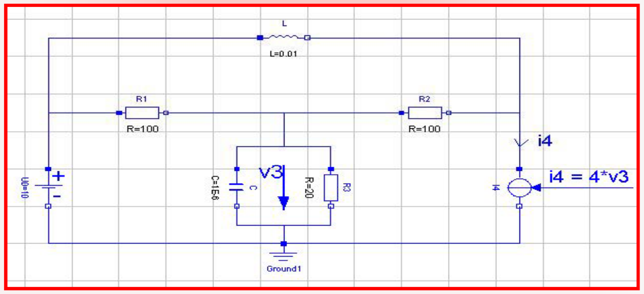
1st Homework Problem

- We wish to analyze the following electrical circuit.



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Mathematical Modeling of Physical Systems

Problems I

1. Write down all of the equations that you need to describe this circuit mathematically. You may use either the technique involving potentials or that involving mesh equations.
2. Sort the equation system horizontally.
3. Sort the horizontally sorted equation system vertically.
4. Convert the horizontally and vertically sorted equation system to state-space form.

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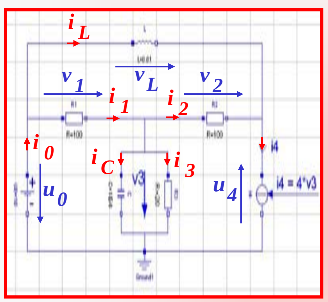

Component equations:

$$u_0 = f_1(t) \qquad v_L = L \cdot \frac{di_L}{dt}$$
$$v_1 = R_1 \cdot i_1 \qquad i_C = C \cdot \frac{dv_3}{dt}$$
$$v_2 = R_2 \cdot i_2$$
$$v_3 = R_3 \cdot i_3 \qquad i_4 = 4 \cdot v_3$$

Mesh equations:

$$u_0 = v_1 + v_3 \qquad v_2 = u_4 + v_3$$
$$v_L = v_1 + v_2$$

Node equations:

$$i_0 = i_1 + i_L \qquad i_4 = i_L + i_2$$
$$i_1 = i_2 + i_3 + i_C$$

$\Rightarrow$ *13 equations in 13 unknowns*

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Problems II

1. Write down all of the equations that you need to describe this circuit mathematically. You may use either the technique involving potentials or that involving mesh equations.
2. Sort the equation system horizontally.
3. Sort the horizontally sorted equation system vertically.
4. Convert the horizontally and vertically sorted equation system to state-space form.

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Horizontal Sorting I

$u_0 = f_1(t)$	$u_0 = v_1 + v_3$
$v_1 = R_1 \cdot i_1$	$v_L = v_1 + v_2$
$v_2 = R_2 \cdot i_2$	$v_2 = u_4 + v_3$
$v_3 = R_3 \cdot i_3$	$i_0 = i_1 + i_L$
$v_L = L \cdot \frac{di_L}{dt}$	$i_1 = i_2 + i_3 + i_C$
$i_C = C \cdot \frac{dv_3}{dt}$	$i_4 = i_L + i_2$
$i_4 = 4 \cdot v_3$	

$\Rightarrow$

$u_0 = f_1(t)$	$u_0 = v_1 + v_3$
$v_1 = R_1 \cdot i_1$	$v_L = v_1 + v_2$
$v_2 = R_2 \cdot i_2$	$v_2 = u_4 + v_3$
$v_3 = R_3 \cdot i_3$	$i_0 = i_1 + i_L$
$v_L = L \cdot \frac{di_L}{dt}$	$i_1 = i_2 + i_3 + i_C$
$i_C = C \cdot \frac{dv_3}{dt}$	$i_4 = i_L + i_2$
$i_4 = 4 \cdot v_3$	

Horizontal Sorting II

$u_0 = f_1(t)$	$u_0 = v_1 + v_3$
$v_1 = R_1 \cdot i_1$	$v_L = v_1 + v_2$
$v_2 = R_2 \cdot i_2$	$v_2 = u_4 + v_3$
$v_3 = R_3 \cdot i_3$	$i_0 = i_1 + i_L$
$v_L = L \cdot \frac{di_L}{dt}$	$i_1 = i_2 + i_3 + i_C$
$i_C = C \cdot \frac{dv_3}{dt}$	$i_4 = i_L + i_2$
$i_4 = 4 \cdot v_3$	

$\Rightarrow$

$u_0 = f_1(t)$	$u_0 = v_1 + v_3$
$v_1 = R_1 \cdot i_1$	$v_L = v_1 + v_2$
$v_2 = R_2 \cdot i_2$	$v_2 = u_4 + v_3$
$v_3 = R_3 \cdot i_3$	$i_0 = i_1 + i_L$
$v_L = L \cdot \frac{di_L}{dt}$	$i_1 = i_2 + i_3 + i_C$
$i_C = C \cdot \frac{dv_3}{dt}$	$i_4 = i_L + i_2$
$i_4 = 4 \cdot v_3$	

Horizontal Sorting III

$u_0 = f_1(t)$	$u_0 = v_1 + v_3$
$v_1 = R_1 \cdot i_1$	$v_L = v_1 + v_2$
$v_2 = R_2 \cdot i_2$	$v_2 = u_4 + v_3$
$v_3 = R_3 \cdot i_3$	$i_0 = i_1 + i_L$
$v_L = L \cdot \frac{di_L}{dt}$	$i_1 = i_2 + i_3 + i_C$
$i_C = C \cdot \frac{dv_3}{dt}$	$i_4 = i_L + i_2$
$i_4 = 4 \cdot v_3$	

$\Rightarrow$

$u_0 = f_1(t)$	$u_0 = v_1 + v_3$
$v_1 = R_1 \cdot i_1$	$v_L = v_1 + v_2$
$v_2 = R_2 \cdot i_2$	$v_2 = u_4 + v_3$
$v_3 = R_3 \cdot i_3$	$i_0 = i_1 + i_L$
$v_L = L \cdot \frac{di_L}{dt}$	$i_1 = i_2 + i_3 + i_C$
$i_C = C \cdot \frac{dv_3}{dt}$	$i_4 = i_L + i_2$
$i_4 = 4 \cdot v_3$	

Horizontal Sorting IV

$u_0 = f_1(t)$	$u_0 = v_1 + v_3$
$v_1 = R_1 \cdot i_1$	$v_L = v_1 + v_2$
$v_2 = R_2 \cdot i_2$	$v_2 = u_4 + v_3$
$v_3 = R_3 \cdot i_3$	$i_0 = i_1 + i_L$
$v_L = L \cdot \frac{di_L}{dt}$	$i_1 = i_2 + i_3 + i_C$
$i_C = C \cdot \frac{dv_3}{dt}$	$i_4 = i_L + i_2$
$i_4 = 4 \cdot v_3$	

$\Rightarrow$

$u_0 = f_1(t)$	$u_0 = v_1 + v_3$
$v_1 = R_1 \cdot i_1$	$v_L = v_1 + v_2$
$v_2 = R_2 \cdot i_2$	$v_2 = u_4 + v_3$
$v_3 = R_3 \cdot i_3$	$i_0 = i_1 + i_L$
$v_L = L \cdot \frac{di_L}{dt}$	$i_1 = i_2 + i_3 + i_C$
$i_C = C \cdot \frac{dv_3}{dt}$	$i_4 = i_L + i_2$
$i_4 = 4 \cdot v_3$	

Horizontal Sorting V

$u_0 = f_1(t)$	$u_0 = v_1 + v_3$	$\Rightarrow$	$u_0 = f_1(t)$	$u_0 = v_1 + v_3$
$v_1 = R_1 \cdot i_1$	$v_L = v_1 + v_2$		$v_1 = R_1 \cdot i_1$	$v_L = v_1 + v_2$
$v_2 = R_2 \cdot i_2$	$v_2 = u_4 + v_3$		$v_2 = R_2 \cdot i_2$	$v_2 = u_4 + v_3$
$v_3 = R_3 \cdot i_3$	$i_0 = i_1 + i_L$		$v_3 = R_3 \cdot i_3$	$i_0 = i_1 + i_L$
$v_L = L \cdot \frac{di_L}{dt}$	$i_1 = i_2 + i_3 + i_C$		$v_L = L \cdot \frac{di_L}{dt}$	$i_1 = i_2 + i_3 + i_C$
$i_C = C \cdot \frac{dv_3}{dt}$	$i_4 = i_L + i_2$		$i_C = C \cdot \frac{dv_3}{dt}$	$i_4 = i_L + i_2$
$i_4 = 4 \cdot v_3$			$i_4 = 4 \cdot v_3$	

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Horizontal Sorting VI

$u_0 = f_1(t)$	$u_0 = v_1 + v_3$	$\Rightarrow$	$u_0 = f_1(t)$	$u_0 = v_1 + v_3$
$v_1 = R_1 \cdot i_1$	$v_L = v_1 + v_2$		$v_1 = R_1 \cdot i_1$	$v_L = v_1 + v_2$
$v_2 = R_2 \cdot i_2$	$v_2 = u_4 + v_3$		$v_2 = R_2 \cdot i_2$	$v_2 = u_4 + v_3$
$v_3 = R_3 \cdot i_3$	$i_0 = i_1 + i_L$		$v_3 = R_3 \cdot i_3$	$i_0 = i_1 + i_L$
$v_L = L \cdot \frac{di_L}{dt}$	$i_1 = i_2 + i_3 + i_C$		$v_L = L \cdot \frac{di_L}{dt}$	$i_1 = i_2 + i_3 + i_C$
$i_C = C \cdot \frac{dv_3}{dt}$	$i_4 = i_L + i_2$		$i_C = C \cdot \frac{dv_3}{dt}$	$i_4 = i_L + i_2$
$i_4 = 4 \cdot v_3$			$i_4 = 4 \cdot v_3$	

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Horizontal Sorting VII

$u_0 = f_1(t)$	$u_0 = v_1 + v_3$	$\Rightarrow$	$u_0 = f_1(t)$	$v_1 = u_0 - v_3$
$v_1 = R_1 \cdot i_1$	$v_L = v_1 + v_2$		$i_1 = v_1 / R_1$	$v_L = v_1 + v_2$
$v_2 = R_2 \cdot i_2$	$v_2 = u_4 + v_3$		$v_2 = R_2 \cdot i_2$	$u_4 = v_2 - v_3$
$v_3 = R_3 \cdot i_3$	$i_0 = i_1 + i_L$		$i_3 = v_3 / R_3$	$i_0 = i_1 + i_L$
$v_L = L \cdot \frac{di_L}{dt}$	$i_1 = i_2 + i_3 + i_C$		$\frac{di_L}{dt} = v_L / L$	$i_C = i_1 - i_2 - i_3$
$i_C = C \cdot \frac{dv_3}{dt}$	$i_4 = i_L + i_2$		$\frac{dv_3}{dt} = i_C / C$	$i_2 = i_4 - i_L$
$i_4 = 4 \cdot v_3$			$i_4 = 4 \cdot v_3$	

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Problems III

1. Write down all of the equations that you need to describe this circuit mathematically. You may use either the technique involving potentials or that involving mesh equations.
2. Sort the equation system horizontally.
3. Sort the horizontally sorted equation system vertically.
4. Convert the horizontally and vertically sorted equation system to state-space form.

Vertical Sorting I

$u_0 = f_1(t)$	$v_1 = u_0 - v_3$
$i_1 = v_1 / R_1$	$v_L = v_1 + v_2$
$v_2 = R_2 \cdot i_2$	$u_4 = v_2 - v_3$
$i_3 = v_3 / R_3$	$i_0 = i_1 + i_L$
$\frac{di_L}{dt} = v_L / L$	$i_C = i_1 - i_2 - i_3$
$\frac{dv_3}{dt} = i_C / C$	$i_2 = i_4 - i_L$
$i_4 = 4 \cdot v_3$	

$\Rightarrow$

$u_0 = f_1(t)$	$v_1 = u_0 - v_3$
$i_3 = v_3 / R_3$	$v_L = v_1 + v_2$
$i_4 = 4 \cdot v_3$	$u_4 = v_2 - v_3$
$i_1 = v_1 / R_1$	$i_0 = i_1 + i_L$
$v_2 = R_2 \cdot i_2$	$i_C = i_1 - i_2 - i_3$
$\frac{di_L}{dt} = v_L / L$	$i_2 = i_4 - i_L$
$\frac{dv_3}{dt} = i_C / C$	

Vertical Sorting II

$u_0 = f_1(t)$	$v_1 = u_0 - v_3$
$i_3 = v_3 / R_3$	$v_L = v_1 + v_2$
$i_4 = 4 \cdot v_3$	$u_4 = v_2 - v_3$
$i_1 = v_1 / R_1$	$i_0 = i_1 + i_L$
$v_2 = R_2 \cdot i_2$	$i_C = i_1 - i_2 - i_3$
$\frac{di_L}{dt} = v_L / L$	$i_2 = i_4 - i_L$
$\frac{dv_3}{dt} = i_C / C$	

$\Rightarrow$

$u_0 = f_1(t)$	$\frac{di_L}{dt} = v_L / L$
$i_3 = v_3 / R_3$	$\frac{dv_3}{dt} = i_C / C$
$i_4 = 4 \cdot v_3$	$v_L = v_1 + v_2$
$v_1 = u_0 - v_3$	$u_4 = v_2 - v_3$
$i_2 = i_4 - i_L$	$i_0 = i_1 + i_L$
$i_1 = v_1 / R_1$	$i_C = i_1 - i_2 - i_3$
$v_2 = R_2 \cdot i_2$	

Vertical Sorting III

$u_0 = f_1(t)$	$\frac{di_L}{dt} = v_L / L$
$i_3 = v_3 / R_3$	$\frac{dv_3}{dt} = i_C / C$
$i_4 = 4 \cdot v_3$	$v_L = v_1 + v_2$
$v_1 = u_0 - v_3$	$u_4 = v_2 - v_3$
$i_2 = i_4 - i_L$	$i_0 = i_1 + i_L$
$i_1 = v_1 / R_1$	$i_C = i_1 - i_2 - i_3$
$v_2 = R_2 \cdot i_2$	

$\Rightarrow$

$u_0 = f_1(t)$	$\frac{di_L}{dt} = v_L / L$
$i_3 = v_3 / R_3$	$\frac{dv_3}{dt} = i_C / C$
$i_4 = 4 \cdot v_3$	$v_L = v_1 + v_2$
$v_1 = u_0 - v_3$	$u_4 = v_2 - v_3$
$i_2 = i_4 - i_L$	$i_0 = i_1 + i_L$
$i_1 = v_1 / R_1$	$i_C = i_1 - i_2 - i_3$
$v_2 = R_2 \cdot i_2$	

Vertical Sorting IV

$u_0 = f_1(t)$	$\frac{di_L}{dt} = v_L / L$
$i_3 = v_3 / R_3$	$\frac{dv_3}{dt} = i_C / C$
$i_4 = 4 \cdot v_3$	$v_L = v_1 + v_2$
$v_1 = u_0 - v_3$	$u_4 = v_2 - v_3$
$i_2 = i_4 - i_L$	$i_0 = i_1 + i_L$
$i_1 = v_1 / R_1$	$i_C = i_1 - i_2 - i_3$
$v_2 = R_2 \cdot i_2$	

$\Rightarrow$

$u_0 = f_1(t)$	$v_L = v_1 + v_2$
$i_3 = v_3 / R_3$	$u_4 = v_2 - v_3$
$i_4 = 4 \cdot v_3$	$i_0 = i_1 + i_L$
$v_1 = u_0 - v_3$	$i_C = i_1 - i_2 - i_3$
$i_2 = i_4 - i_L$	$\frac{di_L}{dt} = v_L / L$
$i_1 = v_1 / R_1$	$\frac{dv_3}{dt} = i_C / C$
$v_2 = R_2 \cdot i_2$	

Vertical Sorting V

$u_0 = f_1(t)$	$v_L = v_1 + v_2$
$i_3 = v_3 / R_3$	$u_4 = v_2 - v_3$
$i_4 = 4 \cdot v_3$	$i_0 = i_1 + i_L$
$v_1 = u_0 - v_3$	$i_C = i_1 - i_2 - i_3$
$i_2 = i_4 - i_L$	$\frac{di_L}{dt} = v_L / L$
$i_1 = v_1 / R_1$	$\frac{dv_3}{dt} = i_C / C$
$v_2 = R_2 \cdot i_2$	

$\Rightarrow$

$u_0 = f_1(t)$	$v_L = v_1 + v_2$
$i_3 = v_3 / R_3$	$u_4 = v_2 - v_3$
$i_4 = 4 \cdot v_3$	$i_0 = i_1 + i_L$
$v_1 = u_0 - v_3$	$i_C = i_1 - i_2 - i_3$
$i_2 = i_4 - i_L$	$\frac{di_L}{dt} = v_L / L$
$i_1 = v_1 / R_1$	$\frac{dv_3}{dt} = i_C / C$
$v_2 = R_2 \cdot i_2$	

Problems IV

1. Write down all of the equations that you need to describe this circuit mathematically. You may use either the technique involving potentials or that involving mesh equations.
2. Sort the equation system horizontally.
3. Sort the horizontally sorted equation system vertically.
4. Convert the horizontally and vertically sorted equation system to state-space form.

State-space Description I

$u_0 = f_1(t)$	$v_L = v_1 + v_2$
$i_3 = v_3 / R_3$	$u_4 = v_2 - v_3$
$i_4 = 4 \cdot v_3$	$i_0 = i_1 + i_L$
$v_1 = u_0 - v_3$	$i_C = i_1 - i_2 - i_3$
$i_2 = i_4 - i_L$	$\frac{di_L}{dt} = v_L / L$
$i_1 = v_1 / R_1$	$\frac{dv_3}{dt} = i_C / C$
$v_2 = R_2 \cdot i_2$	

$$\begin{aligned}
\frac{di_L}{dt} &= v_L / L \\
&= (v_1 + v_2) / L \\
&= v_1 / L + v_2 / L \\
&= (u_0 - v_3) / L + (R_2 / L) \cdot i_2 \\
&= u_0 / L - v_3 / L + (R_2 / L) \cdot (i_4 - i_L) \\
&= u_0 / L - v_3 / L + (4R_2 / L) \cdot v_3 \\
&\quad - (R_2 / L) \cdot i_L \\
&= -(R_2 / L) \cdot i_L + (4R_2 / L - 1 / L) \cdot v_3 \\
&\quad + (1/L) \cdot u_0
\end{aligned}$$

State-space Description II

$u_0 = f_1(t)$	$v_L = v_1 + v_2$
$i_3 = v_3 / R_3$	$u_4 = v_2 - v_3$
$i_4 = 4 \cdot v_3$	$i_0 = i_1 + i_L$
$v_1 = u_0 - v_3$	$i_C = i_1 - i_2 - i_3$
$i_2 = i_4 - i_L$	$\frac{di_L}{dt} = v_L / L$
$i_1 = v_1 / R_1$	$\frac{dv_3}{dt} = i_C / C$
$v_2 = R_2 \cdot i_2$	

$$\begin{aligned}
\frac{dv_3}{dt} &= i_C / C \\
&= (i_1 - i_2 - i_3) / C \\
&= v_1 / (R_1 \cdot C) - (i_4 - i_L) / C \\
&\quad - v_3 / (R_3 \cdot C) \\
&= (u_0 - v_3) / (R_1 \cdot C) - (4 / C) \cdot v_3 \\
&\quad + (1/C) \cdot i_L - v_3 / (R_3 \cdot C) \\
&= (1/C) \cdot i_L \\
&\quad - (1/(R_1 \cdot C) + 1/(R_3 \cdot C) + 4 / C) \cdot v_3 \\
&\quad + 1/(R_1 \cdot C) \cdot u_0
\end{aligned}$$

State-space Description III

$$\frac{di_L}{dt} = -(R_2/L)\cdot i_L + (4R_2/L - 1/L)\cdot v_3 + (1/L)\cdot u_0$$

$$\frac{dv_3}{dt} = (1/C)\cdot i_L - (1/(R_1\cdot C) + 1/(R_3\cdot C) + 4/C)\cdot v_3 + 1/(R_1\cdot C)\cdot u_0$$

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_3}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_2}{L} & \left[\frac{4R_2}{L} - \frac{1}{L}\right] \\ \frac{1}{C} & -\left[\frac{1}{R_1\cdot C} + \frac{1}{R_3\cdot C} + \frac{4}{C}\right] \end{bmatrix} \cdot \begin{bmatrix} i_L \\ v_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ \frac{1}{R_1\cdot C} \end{bmatrix} \cdot u_0$$

State-space Description IV

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_3}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_2}{L} & \left[\frac{4R_2}{L} - \frac{1}{L}\right] \\ \frac{1}{C} & -\left[\frac{1}{R_1\cdot C} + \frac{1}{R_3\cdot C} + \frac{4}{C}\right] \end{bmatrix} \cdot \begin{bmatrix} i_L \\ v_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ \frac{1}{R_1\cdot C} \end{bmatrix} \cdot u_0$$

$$v_3 = \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} i_L \\ v_3 \end{bmatrix}$$

Problems V

5. Program the state-space model in Matlab.
6. Simulate the circuit across *50 μsec* of simulated time, und plot the variable v_3 as a function of time. You may assume that the initial Voltage across the capacitance and the initial current through the inductivity are equal to *0.0*.

C:\Cellier\Classes\Ece449\Hw_1\mmps_hw1.m

```
R1 = 100;  R2 = 100;  R3 = 20;
C = 1e-6;  L = 0.01;
R1C = 1/(R1*C);  R3C = 1/(R3*C);
Linv = 1/L; Cinv = 1/C;
a11 = -R2*Linv;
a12 = (4*R2 - 1)*Linv;
a21 = Cinv;
a22 = -(R1C + R3C + 4*Cinv);
A = [ a11 , a12 ; a21 , a22 ];
b = [ Linv ; R1C ];
c = [ 0 , 1 ];
d = 0;
S = ss(A,b,c,d);
t = [ 0 : 5e-8 : 5e-5 ];
u = 10*ones(size(t));
x0 = zeros(2,1);
y = lsim(S,u,t,x0);
plot(t,y)
grid on
return
```