Example:

$$\dot{\underline{x}} = \begin{bmatrix} 53 & -22 \\ 135 & -56 \end{bmatrix} \underline{x} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 & -2 \end{bmatrix} \underline{x}$$

(a) Input decoupling:

$$Q_c = [\underline{b}, A\underline{b}] = \begin{bmatrix} 2 & -4 \\ 5 & -10 \end{bmatrix}$$

$\Rightarrow \det(Q_c) = \phi$

$\Rightarrow \text{Rank}(Q_c) = 1$

$\Rightarrow$ One state is controllable, the other is not controllable.

$$\Rightarrow \hat{Q}_c = \left[\begin{array}{c:c} 2 & -1 \\ 5 & -2 \end{array}\right] \Rightarrow \det(\hat{Q}_c) = 1$$

$$\Rightarrow T = \hat{Q}_c^{-1} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$$

$$\Rightarrow \hat{A} = T \cdot A / T = \begin{bmatrix} -2 & -5 \\ \phi & -1 \end{bmatrix} ; \; \hat{\underline{b}} = \begin{bmatrix} 1 \\ \phi \end{bmatrix}$$

$$\hat{\underline{c}}' = [\phi \quad 1]$$

$$\Rightarrow \left| \begin{aligned} \dot{\underline{\xi}} &= \begin{bmatrix} -2 & -5 \\ \phi & -1 \end{bmatrix} \underline{\xi} + \begin{bmatrix} 1 \\ \phi \end{bmatrix} u \\ y &= [\,\phi \quad 1\,]\,\underline{\xi} \end{aligned} \right| \leftarrow \text{not controllable}$$

$$\Rightarrow \left| \begin{aligned} \dot{\underline{x}}_c &= [-2]\,\underline{x}_c + [1]\,u \\ y &= [\,\phi\,]\,\underline{x}_c \end{aligned} \right|$$

is the controllable subsystem.

The uncontrollable mode is at $\underline{\underline{-1}}$ and thus stable.

(b) output decoupling:

$$Q_o = [\hat{\underline{c}}'] \equiv [\phi] \Rightarrow \text{Rank}(Q_o) = \phi$$

$\Rightarrow$ There is one unobservable mode at $\underline{\underline{-2}}$, thus stable.

$$\Rightarrow \underline{\underline{G(s) \equiv \phi}}$$

There was no subsystem S_1.

There was a 1^{st}-order subsystem S_2 with:

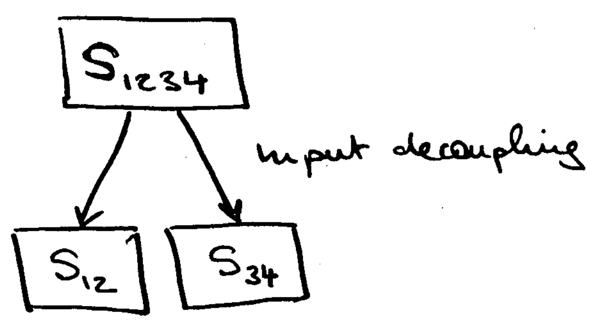
$$\underline{\underline{A_{c\bar{o}} = [-2]}}\,.$$

There was a 1^{st}-order subsystem S_3 with:

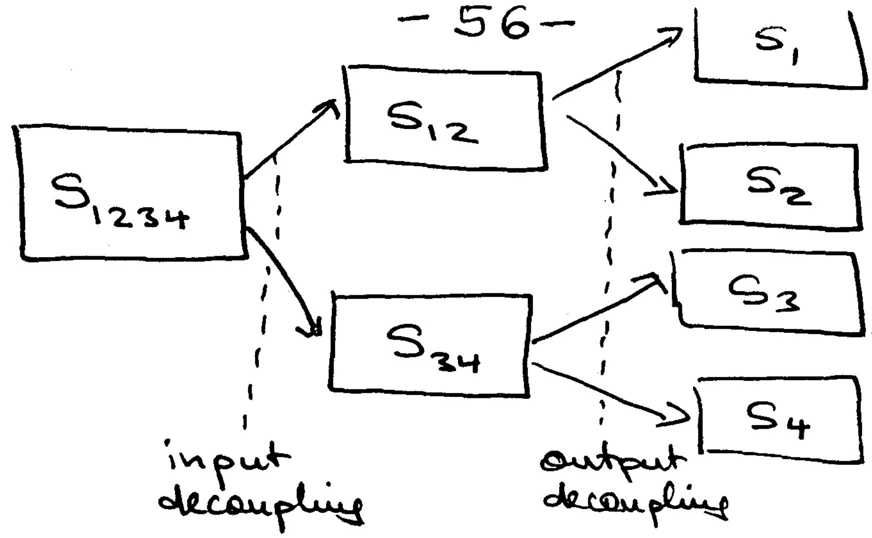
$$\underline{\underline{A_{\bar{c}o} = [-1]}}\,.$$

There was no subsystem S_4.

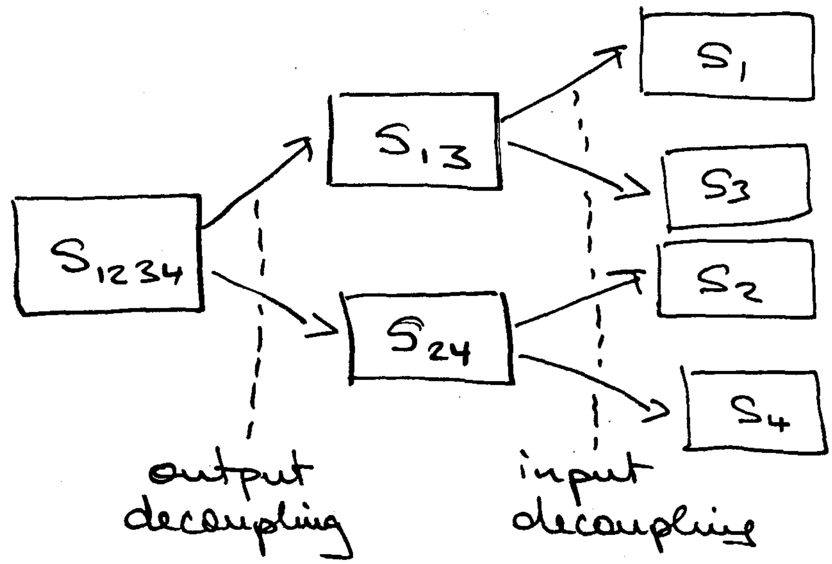
<u>Warning</u>: In the literature, one can often read that input decoupling separates the sontrollable from the uncontrollable subsystem, thus:



thus, one can do the following:



or:



This observation is unfortunately incorrect. It can be proven that Input/output decoupling cannot be used for a complete Kalman decomposition.

Instead, it is true that Input decoupl can be used to extract:

(1) the controllable subsystem

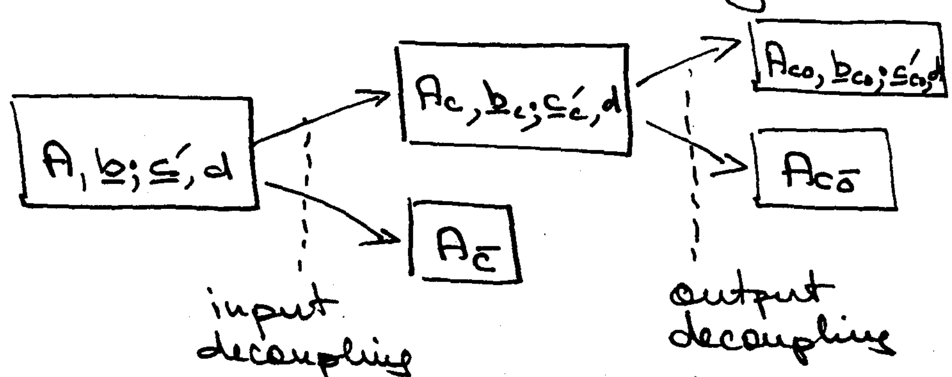
(2) the system matrix of the uncontrollable subsystem,

and output decoupling can be used to extract:

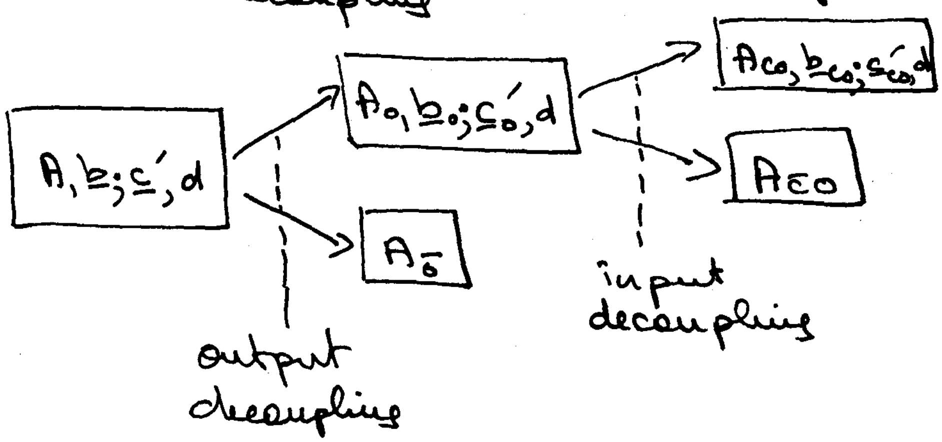
(1) the observable subsystem

(2) the system matrix of the unobservable subsystem.

$\Rightarrow \quad \Sigma_1 = \{ \mathrm{eig}(A) \}$ ∴ set of all eigenvalues



<u>and</u>:



$$\Rightarrow \quad \Sigma_{co} = \{ \mathrm{eig}(A_{co}) \}$$

$$\Sigma_{c\bar{o}} = \{ \mathrm{eig}(A_{c\bar{o}}) \}$$

$$\Sigma_{\bar{c}o} = \{ \mathrm{eig}(A_{\bar{c}o}) \}$$

$$\Sigma_{\bar{c}\bar{o}} = \Sigma - \left(\Sigma_{co} + \Sigma_{c\bar{o}} + \Sigma_{\bar{c}o} \right)$$

However, this does <u>not</u> give you the appropriate $\underline{b}_i$, $\underline{c}'_i$ and A_{ij} coupling elements to generate a complete Kalman decomposition of the form:

$$\left|\begin{array}{l}\dot{\underline{x}} = \left[\begin{array}{c|c|c|c} A_{co} & \phi & A_{13} & \phi \\ \hline A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ \hline \phi & \phi & A_{\bar{c}o} & \phi \\ \hline \phi & \phi & A_{43} & A_{\bar{c}\bar{o}} \end{array}\right] \underline{x} + \left[\begin{array}{c} \underline{b}_{co} \\ \hline \underline{b}_{c\bar{o}} \\ \hline \phi \\ \hline \phi \end{array}\right] u \\ y = \left[\begin{array}{c|c|c|c} \underline{c}'_{co} & \phi & \underline{c}'_{\bar{c}o} & \phi \end{array}\right] \underline{x} + [d]\, u \end{array}\right|$$

This is however not so important, as we usually want to know only:

(1) a minimal realization

(2) the location of the uncontrollable and/or unobservable modes (for reasons of their stability).

The Duality Principle:

Given a system:

$$S_1 = [\, A \,,\, \underline{b} \,;\, \underline{c}' \,,\, d \,]$$

We can build another system:

$$\bar{S} = [\, A' \,,\, \underline{c} \,;\, \underline{b}' \,,\, d' \,] \equiv S' \quad \begin{matrix}(d'=d \\ \text{for SISO}\end{matrix}$$

$\bar{S}$ is called the <u>dual system</u> of S.

- Let us find the transfer function of the dual system:

$$\begin{aligned}
\bar{G}(s) &= \underline{b}'\,(sI - A')^{-1}\,\underline{c} + d' \\
&= \underline{b}'\,(sI' - A')^{-1}\,\underline{c} + d' \\
&= \underline{b}'\,[(sI-A)']^{-1}\,\underline{c} + d' \\
&= \underline{b}'\,[(sI-A)^{-1}]'\,\underline{c} + d' \\
&= [\,\underline{c}'\,(sI-A)^{-1}\,\underline{b}\,]' + d' \\
&= [\,\underline{c}'\,(sI-A)^{-1}\,\underline{b} + d\,]' = G'(s) \underset{\substack{\uparrow \\ \text{SISO}}}{=} G(s)
\end{aligned}$$

- In the SISO case, S and $\bar{S}$ have the same transfer functions, and thus the same minimal realizatio

- Some interesting properties of dual syste

(1) $\Sigma = \{ eig(A) \} \equiv \{ eig(A') \} = \bar{\Sigma}$

(2) $\Sigma_{co} = \bar{\Sigma}_{co}$ (same $G(s)$)

(3) $\Sigma_{c\bar{o}} = \bar{\Sigma}_{\bar{c}o}$

(4) $\Sigma_{\bar{c}o} = \bar{\Sigma}_{c\bar{o}}$

(5) $\Sigma_{\bar{c}\bar{o}} = \bar{\Sigma}_{\bar{c}\bar{o}}$

$\Rightarrow$ In the <u>duality</u> <u>transformation</u>, the properties controllability and observabili become interchanged.

- $$\boxed{S_{1234}} \xrightarrow[\text{decoupling}]{\text{output}} \boxed{S_{13}}$$

$$|||$$

$$\boxed{S_{1234}} \xrightarrow[\text{transf.}]{\text{duality}} \boxed{\bar{S}_{1234}} \xrightarrow[\text{decoupl.}]{\text{input}} \boxed{\bar{S}_{12}} \xrightarrow[\text{transf.}]{\text{duality}} \boxed{S_{1}}$$

and vice-versa.

The Observer - canonical form:

The dual form to the controller-canonical form is the observer-canonical form:

$$
\left|\begin{aligned}
\dot{\underline{x}} &= \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & \vdots \\ \emptyset & & & 1 & -a_{n-1} \end{bmatrix} \underline{x} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \\ \hdashline \emptyset \end{bmatrix} u \quad \leftarrow \text{if } m<n-1 \\
y &= [\,\emptyset \cdots\cdots \emptyset \;\vdots\; 1\,]\,\underline{x} + [\,d\,]\,u
\end{aligned}\right|
$$

<u>that is</u>:

$$
\left|\begin{aligned}
A_{OCF} &\equiv A'_{CCF} \\
\underline{b}_{OCF} &\equiv \underline{c}_{CCF} \\
\underline{c}'_{OCF} &= \underline{b}'_{CCF} \\
d_{OCF} &= d'_{CCF}
\end{aligned}\right|
$$

An algorithm to get into this form from any given state-space representation is the following:

Recipe:

(1) Compute the observability matrix:

$$Q_0 = [\underline{c}' ; \underline{c}'A ; \dots ; \underline{c}'A^{n-1}$$

(2) Take its inverse:

$$Q_0^{-1} = \text{inv}(Q_0)$$

and extract its last column

$$\underline{q} = \text{Qoin}(:, n)$$

(3) Compute the controllability mat assuming $\underline{q}$ to be the input vector:

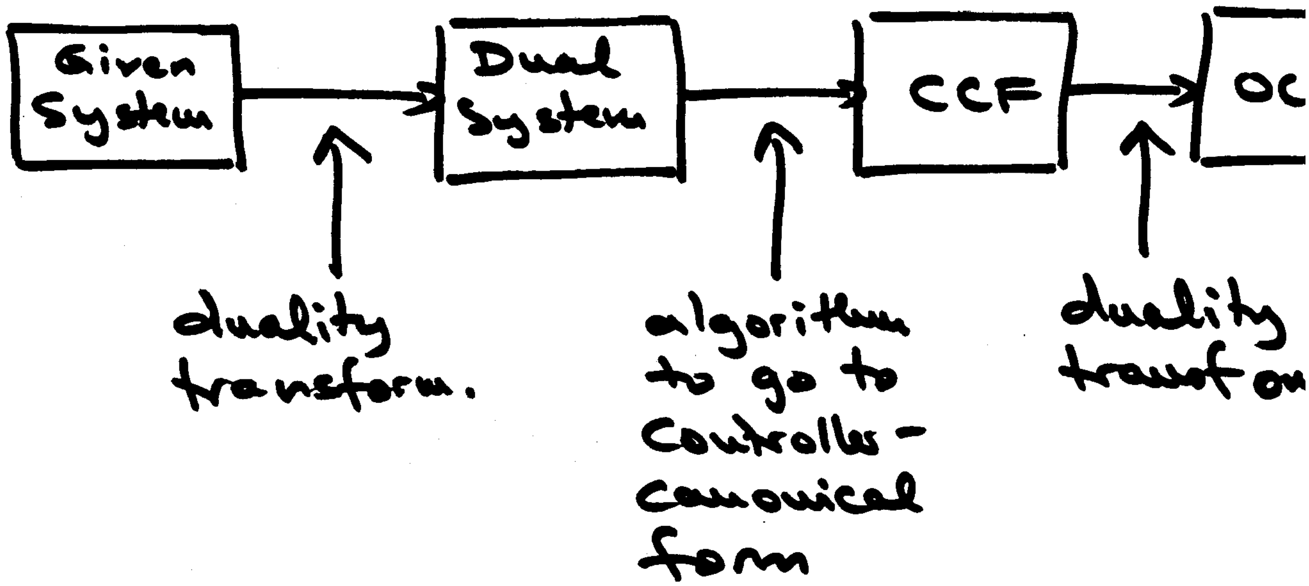
$$P = [\underline{q}, A\underline{q}, \dots, A^{n-1}\underline{q}$$

(4) Calculate its inverse:

$$T = P^{-1}$$

and use T for a similarity transformation.

Of course, an alternative algori
would be:



- While the transformation to controller-canonical form was only possible when the Q_c matrix was nonsingular (no un controllable modes), the trans= formation to observer-canonica form is only possible when the Q_o-matrix is nonsingular (no unobservable modes).
- Both are equally good to determine the transfer function