General Compensation Techniques

Linear state feedback has been expressed as:

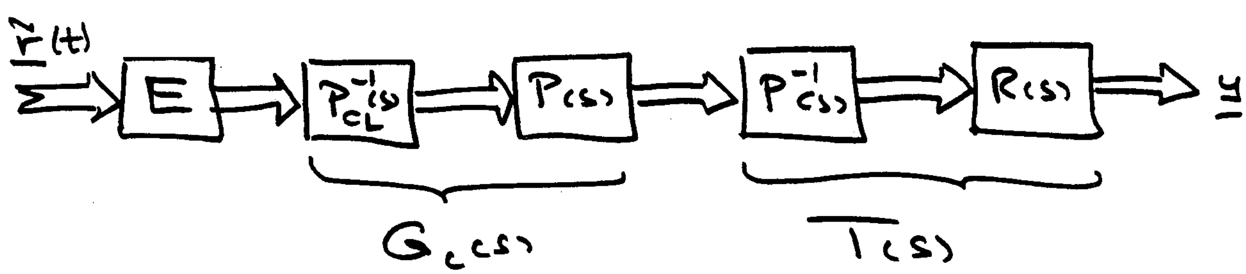
$$G_{tot}(s) = R(s) \cdot P_{CL}^{-1}(s) \cdot E$$
$$= T_{tot}(s) \cdot E$$

We can add E also into the open-loop system:

$$G(s) = R(s) \cdot P^{-1}(s) \cdot E$$
$$= T(s) \cdot E$$

The total transfer function can be realized by a simple series compensation:

$$G_{tot}(s) = T(s) \cdot G_c(s) \cdot E$$



- Obviously, any linear state feedback can be realized as a series compensation with a proper transfer function $G_c(s)$.

- This is not desirable since we lose all the nice properties of feedback control (reduced sensitivity), and since we cancel unstable open-loop poles with unstable zeros which is utterly dubious.

Question: Can we do the opposite, i.e., can we first design a series compensator (which is easy), and then try to realize as large a part of this compensator as possible through an equivalent feedback loop?

"Division" of PM's :

Analogy: $7/3 = \underbrace{2.333\ldots}_{\text{not integer}}$

We can write :

$$7 = \underbrace{2}_{\text{divisor}} \cdot 3 + \underbrace{1}_{\text{remainder}}$$

⇒ Given two polynomial matrices:

$\mathcal{J}(s) \in \mathbb{R}_p^{p \times m}$ and $\bar{\mathcal{J}}(s) \in \mathbb{R}_p^{q \times m}$

where $\rho(\mathcal{J}(s)) = m$

⇒ It is always possible to find two polynomial matrices:

$M(s) \in \mathbb{R}_p^{q \times p}$ and $N(s) \in \mathbb{R}_p^{q \times m}$

such that :

$$\bar{\mathcal{J}}(s) = \underbrace{M(s)}_{\text{divisor}} \cdot \mathcal{J}(s) + \underbrace{N(s)}_{\text{remainder}}$$

Algorithm:

We define $\hat{J}(s)$ to be a $m \times m$ Polynomial matrix consisting of m linearly independent rows of $J(s)$

$\Rightarrow$ $\bar{J}(s) \cdot \hat{J}^{-1}(s)$ is then a rational transfer function matrix (which however does not need to be proper)

$$\bar{J}(s) \cdot \hat{J}^{-1}(s) \in \mathbb{R}_G^{q \times m}$$

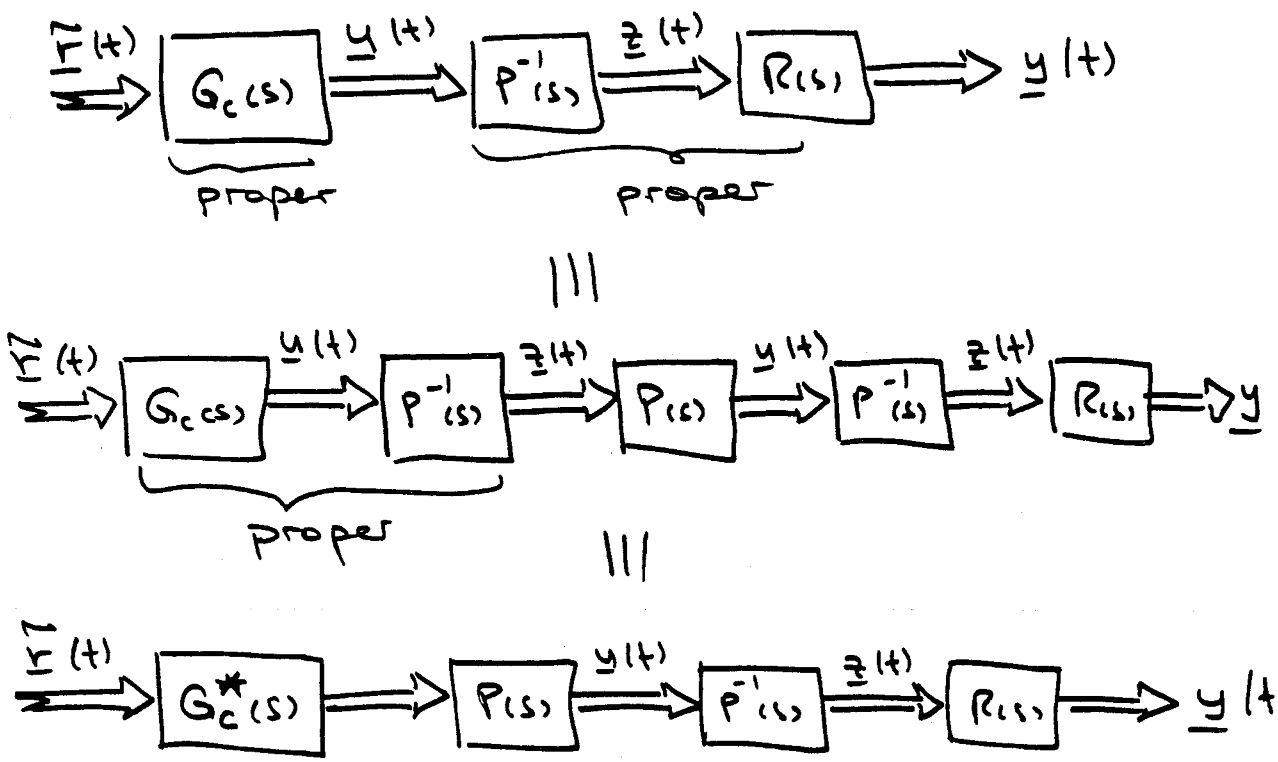
We divide through:

$$\bar{J}(s) \cdot \hat{J}^{-1}(s) = \hat{M}(s) + \underbrace{N(s) \cdot \hat{J}^{-1}(s)}_{\text{strictly proper}}$$

$$\Rightarrow \bar{J}(s) = \hat{M}(s) \cdot \hat{J}(s) + N(s)$$

$\Rightarrow$ Fill the eliminated rows of J back in, and add "zero"-columns in $\hat{M}(s)$

$$\Rightarrow \bar{J}(s) = M(s) \cdot J(s) + N(s)$$



$$G_c^*(s) = P^{-1}(s) \cdot G_c(s)$$

We regroup terms, and decompose $G_c^*(s)$ as:

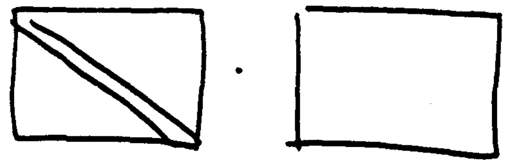
$$G_c^*(s) = P_c^{-1}(s) \cdot L_c(s)$$

where: $\{P_c(s), L_c(s)\}$ are relative left prime (e.g. observer-canonical decomposition). Furthermore, we get $L_c(s)$ into upper-triangular form.

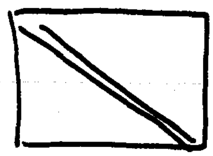
Algorithm:

$$G_c^*(s) = P_c^{*^{-1}}(s) \cdot L_c^*(s)$$

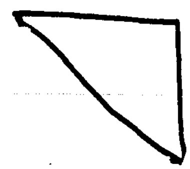
observer-canonical decomposition



$$\Rightarrow \quad G_c^*(s) = P_c^{*^{-1}}(s) \cdot U_L(s) \cdot L_c(s)$$



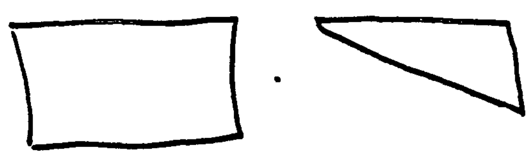
$\cdot \{\text{unimodular}\} \cdot$



$$= P_c^{*^{-1}}(s) \cdot \left(U_L^{-1}(s)\right)^{-1} \cdot L_c(s)$$

$$= \left[\underbrace{U_L^{-1}(s) \cdot P_c^*(s)}_{P_c(s)}\right]^{-1} \cdot L_c(s)$$

$$\Rightarrow \quad G_c^*(s) = P_c(s) \cdot L_c(s)$$



$$\underline{r}(t) \Rightarrow \boxed{L_c(s)} \Rightarrow \boxed{P_c^{-1}(s)} \Rightarrow \boxed{P(s)} \xRightarrow{\underline{u}(t)} \boxed{\bar{P}^{-1}(s)} \xRightarrow{\underline{z}(t)} \boxed{R(s)} \Rightarrow \underline{y}(t)$$

We expand once more:

$$\underline{r}(t) \Rightarrow \boxed{L_c(s)} \Rightarrow \boxed{T_L(s)} \Rightarrow \boxed{T_L^{-1}(s)} \Rightarrow \boxed{P_c^{-1}(s)} \Rightarrow \boxed{P(s)} \xRightarrow{\underline{u}(t)} \boxed{\bar{P}^{-1}(s)} \xRightarrow{\underline{z}(t)} \boxed{R(s)} \Rightarrow \underline{y}$$

We choose $T_L(s)$ such that:

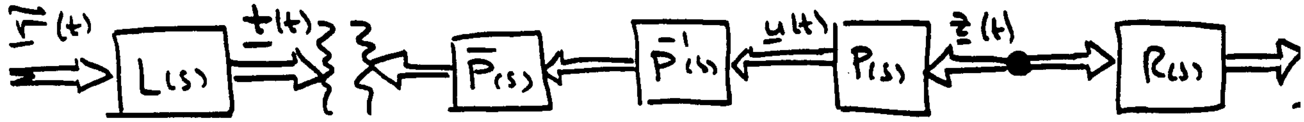
- Roots $\{ |T_L(s)| \}$ are stable
- $P(s) \cdot P_c^{-1}(s) \cdot T_L^{-1}(s)$ is proper

If $P(s) \cdot P_c^{-1}(s)$ is already proper $\Rightarrow$ omit the step ($T_L(s) = I$).
It is always possible to make $T_L(s)$ diagonal.

$$\underline{r}(t) \Rightarrow \boxed{L(s)} \Rightarrow \underbrace{\boxed{\bar{P}^{-1}(s)} \Rightarrow \boxed{P(s)}}_{\text{proper}} \xRightarrow{\underline{u}(t)} \boxed{\bar{P}^{-1}(s)} \xRightarrow{\underline{z}(t)} \boxed{R(s)} \Rightarrow \underline{y}$$

$$\boxed{\begin{aligned} L(s) &= T_L(s) \cdot L_c(s) \\ \bar{P}(s) &= T_L(s) \cdot P_c(s) \end{aligned}}$$

We redraw:

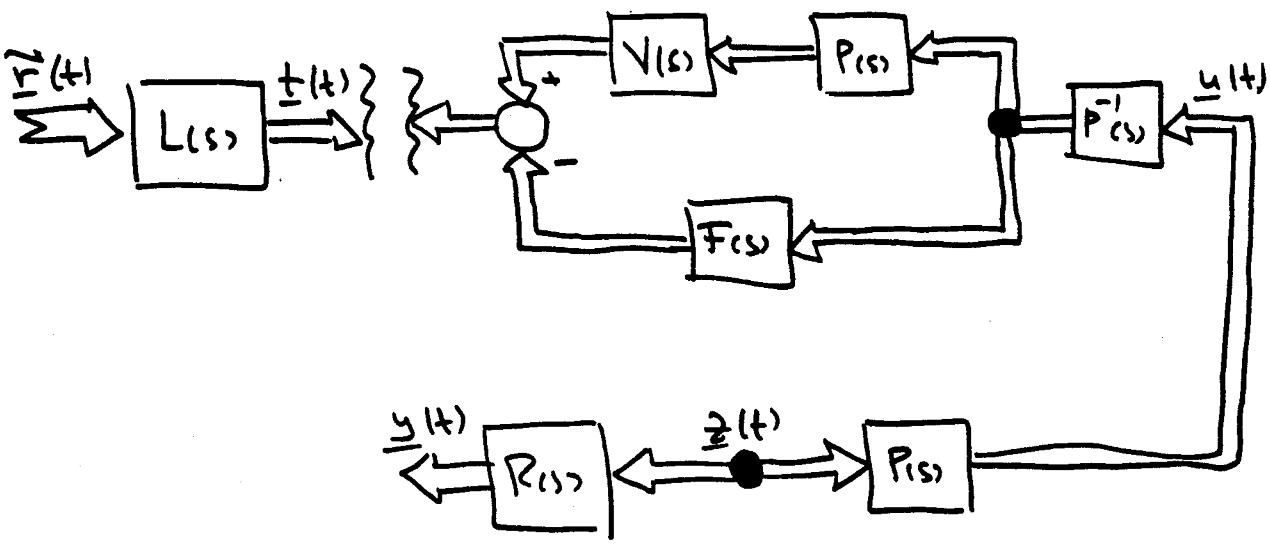


We use the division algorithm:

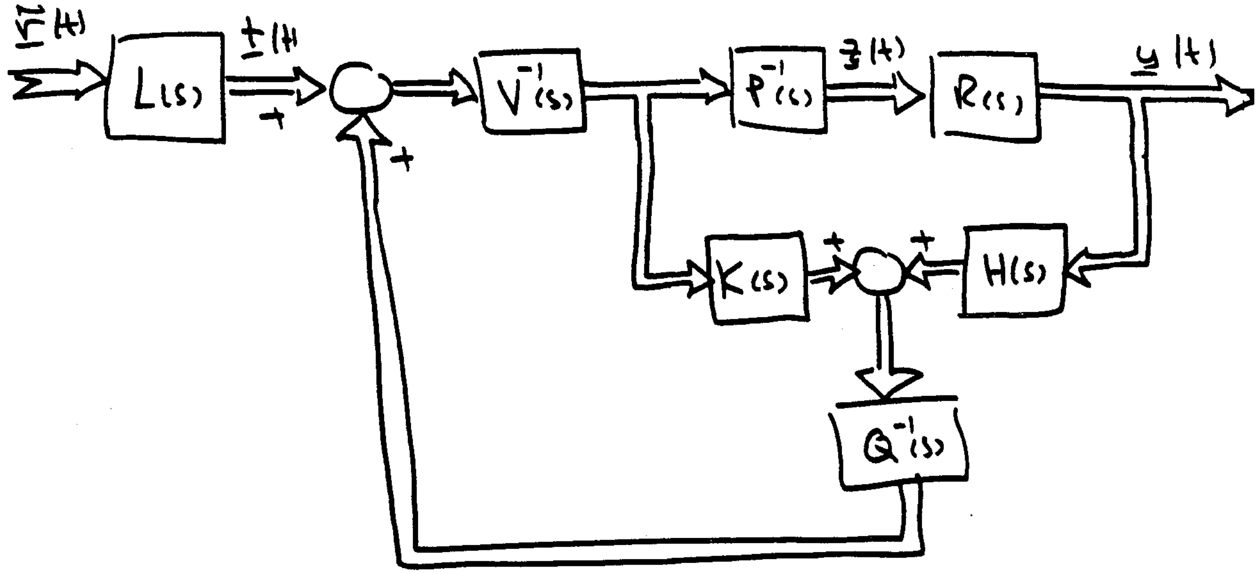
$$\bar{P}(s) = V(s) \cdot P(s) - F(s)$$

where: $V(s), F(s) \in \mathbb{R}_p^{m \times m}$

$$\partial c_i [F(s)] < \partial c_i [P(s)]$$



Let us compare what we got with what we wish to get:



$$\Rightarrow \quad \underline{t}(t) = \left[V(s) \cdot P(s) - F(s) \right] \underline{z}(s)$$

$$\equiv \left[V(s) \cdot P(s) - Q^{-1}(s) \left(K(s) \cdot P(s) + H(s) \cdot R(s) \right) \right] \cdot \underline{z}(s)$$

$$\Rightarrow \boxed{Q(s) \cdot F(s) = B(s) = K(s) P(s) + H(s) R(s)}$$

We already showed how to find $\{ Q(s), K(s), H(s) \}$ from this equation.

Example: Given the system

$$G(s) = \begin{bmatrix} \frac{s+1}{s^2} & \frac{s+2}{s^2+1} \\ \frac{2}{s} & \frac{2s+3}{s^2+1} \end{bmatrix} = \underbrace{\begin{bmatrix} (s+1) & 1 \\ 2s & 3 \end{bmatrix}}_{R(s)} \cdot \underbrace{\begin{bmatrix} s & -s^2 \\ \phi & (s^2+1) \end{bmatrix}}_{P(s)}$$

$\{R(s), P(s)\}$ are relative right prime.

We want:

$$G_{tot}(s) = \begin{bmatrix} \frac{1}{s^2+3s+2} & \phi \\ \phi & \frac{1}{s^2+3s+2} \end{bmatrix}$$

$$\Rightarrow G_{tot}(s) = G(s) \cdot G_c(s) = R(s) \cdot P^{-1}(s) \cdot G_c(s)$$

$$\Rightarrow G_c(s) = P(s) \cdot R^{-1}(s) \cdot G_{tot}(s)$$

$$\Rightarrow G_c(s) = \begin{bmatrix} \frac{2s^3+3s^2}{s^3+6s^2+11s+6} & \frac{-s^3-2s^2}{s^3+6s^2+11s+6} \\ \frac{-2s^3-2s}{s^3+6s^2+11s+6} & \frac{s^3+s^2+s+1}{s^3+6s^2+11s+6} \end{bmatrix}$$

Is the series compensator.

(i)
$$P^{-1}(s)\cdot G_c(s) = \frac{\begin{bmatrix} 3 & -1 \\ -2s & (s+1) \end{bmatrix}}{s^3+6s^2+11s+6}$$

$$\equiv P_c^{-1}(s)\cdot L_c(s) = \underbrace{\begin{bmatrix} (s^3+6s^2+11s+6) & \phi \\ (2s^3+6s^2+4s) & (3s^2+9s+6) \end{bmatrix}}_{P_c(s)}^{-1} \cdot \underbrace{\begin{bmatrix} 3 & \cdot \\ \phi & \end{bmatrix}}_{L_c}$$

$L_c(s) =$; $\{P_c(s), L_c(s)\}$ are relative left prime.

(ii) $P(s)\cdot P_c^{-1}(s)$ is proper $\Rightarrow T_L(s) = I$

$$\Rightarrow \bar{P}(s) \equiv P_c(s) \quad ; \quad L(s) \equiv L_c(s)$$

(iii)
$$\bar{P}(s)\cdot P^{-1}(s) = \underbrace{\begin{bmatrix} (s+6) & (s+6) \\ (2s+6) & (2s+9) \end{bmatrix}}_{V(s)} - \underbrace{\begin{bmatrix} \frac{-11s-6}{s^2} & \frac{-10s}{s^2+1} \\ \frac{-4}{s} & \frac{-11s+3}{s^2+1} \end{bmatrix}}_{F(s)\cdot P^{-1}(s)}$$

$$= \begin{bmatrix} (-11s-6) & (s+6) \\ -4s & (-7s+3) \end{bmatrix}$$

$$V^{-1}(s) = \begin{bmatrix} \frac{\frac{2}{3}s+3}{s+6} & \frac{-1}{3} \\ \frac{-\frac{2}{3}s-2}{s+6} & \frac{1}{s} \end{bmatrix} \quad ; \quad L(s) = \begin{bmatrix} 3 & -1 \\ \phi & 1 \end{bmatrix}$$

$\Rightarrow$ Find $\{H(s), K(s), Q(s)\}$ according to the last chapter, where

$$\partial\left[\,|Q(s)|\,\right] = 2.$$

Decoupling and Model Reduction

"Inversion" of Dynamical Systems.

Def: $G_{LI}(s)$ is called left-inverse to $G(s)$ iff:

$$G_{LI}(s) \cdot G(s) = I^{(m)}$$

Def: $G_{RI}(s)$ is called right-inverse to $G(s)$ iff:

$$G(s) \cdot G_{RI}(s) = I^{(p)}$$

Theorem: • A system has only a left-inverse if

$$\rho\left[G(s)\right] = m$$

• A system has only a right-inverse if

$$\rho\left[G(s)\right] = p \quad .$$

Note: • In general, $G_{LI}(s)$ and $G_{RI}(s)$ are not proper.

• If $m = p$, and $G(s)$ is non-singular
$\Rightarrow G_{LI}(s) = G_{RI}(s) = G^{-1}(s)$

• If $m < p \Rightarrow G_{LI}(s)$ is not uniq
$G_{RI}(s)$ does not exist

• If $m > p \Rightarrow G_{RI}(s)$ does not exist
$G_{RI}(s)$ is not uniqu

Decoupling:

Theorem: A system $G(s) \in R_G^{p \times m}$ can be decoupled only if it is right-invertible, i.e.
$p \equiv \mathfrak{s}[G(s)]$

<u>Proof</u>:

(a) We find $G_{RI}(s)$ such that:

$$G(s) \cdot G_{RI}(s) = I^{(p)}.$$

Then we construct:

$$G_c(s) = G_{RI}(s) \cdot \left[\text{diag}\{p_i(s)\}\right]^{-}$$

by choosing $p_i(s)$ such that $G_c(s)$ is proper.

$\Rightarrow G_c(s)$ is a series-compensato[r] which decouples the system

$$G_{tot}(s) = G(s) \cdot G_c(s) = G(s) \cdot G_{RI}(s) \cdot \left[\text{diag}\{p_i(s)\}\right]^{-1} \equiv \left[\text{diag}\{p_i(s)\}\right]^{-}$$

q.e.d.

(b) If $G(s)$ can be decoupled with $G_c(s)$

$$\Rightarrow G(s) \cdot G_c(s) = G_d(s)$$

diagonal, non-singular

$$\Rightarrow G(s) \cdot \underbrace{G_c(s) \cdot G_d^{-1}(s)}_{G_{RI}(s)} = I^{(p)}$$

q.e.d.

Warnings: We obtain immediately an algorithm for dynamical decoupling. However, there remain a number of questions to be answered:

(1) How many poles can be chosen freely beside from decoupling ?

(2) Are there invariant poles ?

(3) What is the minimal order of the decoupling compensator ?

(4) Can state-feedback alone decouple the system ?

Answers:

- There exist already algorithms to design compensators which minimize the dynamics and allow to move as many poles as possible (non-trivial!)

- There can indeed exist invariant poles which appear in the decoupled system as unobservable modes. These must (of course) be stable.

- These algorithms comput the minimal order of the compensator.

Static Decoupling:

A somewhat simpler problem is the static decoupling problem. Here, we do not request that one input influences only one output at all times, but that, in the steady-state, a change in one input has only a permanent effect on not more than one output, and that every output is affected by exactly one input.

Usually:

$$\lim_{s \to 0} G_{tot}(s) = I^{(p \times m)}$$

e.g. $G_{tot}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Obviously, this is only possible if $m \geq p$.

<u>Def</u>: $G_{tot}(s)$ is statically decoupled iff:

(i) $G_{tot}(s)$ is asymptotically stable

(ii) $\lim_{s \to \phi} G_{tot}(s)$ is diagonal, non-singul

<u>Algorithm</u>:

We use state feedback:

$$\underline{u}(t) = F \cdot \underline{x}(t) + E \cdot \underline{r}(t)$$

$$\Rightarrow G_{tot}(s) = R(s) \left[P(s) - F(s) \right]^{-1} \cdot E$$

Choose F such that:

- $|P(s) - F(s)|$ has all poles in the left half plane
- $|P(\phi) - F(\phi)|$ is non-singular

Choose E such that:

$$\lim_{s \to \phi} G_{tot}(s) = R(\phi) \left[P(\phi) - F(\phi) \right]^{-1} \cdot E$$

$$= \Lambda \text{, diagonal} \in \mathbb{R}^{p \times p}$$

Obviously, this is only possible if:

$$\rho[R(0)] = p$$

<u>Theorem</u>: The static decoupling problem is solvable iff

$$\rho[R(0)] = p .$$

<u>Note</u>:
- F is only used for stabilization. Static decoupling is achieved through the choice of E.
- Obviously, static decoupling is only possible if:

$$m \geq p$$

- If $m = p \Rightarrow E$ is unique
- If $m > p \Rightarrow E$ is <u>not</u> unique.

Model Reconstruction:

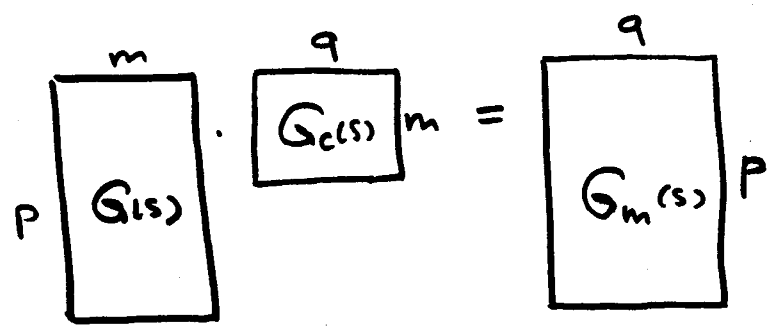
Problem: Given a system with $G(s)$ and a model of that system: $G_m(s)$. We would like to find $G_c(s)$ such that:

$$G(s) \cdot G_c(s) \equiv G_{tot}(s) \equiv G_m(s)$$

Two cases:

(a) $p \geq m$, $\rho[G(s)] = m$

$\Rightarrow$ $G(s)$ is left-invertible.



Algorithm:

We construct $\hat{G}(s)$ from any m linearly independent rows of $G(s)$. We construct $\hat{G}_m(s)$ from the same rows of $G_m(s)$.

$$\Rightarrow \hat{G}(s)\cdot G_c(s) = \hat{G}_m(s)$$

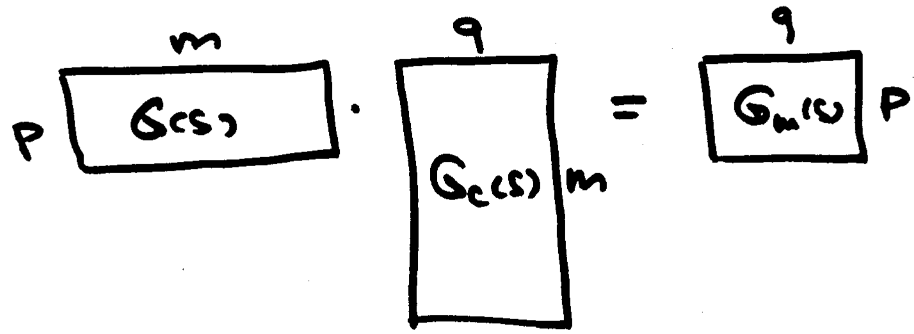
$$\Rightarrow G_c(s) = \hat{G}^{-1}(s)\cdot \hat{G}_m(s)$$

$$\Rightarrow G_m(s) = G(s)\cdot G_c(s) = G(s)\cdot \hat{G}^{-1}(s)\cdot \hat{G}_m(s)$$

$\Rightarrow$ We can choose only m rows of $G_m(s)$ freely. A further restriction comes from the fact that $G_c(s)$ should be proper.

(b) $m \geq p$, $\rho[G(s)] = p$

$\Rightarrow G(s)$ is right-invertible



Same algorithm as before. However, here the solution is *not* unique. We can use the additional freedom to:

(i) ensure that $G_c(s)$ is proper

(ii) the compensated system has stable poles only

(iii) the additional system order is minimal.