$$

but: $\underline{z}(t) = T\cdot\underline{x}(t) + H(s)\cdot\underline{u}(t)$

$$\Rightarrow \quad \dot{\underline{z}}(t) = T\cdot\dot{\underline{x}}(t) + H(s)\cdot\dot{\underline{u}}(t) + sH(s)\cdot\underline{u}(t)$$
$$= TA\,\underline{x}(t) + T\cdot B\,\underline{u}(t) + H(s)\cdot\dot{\underline{u}}(t) + sH(s)\cdot$$
$$\equiv \hat{A}T\underline{x}(t) + \hat{A}\cdot H(s)\,\underline{u}(t) + \hat{B}\,\underline{u}(t)$$

Comparison of coefficients:

$$\Rightarrow H(s) = \emptyset$$
$$TA = \hat{A}T \quad \Rightarrow \quad \hat{A} = TAT^{-1}$$
$$TB = \hat{B} \quad \Rightarrow \quad \hat{B} = TB$$

$\Rightarrow$ is the standard equivalence relation between state-space representations.

Def: Two systems represented by polynomial matrices are equivalent iff they are equivalent to the same state-space representation.

Remark:

The above definition does not request that $\dim(\underline{z}_1(t)) \equiv \dim(\underline{z}_2(t))$

However, in the special case where:

$$\dim\{\underline{z}_2(t)\} \equiv \dim\{\underline{z}_1(t)\}$$

$$\Longleftrightarrow \left| \begin{array}{l} \hat{P}(s) = U_L(s) \cdot P(s) \cdot U_R(s) \\ \hat{Q}(s) = U_L(s) \cdot Q(s) \\ \hat{R}(s) = R(s) \cdot U_R(s) \\ \hat{W}(s) = W(s) \end{array} \right|$$

cf: Int. J. of Control, 25 (1977), pp 1-3