# Compensator Design in the Frequency Domain:

## (A) Interpretation of Linear State Feedback in the Frequency Domain

Given: $S = \{A, B; C, D\}$

Let us transform this system into controller-canonical form:

$$S_{CCF} = \{\hat{A}, \hat{B}; \hat{C}, D\}$$

We apply state-feedback:

$$\underline{u} = E \cdot \underline{r} + \hat{F} \cdot \underline{x}$$

$$\Rightarrow \quad S_{CL} = \{(\hat{A} + \hat{B}\hat{F}), \hat{B}E; (\hat{C} + D\hat{F}), DE\}$$

We can compute the total transfer function:

$$G_{tot}(s) = (\hat{C} + D\hat{F}) \cdot (sI - \hat{A} - \hat{B}\hat{F})^{-1} \cdot \hat{B}E + D$$

We can use the structure theorem:

$$G_{tot}(s) = (\hat{C}+D\hat{F})\,\Sigma(s)\,\delta_{CL}^{-1}(s)\cdot\hat{B}_m\,E + DE$$

$$= \left[(\hat{C}+D\hat{F})\Sigma(s) + D\hat{B}_m^{-1}\delta_{CL}(s)\right]\cdot\left[\hat{B}_m^{-1}\delta_{CL}(s)\right]^{-1}\cdot E$$

where:

$$\delta_{CL}(s) = \begin{bmatrix} s^{d_1} & & \emptyset \\ & s^{d_2} & \\ \emptyset & \ddots & s^{d_m} \end{bmatrix} - (\hat{A}_m + \hat{B}_m\hat{F})\cdot\Sigma(s)$$

However, we can apply the structure theorem also to the open-loop system:

$$G(s) = \left[\hat{C}\,\Sigma(s) + D\hat{B}_m^{-1}\delta(s)\right]\cdot\left[\hat{B}_m^{-1}\delta(s)\right]^{-1}$$

where:

$$\delta(s) = \begin{bmatrix} s^{d_1} & & \emptyset \\ & s^{d_2} & \\ \emptyset & \ddots & s^{d_m} \end{bmatrix} - \hat{A}_m\,\Sigma(s)$$

Comparison:

$$\delta_{CL}(s) = \delta(s) - \hat{B}_m \hat{F} \Sigma(s)$$

$$\Rightarrow D\hat{B}_m^{-1}\delta_{CL}(s) = D\hat{B}_m^{-1}\delta(s) - D\hat{F}\Sigma(s)$$

$$\Rightarrow G_{tot}(s) = \underbrace{\left[\hat{C}\cdot\Sigma(s) + D\hat{B}_m^{-1}\delta(s)\right]}_{R(s)} \cdot \underbrace{\left[\hat{B}_m^{-1}\cdot\delta_{CL}(s)\right]^{-1}}_{P_{CL}(s)} \cdot F$$

$$G(s) = \underbrace{\left[\hat{C}\cdot\Sigma(s) + D\hat{B}_m^{-1}\delta(s)\right]}_{R(s)} \cdot \underbrace{\left[\hat{B}_m^{-1}\delta(s)\right]^{-1}}_{P(s)}$$

Also:

$$P_{CL}(s) = P(s) - \underbrace{\hat{F}\cdot\Sigma(s)}_{F(s)}$$

$$\Rightarrow \boxed{P_{CL}(s) = P(s) - F(s)}$$

Conclusions:

(1) $R(s)$ is <u>not</u> modified by state feedback. $\Rightarrow$ State feedback moves the poles, but leaves the zeros unchanged.

(2) $P_{CL}(s)$ remains column-proper

$$\Gamma_c\left[P_{CL}(s)\right] \equiv \Gamma_c\left[P(s)\right] \equiv \hat{B}_m^{-1}$$

and: $\partial_{c_i}\left[P_{CL}(s)\right] \equiv \partial_{c_i}\left[P(s)\right] \equiv d_i$

(3) Up to these two conditions, the polynomial matrix $P_{CL}(s)$ can be chosen freely by selecting $F(s)$ (or $\hat{F}$).

(4) $E$ can be selected freely.

(5) If the system is not completely controllable, use the modified structure theorem

## (B) Compensator Design for SISO:

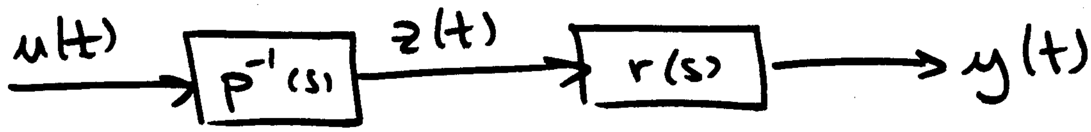
$$y(s) = g(s) \cdot u(s)$$

$$g(s) = r(s) \cdot p^{-1}(s) \; ; \; \partial[p(s)] = n$$

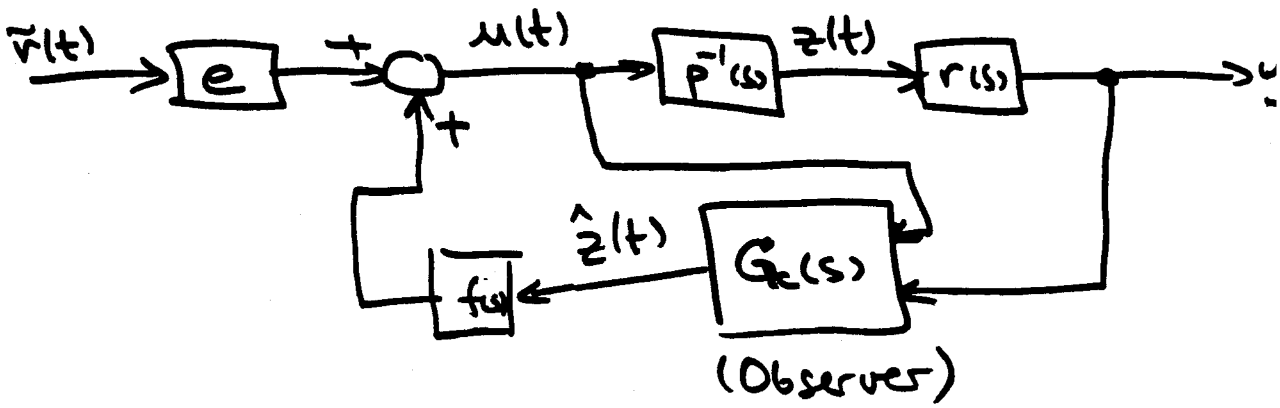
Assume: System is proper, thus:

$$\partial[r(s)] \leq \partial[p(s)]$$

$$\Rightarrow \quad \left| \begin{array}{l} p(s) \cdot z(t) = u(t) \\ y(t) = r(s) \cdot z(t) \end{array} \right|$$




We wish to design a compensator with the following structure:



We know that we can make the observer poles uncontrollable thus:

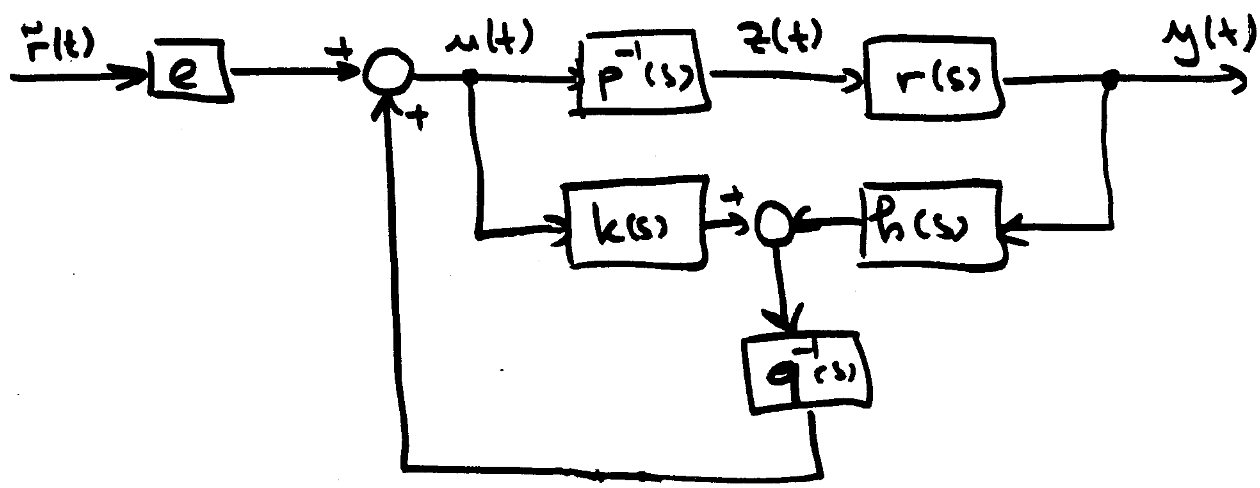
$$g_{tot}(s) = r(s)\cdot p_{CL}^{-1}(s)\cdot e \quad ; \quad p_{CL}(s) = p(s) - f($$

$f(s)\,G_c(s)$ is a two-input/one-output system. Let us write this in observer-canonical form:

$$f(s)\,G_c(s) = \left[ g_{c_1}(s) \;\vdots\; g_{c_2}(s) \right] \equiv q^{-1}(s)\cdot \left[ k(s) \;\vdots\; h \right.$$

$q(s)$, $k(s)$, and $h(s)$ are polynomi

$$\partial[k(s)],\ \partial[h(s)] \le \partial[q(s)] \equiv n-$$



$$
\begin{aligned}
u(t) &= p(s)\cdot z(t) \\
&= e\cdot\tilde{r}(t) + q^{-1}(s)\cdot\left[k(s)\cdot p(s) + h(s)\cdot r(s)\right]z \\
&\equiv e\cdot\tilde{r}(t) + f(s)\cdot z(t)
\end{aligned}
$$

$$\Rightarrow \boxed{k(s)\cdot p(s) + h(s)\cdot r(s) = q(s)\cdot f(s)}$$

$$\Rightarrow u(t) = e\cdot\tilde{r}(t) + \underbrace{q^{-1}(s)\cdot q(s)}_{\substack{\text{observer poles}\\ \text{are uncontrollable}}}\cdot f(s)\cdot z(t)$$

$\Rightarrow$ make $q(s)$ stable !!

$$u(t) = e\cdot\tilde{r}(t) + f(s)\cdot z(t) \equiv p(s)\cdot z(t)$$

$$\Rightarrow [p(s) - f(s)]\cdot z(t) = e\cdot\tilde{r}(t)$$

$$\Rightarrow z(t) = (\underbrace{p(s) - f(s)}_{P_{CL}(s)})^{-1}\cdot e\cdot\tilde{r}(t)$$

Algorithm:

Given: $p(s)$ and $r(s)$ relative prime

(i) Choose $f(s)$ such that:

$$p_{CL}(s) = p(s) - f(s)$$

has the desired poles.

(ii) Choose $q(s)$ as a stable polynomial of order $(n-1)$. Its roots are the observer poles $\Rightarrow$ slightly faster than $p_{CL}(s)$.

(iii) Determine $k(s)$ and $h(s)$

Given:

$$p(s) = p_0 + p_1 s + \cdots + p_n s^n \; ; \; p_n =$$
$$r(s) = r_0 + r_1 s + \cdots + r_n s^n$$

We construct the eliminant matrix

$$M_e = \begin{bmatrix} r_0 & r_1 & \cdots & r_{n-1} & r_n & \emptyset & \emptyset & \cdots & \emptyset \\ \emptyset & r_0 & \cdots & r_{n-2} & r_{n-1} & r_n & \emptyset & \cdots & \emptyset \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \emptyset \\ \emptyset & \emptyset & \cdots & r_0 & r_1 & r_2 & r_3 & \cdots & r_n \\ p_0 & p_1 & \cdots & p_{n-1} & p_n & \emptyset & \emptyset & \cdots & \emptyset \\ \emptyset & p_0 & \cdots & p_{n-2} & p_{n-1} & p_n & \emptyset & \cdots & \emptyset \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \emptyset \\ \emptyset & \emptyset & \cdots & p_0 & p_1 & p_2 & p_3 & \cdots & p_n \end{bmatrix}$$

$$M_e \in \mathbb{R}^{2n \times 2n}$$

is a double upper Toeplitz matrix.

<u>Theorem</u>: If $p(s)$ and $r(s)$ are relative prime, $\Longleftrightarrow$ $M_e$ is nonsingular. (without proof).

<u>Let</u>: $\sigma(s) = \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^{2n-1} \end{bmatrix}$

$\Rightarrow \quad q(s) \cdot f(s) = \beta(s) = \underline{\beta}' \cdot \sigma(s) = [\underline{h}' \,\vdots\, \underline{k}'] \cdot M_e$

$$\Rightarrow \boxed{[\underline{h}' \,\vdots\, \underline{k}'] = \underline{\beta}' \cdot M_e^{-1}}$$

finds the coefficients of $h(s)$ and $k(s)$

## (c) Observer for MIMO - Systems:

**Given:** $G(s) = R(s) \cdot P^{-1}(s)$

**where:** $P(s), R(s)$ are relative right pri

$$\partial\left[\,|P(s)|\,\right] = n \;;\; P(s) \text{ is column-prop}$$

$$\partial c_j\left[R(s)\right] \leq \partial c_j\left[P(s)\right] = \bar{d}_j$$

**Achieve:** $G_{tot}(s) = R(s) \cdot P_{CL}^{-1}(s) \cdot E$

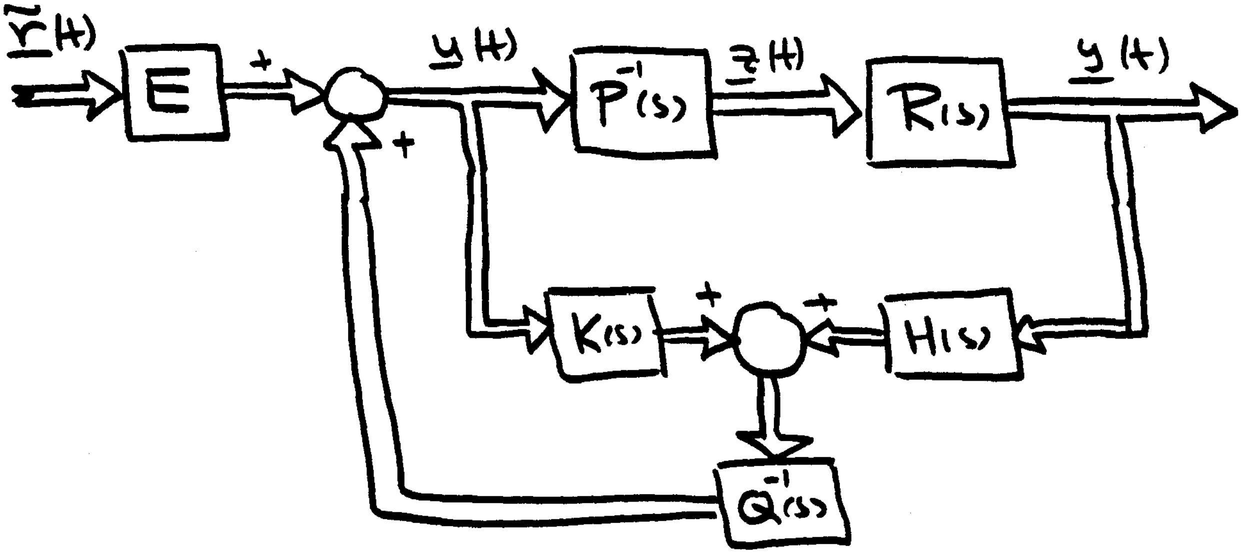
**such that:** $P_{CL}(s) = P(s) - F(s)$

$$F(s) = \hat{F} \cdot \Sigma(s)$$

$\hat{F}$ is free upto:

$$\Gamma_c\left[P_{CL}(s)\right] = \Gamma_c\left[P(s)\right] \;;\; \partial c_j\left[P_{CL}(s)\right] = \bar{d}_j$$

We use exactly the same approach as in the SISO case:




(i) $K(s) \cdot P(s) + H(s) \cdot R(s) = Q(s) \cdot F(s)$

$F(s)$ is free upto $\partial c_i [F(s)] < \partial c_i [P(s)]$

(ii) The zeros of $|Q(s)|$ are stable

(iii) $Q^{-1}(s) \cdot K(s)$ and $Q^{-1}(s) \cdot H(s)$ are proper.

Algorithm:

(i) Choose $F(s)$ such that:

$$P_{CL}(s) = P(s) - F(s)$$

has the desired poles

(ii) Choose $Q(s)$ stable and faster than $P_{CL}(s)$.

(iii) We write:

$$\begin{array}{c} p\cdot k\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ m\cdot k\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \end{array}\begin{bmatrix} R(s) \\ s\cdot R(s) \\ \vdots \\ s^{k-1}\cdot R(s) \\ \hdashline P(s) \\ s\cdot P(s) \\ \vdots \\ s^{k-1}\cdot P(s) \end{bmatrix} = M_k \cdot \underbrace{\begin{bmatrix} \begin{matrix} 1 \\ s \\ \vdots \\ s^{d_1+k-1} \end{matrix} & & \\ & \begin{matrix} 1 \\ s \\ \vdots \\ s^{d_2+k-1} \end{matrix} & \\ & & \ddots \\ & & & \begin{matrix} 1 \\ s \\ \vdots \\ s^{d_m+k-1} \end{matrix} \end{bmatrix}}_{\Sigma_k(s)}$$

$$M_k \in \mathbb{R}^{(pk+mk)\times(n+mk)}$$

$$\Sigma_k(s) \in \mathbb{R}_p^{(n+mk)\times m}$$

k is a constant which is currently undetermined.

Example:

$$G(s) = \begin{bmatrix} \frac{s+1}{s^2} & \phi \\ \frac{1}{s(s-1)} & \frac{1}{-s+1} \end{bmatrix} = \underbrace{\begin{bmatrix} (s+1) & \phi \\ 1 & 1 \end{bmatrix}}_{R(s)} \cdot \underbrace{\begin{bmatrix} s^2 & \phi \\ 1 & (-s+1) \end{bmatrix}}_{P(s)}$$

- $P(s)$ and $R(s)$ are relative right pri
- $m = p = 2$, $d_1 = 2$, $d_2 = 1$ $\Rightarrow$ $n = 3$
- $R(s)$ and $P(s)$ have the desired properties.

We choose: $\underline{\underline{k=2}}$:

$$\begin{bmatrix} R(s) \\ sR(s) \\ P(s) \\ sP(s) \end{bmatrix} = \begin{bmatrix} (s+1) & \phi \\ 1 & 1 \\ (s^2+s) & \phi \\ s & s \\ s^2 & \phi \\ 1 & (-s+1) \\ s^3 & \phi \\ s & (-s^2+s) \end{bmatrix} = \underbrace{\left[\begin{array}{cccc|ccc} 1 & 1 & \phi & \phi & \phi & \phi & \phi \\ 1 & \phi & \phi & \phi & 1 & \phi & \phi \\ \hline \phi & 1 & 1 & \phi & \phi & \phi & \phi \\ \phi & 1 & \phi & \phi & \phi & 1 & \phi \\ \hline \phi & \phi & 1 & \phi & \phi & \phi & \phi \\ 1 & \phi & \phi & \phi & 1 & -1 & \phi \\ \hline \phi & \phi & \phi & 1 & \phi & \phi & \phi \\ \phi & 1 & \phi & \phi & \phi & 1 & -1 \end{array}\right]}_{M_k} \cdot \Sigma_k$$

- We notice that with $k \geq \frac{n}{p}$, $M_k$ obtains more or equal rows than columns. The larger we choose $k$, the more rows we get.

- We choose $k = \nu$ such that $k$ is the smallest value for which $M_k$ has the full rank.

Theorem:

If $P(s)$ and $R(s)$ are relative right prime $\Longleftrightarrow$ $\exists k$, $M_k$ has the full rank.

(without proof)
$\rightarrow$ cf. Wolovich

Theorem: For the controller - canonic realization:

$$G(s) = R(s) \cdot P^{-1}(s)$$

the value of $\nu$ is the observability index of the corresponding $\{A, C\}$

(without proof)

$\longrightarrow$ cf. Wolovich

Note: More details about eliminant matrices can be found in:

Proc. DCC'76 (Decision and Control Conf.)

papers by:
(1) Kung, Kailath, Morf
(2) Anderson, Jury
(3) Emre, Silverman

Example continued:

$$k=2 \Rightarrow \rho[M_2]=7 \rightarrow \text{full rank}$$

$$\rightarrow \underline{\underline{\nu=2}}.$$

We write:

$$Q(s)\cdot F(s) = B(s) = H(s)R(s) + K(s)\cdot P(s)$$

$$= B\cdot\Sigma_\nu(s) = [H \vdots K]\cdot M_\nu\cdot\Sigma_\nu(s)$$

$$\Rightarrow \qquad [H \vdots K] = B\cdot\underbrace{M_\nu^{-1}}_{\text{pseudoinverse}}$$

Alternatively: Choose any linearly independent set of rows of $M_\nu$ such that $\rho(\hat{M}_\nu) \equiv \rho(M_\nu)$

$$\rho(\underbrace{\hat{M}_\nu}_{\substack{\text{reduced}\\\text{set of}\\\text{rows}}}) \equiv \rho(\underbrace{M_\nu}_{\substack{\text{full set}\\\text{of rows}\\M_\nu \text{ is rect.}}})$$

$\hat{M}_\nu$ is square

$$\Rightarrow [\hat{H} \vdots \hat{K}] = B \cdot \hat{M}_\nu^{-1}$$

Make $[H \vdots K]$ out of $[\hat{H} \vdots \hat{K}]$ by filling with zero columns where rows of $M_\nu$ were eliminated.

$\Rightarrow$ The problem is underdetermined.

(Remember: In the MIMO case, we have some choice beyond the pole placement.)

Example continued:

Let us assume that:

$$B(s) = \begin{bmatrix} (-2s^2+3s+10) & 10 \\ (s^2+6s+6) & (2s+4) \end{bmatrix}$$

$$\Rightarrow B(s) = \underbrace{\left[\begin{array}{cccc:ccc} 10 & 3 & -2 & 0 & 10 & 0 & 0 \\ \hdashline 6 & 6 & 1 & 0 & 4 & 2 & 0 \end{array}\right]}_{B} \cdot \left[\begin{array}{c:c} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ s^3 & 0 \\ \hdashline 0 & 1 \\ 0 & s \\ 0 & s^2 \end{array}\right]$$

We eliminate e.g. the $6^{th}$ row of $M_2$:

$$\hat{M}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \hat{M}_2^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

$$[\hat{H} \,\vdots\, \hat{K}] = B \cdot \hat{M}_2^{-1} = \left[\begin{array}{cccc:ccc} 0 & 10 & 3 & 0 & -5 & 0 & 0 \\ 2 & 4 & 2 & 2 & -1 & 0 & 0 \end{array}\right]$$

$\uparrow$ eliminated

$$\Rightarrow [H \,\vdots\, K] = \left[\begin{array}{cccc:cccc} 0 & 10 & 3 & 0 & -5 & 0 & 0 & 0 \\ \underbrace{2 \quad 4 \quad 2 \quad 2}_{H} & & & & \underbrace{-1 \quad 0 \quad 0 \quad 0}_{K} \end{array}\right]$$

$$[H \,\vdots\, K] \cdot M_\nu \cdot \Sigma_y(s) = [H \,\vdots\, K] \cdot \begin{bmatrix} R(s) \\ sR(s) \\ P(s) \\ sP(s) \end{bmatrix}$$

$$= \underbrace{H \cdot \begin{bmatrix} 1 & \phi \\ \phi & 1 \\ s & \phi \\ \phi & s \end{bmatrix}}_{H(s)} R(s) + \underbrace{K \begin{bmatrix} 1 & \phi \\ \phi & 1 \\ s & \phi \\ \phi & s \end{bmatrix}}_{K(s)} P(s)$$

$$\Rightarrow H(s) = \begin{bmatrix} 3s & 1\phi \\ (2+2s) & (4+2s) \end{bmatrix} ; K(s) = \begin{bmatrix} -5 & \phi \\ -1 & \phi \end{bmatrix}$$

We still need to answer the question how we determine $Q(s)$ and $B(s)$.

We know: zeros of $|Q(s)|$ must be stable

and: $\partial r_i [K(s)] \leq \partial r_i [Q(s)]$
$\partial r_i [H(s)] \leq \partial r_i [Q(s)]$

We try:

$$Q(s) = \begin{bmatrix} s^{\nu-1} & \phi & \cdots & \phi & q_{1m}(s) \\ -1 & s^{\nu-1} & \cdots & \phi & q_{2m}(s) \\ \phi & -1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & \\ \phi & \phi & \cdots & -1 & (s^{\nu-1} + q_{mm}(s)) \end{bmatrix}$$

where: $q_{im}(s) = \sum_{k=0}^{\nu-2} a_{[(i-1)(\nu-1)+k]} \cdot s^k$

$$\Rightarrow |Q(s)| = a_0 + a_1 s + \dots + a_{[m\nu-m-1]} s^{m\nu-m-1} + s^{m\nu}$$

Example continued:

Design compensator such that:

$$G_{tot}(s) = R(s) \cdot P_{CL}^{-1}(s) \cdot E$$

$$= R(s) \cdot \left[E^{-1} \cdot P_{CL}(s)\right]^{-1}$$

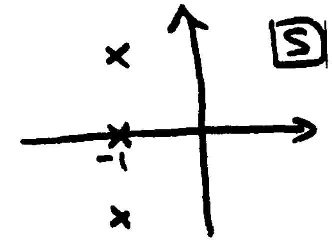
Assume:

$$E^{-1} \cdot P_{CL}(s) = \begin{bmatrix} (s^2+2s+2) & \phi \\ (s+1) & (s+1) \end{bmatrix}$$

In this case:

$$G_{tot}(s) = R(s)\cdot\left[E^{-1}\cdot P_{CL}(s)\right]^{-1}$$

$$= \begin{bmatrix} \dfrac{s+1}{s^2+2s+2} & \phi \\ \phi & \dfrac{1}{s+1} \end{bmatrix}$$




is input/output decoupled.

$\Longrightarrow$ The choice of $P_{CL}(s)$ will be explained later.

<u>Requirements</u>:

- $\Gamma_c\left[P_{CL}(s)\right] \equiv \Gamma_c\left[P(s)\right]$

$$\Gamma_c\left[E^{-1}\cdot P_{CL}(s)\right] = E^{-1}\cdot\Gamma_c\left[P_{CL}(s)\right]$$

$$\equiv E^{-1}\cdot\Gamma_c\left[P(s)\right]$$

$$\Longrightarrow \boxed{E = \Gamma_c\left[P(s)\right]\cdot\Gamma_c^{-1}\left[E^{-1}\cdot P_{CL}(s)\right]}$$

Example continued:

$$E = \begin{bmatrix} 1 & \phi \\ \phi & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \phi \\ \phi & -1 \end{bmatrix}$$

$$\Rightarrow P_{CL}(s) = E \cdot \left[ E^{-1} \cdot P_{CL}(s) \right]$$

$$= \begin{bmatrix} (s^2+2s+2) & \phi \\ (-s-1) & (-s-1) \end{bmatrix}$$

$$\Rightarrow F(s) = P(s) - P_{CL}(s) = \begin{bmatrix} (-2s-2) & \phi \\ (s+2) & 2 \end{bmatrix}.$$

$$Q(s) = \begin{bmatrix} s & a_0 \\ -1 & (s+a_1) \end{bmatrix}$$

$$\Rightarrow |Q(s)| = s^2 + a_1 s + a_0$$

We choose the observer poles at $(-1 \pm 2j)$

$$\Rightarrow |Q(s)| = s^2 + 2s + 5$$

$$\Rightarrow Q(s) = \begin{bmatrix} s & 5 \\ -1 & (s+2) \end{bmatrix}$$

$$\Rightarrow B(s) = Q(s) \cdot F(s) = \begin{bmatrix} (-2s^2+3s+1\emptyset) & 1\emptyset \\ (s^2+6s+6) & (2s+ \end{bmatrix}$$

as anticipated.

<u>Remarks</u>:

- We have now completed the compensator design. Our originally unstable 3rd order system has been turned with $\{E, K(s), H(s), Q(s)\}$ into a stable, decoupled system with a controllable & observable part of order 3, and an uncontroll & stable part of order 2 ($=\nu$).
- The synthesis was done in the frequency domain. The result is equivalent to a linear state feedback compensation in the time domain including an observer.

- As a side product, we obtained yet another technique to determi[ne] the controllability (Rank($M_x$)), and an algorithm to determine the observability index directly in the frequency domain.
- The obtained observer is a reduced order observer, but <u>not</u> necessarily a minimum order observer. In our example:

$$n = 3,\ p = 2 \Longrightarrow n_{m.o.o.} = n - p = 1$$

whereas:

$$n_{comp} = m(\nu - 1) = 2 \cdot 1 = 2$$

This depends on the design. In our design, $n_{comp} = 2$ since we asked:

$$\partial_{c_i}[Q(s)] = (\nu - 1)$$