3rd Homework Solution

- In this homework problem, we wish to exercise the application of the algorithms by Pantelides and Tarjan, as well as the tearing method.
- The problem deals with another simple electrical circuit.

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Mathematical Modeling of Physical Systems

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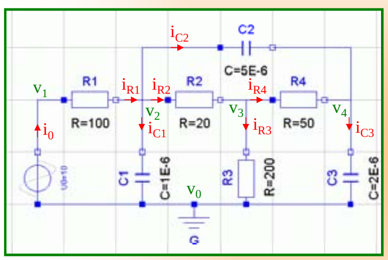
Structural Singularity



Show that the circuit depicted on the left exhibits a ***structural singularity***.

To this end, find a complete set of equations in currents, potentials, and Voltages (ignoring the mesh equations), and draw the digraph of the resulting DAE system.

Subsequently, color the digraph by use of the algorithm by Tarjan, and demonstrate that the system is indeed structurally singular.

Explain the structural singularity by analyzing the mesh that is formed by the three capacitors.

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1: $U_0 = f(t)$	13: $U_0 = v_1 - v_0$
2: $u_{R1} = R_1 \cdot i_{R1}$	14: $u_{R1} = v_1 - v_2$
3: $u_{R2} = R_2 \cdot i_{R2}$	15: $u_{R2} = v_2 - v_3$
4: $u_{R3} = R_3 \cdot i_{R3}$	16: $u_{R3} = v_3 - v_0$
5: $u_{R4} = R_4 \cdot i_{R4}$	17: $u_{R4} = v_3 - v_4$
6: $i_{C1} = C_1 \cdot du_{C1}/dt$	18: $u_{C1} = v_2 - v_0$
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8: $i_{C3} = C_3 \cdot du_{C3}/dt$	20: $u_{C3} = v_4 - v_0$
9: $i_0 = i_{R1}$	21: $v_0 = 0$
10: $i_{R1} = i_{C1} + i_{C2} + i_{R2}$	
11: $i_{R2} = i_{R3} + i_{R4}$	
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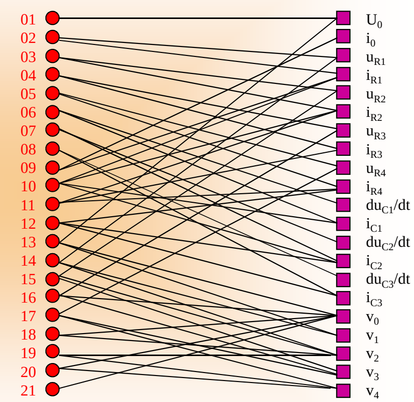

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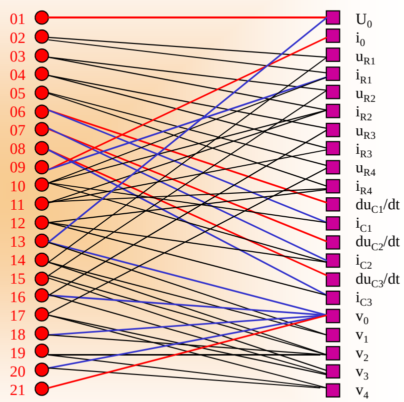

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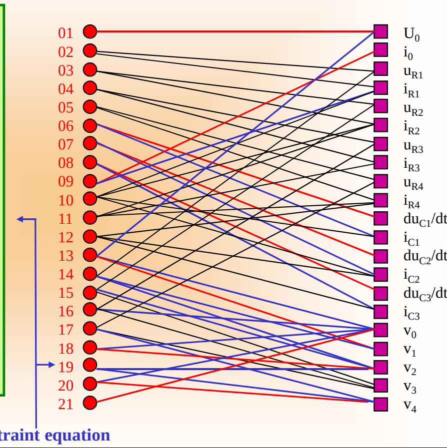

Pantelides Algorithm

- Apply the algorithm by Pantelides to the equation system found before, and determine the resulting DAE system that by now no longer exhibits any structural singularity.
- Find the structure incidence matrix of the resulting implicit DAE system.

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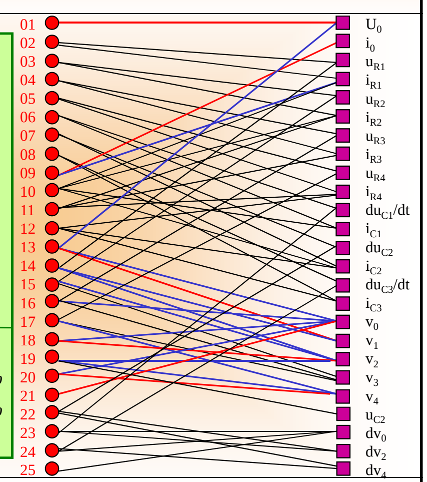
	25: $dv_0 = 0$



S =

	U0	i0	uR1	iR1	uR2	iR2	uR3	iR3	uR4	iR4	duC1/dt	iC1	duC2	iC2	duC3/dt	iC3	v0	v1	v2	v3	v4	uC2	dv0	dv2	dv4
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
9	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
13	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
14	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
15	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
16	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0
17	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1
23	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0
24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0

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Algorithm by Tarjan

- Draw the digraph of the resulting DAE system, and color it by use of the algorithm by Tarjan.
- The colored digraph symbolizes a partially sorted equation system, which however still contains a large algebraic loop. Write down the partially sorted equation system.
- Find the structure incidence matrix of the partially sorted equation system. This is now in block lower triangular form (BLT-Form).

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15: $u_{R2} = v_2 - v_3$

16: $u_{R3} = v_3 - v_0$

17: $u_{R4} = v_3 - v_4$

04: $u_{C1} = v_2 - v_0$

06: $u_{C2} = v_2 - v_4$

05: $u_{C3} = v_4 - v_0$

02: $v_0 = 0$

22: $du_{C2} = dv_2 - dv_4$

23: $du_{C1}/dt = dv_2 - dv_0$

24: $du_{C3}/dt = dv_4 - dv_0$

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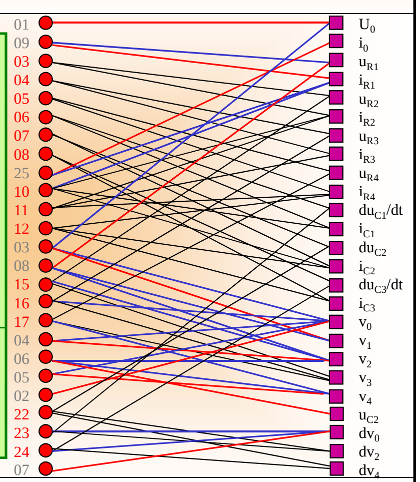

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S =

	U0	v0	v1	v2	v4	uC2	dv0	uR1	iR1	uR2	iR2	uR3	iR3	uR4	iR4	duC1/dt	iC1	duC2	iC2	duC3/dt	iC3	v3	dv2	dv4	i0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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9	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
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10	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0
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17	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	0
23	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0
24	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0
25	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

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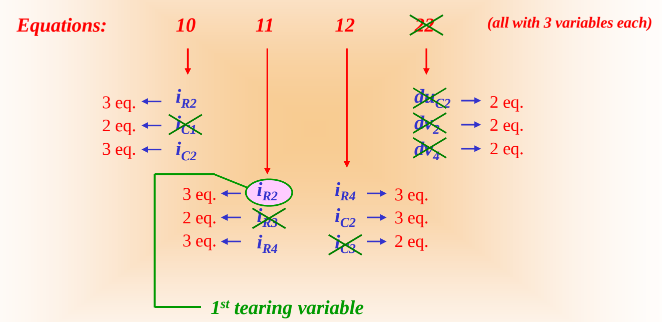
Tearing of the Algebraic Loop

- Find appropriate tearing variables using the following heuristics:
  - In the digraph, determine those equations with the largest number of unknowns.
  - For every one of these equations, find those unknowns that show up most frequently in the not yet used equations.
  - For every one of these variables, determine how many additional equations can be made causal if they are assumed known.
  - Choose the one variable as the next tearing variable, which allows to make the largest number of additional equations causal.

Selection of tearing variables:



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10: $i_{R1} = i_{C1} + i_{C2} + i_{R2}$	22: $du_{C2} = dv_2 - dv_4$
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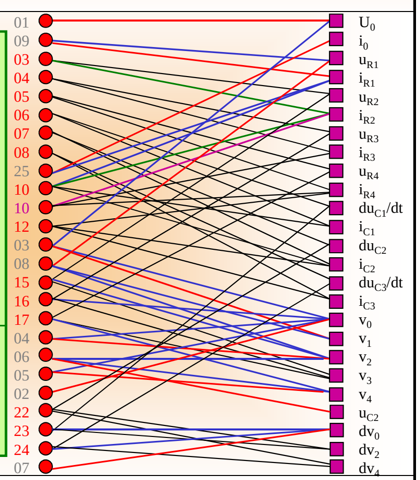

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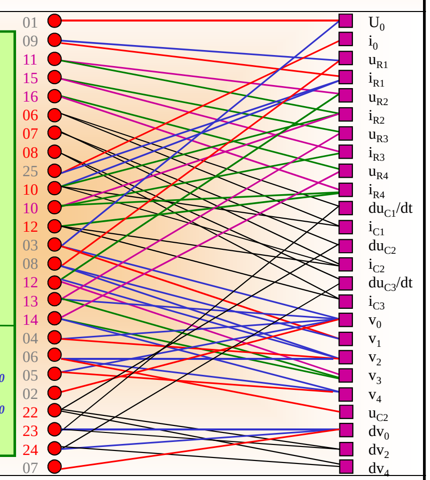

Selection of tearing variables:

Equations: 22 (the only equation with 3 unknowns)

du_{C2} → 2 eq.
dv_2 → 2 eq.
dv_4 → 2 eq.

2^{nd} tearing variable

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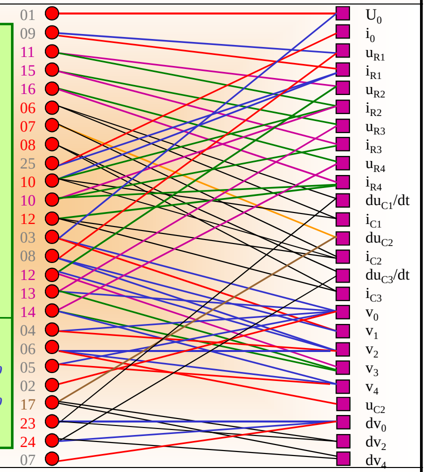
25: $i_0 = i_{R1}$	02: $v_0 = 0$
10: $i_{R1} = i_{C1} + i_{C2} + i_{R2}$	17: $du_{C2} = dv_2 - dv_4$
10: $i_{R2} = i_{R3} + i_{R4}$	23: $du_{C1}/dt = dv_2 - dv_0$
12: $i_{C3} = i_{R4} + i_{C2}$	24: $du_{C3}/dt = dv_4 - dv_0$
	07: $dv_0 = 0$



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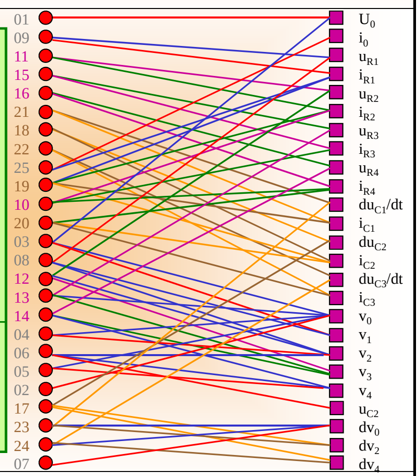
01: $U_0 = f(t)$	03: $U_0 = v_1 - v_0$
09: $u_{R1} = R_1 \cdot i_{R1}$	08: $u_{R1} = v_1 - v_2$
11: $u_{R2} = R_2 \cdot i_{R2}$	12: $u_{R2} = v_2 - v_3$
15: $u_{R3} = R_3 \cdot i_{R3}$	13: $u_{R3} = v_3 - v_0$
16: $u_{R4} = R_4 \cdot i_{R4}$	14: $u_{R4} = v_3 - v_4$
21: $i_{C1} = C_1 \cdot du_{C1}/dt$	04: $u_{C1} = v_2 - v_0$
18: $i_{C2} = C_2 \cdot du_{C2}$	06: $u_{C2} = v_2 - v_4$
22: $i_{C3} = C_3 \cdot du_{C3}/dt$	05: $u_{C3} = v_4 - v_0$
25: $i_0 = i_{R1}$	02: $v_0 = 0$
19: $i_{R1} = i_{C1} + i_{C2} + i_{R2}$	17: $du_{C2} = dv_2 - dv_4$
10: $i_{R2} = i_{R3} + i_{R4}$	23: $du_{C1}/dt = dv_2 - dv_0$
20: $i_{C3} = i_{R4} + i_{C2}$	24: $du_{C3}/dt = dv_4 - dv_0$
	07: $dv_0 = 0$



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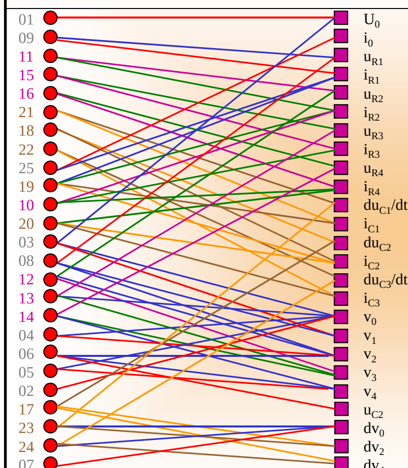

01: $U_0 = f(t)$	14: $u_{R4} = v_3 - v_4$
02: $v_0 = 0$	15: $i_{R3} = u_{R3}/R_3$
03: $v_1 = U_0 - v_0$	16: $i_{R4} = u_{R4}/R_4$
04: $v_2 = u_{C1} - v_0$	17: $du_{C2} = dv_2 - dv_4$
05: $v_4 = u_{C3} - v_0$	18: $i_{C2} = C_2 \cdot du_{C2}$
06: $u_{C2} = v_2 - v_4$	19: $i_{C1} = i_{R1} - i_{C2} - i_{R2}$
07: $dv_0 = 0$	20: $i_{C3} = i_{R4} + i_{C2}$
08: $u_{R1} = v_1 - v_2$	21: $du_{C1}/dt = i_{C1}/C_1$
09: $i_{R1} = u_{R1}/R_1$	22: $du_{C3}/dt = i_{C3}/C_3$
10: $i_{R2} = i_{R3} + i_{R4}$	23: $dv_2 = du_{C1}/dt - dv_0$
11: $u_{R2} = R_2 \cdot i_{R2}$	24: $dv_4 = du_{C3}/dt - dv_0$
12: $v_3 = v_2 - u_{R2}$	25: $i_0 = i_{R1}$
13: $u_{R3} = v_3 - v_0$	

Structure Diagram

- Draw the structure diagram of the causalized algebraic loop.
- It can be seen that two tearing variables are needed to make all equations of the loop causal.
- The two tearing variables decouple the equation system in such a way that there result two separate equation systems in one tearing variable each (this is not always the case, but it happens in the given example).
- Find the structure incidence matrix of the fully causalized DAE system. This now has two diagonal blocks of smaller sizes.

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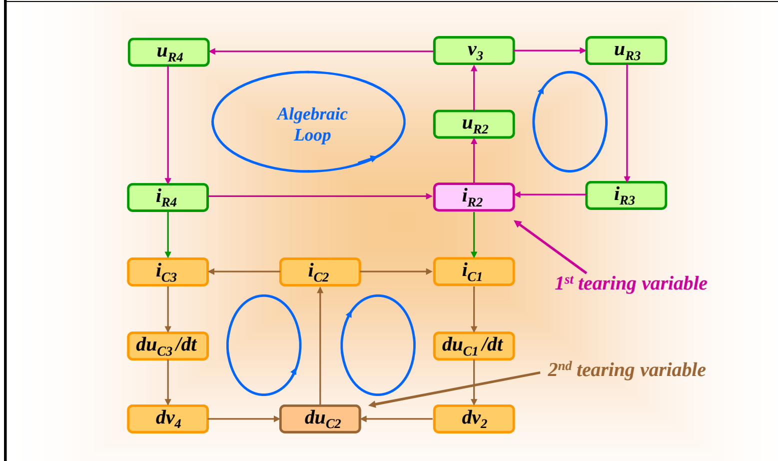

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S =

	U0	v0	v1	v2	v4	uC2	dv0	uR1	iR1	iR2	uR2	v3	uR3	uR4	iR3	iR4	duC2	iC2	iC1	iC3	duC1/dt	duC3/dt	dv2	dv4	i0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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11	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0
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18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
23	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0
24	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0
25	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

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Solving the Coupled Equations

- Solve the two equation systems symbolically, and replace the residual equations of the equation system by the so found explicit equations.
- Draw the digraph of the once more modified equation system, and color it by use of the algorithm by Tarjan.
- Determine the resulting structure incidence matrix. This is now in lower triangular form.

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01: $U_0 = f(t)$
02: $v_0 = 0$
03: $v_1 = U_0 - v_0$
04: $v_2 = u_{C1} - v_0$
05: $v_4 = u_{C3} - v_0$
06: $u_{C2} = v_2 - v_4$
07: $dv_0 = 0$
08: $u_{R1} = v_1 - v_2$
09: $i_{R1} = u_{R1}/R_1$
10: $i_{R2} = i_{R3} + i_{R4}$
11: $u_{R2} = R_2 \cdot i_{R2}$
12: $v_3 = v_2 - u_{R2}$
13: $u_{R3} = v_3 - v_0$
14: $u_{R4} = v_3 - v_4$
15: $i_{R3} = u_{R3}/R_3$
16: $i_{R4} = u_{R4}/R_4$
17: $du_{C2} = dv_2 - dv_4$
18: $i_{C2} = C_2 \cdot du_{C2}$
19: $i_{C1} = i_{R1} - i_{C2} - i_{R2}$
20: $i_{C3} = i_{R4} + i_{C2}$
21: $du_{C1}/dt = i_{C1}/C_1$
22: $du_{C3}/dt = i_{C3}/C_3$
23: $dv_2 = du_{C1}/dt - dv_0$
24: $dv_4 = du_{C3}/dt - dv_0$
25: $i_0 = i_{R1}$

$$\begin{aligned}
i_{R2} &= i_{R3} + i_{R4} \\
&= u_{R3}/R_3 + u_{R4}/R_4 \\
&= (v_3 - v_0)/R_3 + (v_3 - v_4)/R_4 \\
&= (R_3 + R_4)/(R_3 \cdot R_4)\, v_3 - v_0/R_3 - v_4/R_4 \\
&= (R_3 + R_4)/(R_3 \cdot R_4) \cdot (v_2 - u_{R2}) - v_0/R_3 - v_4/R_4 \\
&= -(R_3 + R_4)/(R_3 \cdot R_4) \cdot u_{R2} \\
&\quad + (R_3 + R_4)/(R_3 \cdot R_4) \cdot v_2 - v_0/R_3 - v_4/R_4 \\
&= -R_2 \cdot (R_3 + R_4)/(R_3 \cdot R_4) \cdot i_{R2} \\
&\quad + (R_3 + R_4)/(R_3 \cdot R_4) \cdot v_2 - v_0/R_3 - v_4/R_4
\end{aligned}$$

$$\Rightarrow \quad i_{R2} = \frac{(R_3 + R_4) \cdot v_2 - R_4 \cdot v_0 - R_3 \cdot v_4}{R_2 \cdot R_3 + R_2 \cdot R_4 + R_3 \cdot R_4}$$

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06: $u_{C2} = v_2 - v_4$
07: $dv_0 = 0$
08: $u_{R1} = v_1 - v_2$
09: $i_{R1} = u_{R1}/R_1$
10: $i_{R2} = i_{R3} + i_{R4}$
11: $u_{R2} = R_2 \cdot i_{R2}$
12: $v_3 = v_2 - u_{R2}$
13: $u_{R3} = v_3 - v_0$
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15: $i_{R3} = u_{R3}/R_3$
16: $i_{R4} = u_{R4}/R_4$
17: $du_{C2} = dv_2 - dv_4$
18: $i_{C2} = C_2 \cdot du_{C2}$
19: $i_{C1} = i_{R1} - i_{C2} - i_{R2}$
20: $i_{C3} = i_{R4} + i_{C2}$
21: $du_{C1}/dt = i_{C1}/C_1$
22: $du_{C3}/dt = i_{C3}/C_3$
23: $dv_2 = du_{C1}/dt - dv_0$
24: $dv_4 = du_{C3}/dt - dv_0$
25: $i_0 = i_{R1}$

$$\begin{aligned}
du_{C2} &= dv_2 - dv_4 \\
&= (du_{C1}/dt - dv_0) - (du_{C3}/dt - dv_0) \\
&= du_{C1}/dt - du_{C3}/dt \\
&= i_{C1}/C_1 - i_{C3}/C_3 \\
&= (i_{R1} - i_{C2} - i_{R2})/C_1 - (i_{R4} + i_{C2})/C_3 \\
&= -(C_1 + C_3)/(C_1 \cdot C_3) \cdot i_{C2} \\
&\quad + (i_{R1} - i_{R2})/C_1 - i_{R4}/C_3 \\
&= -C_2 \cdot (C_1 + C_3)/(C_1 \cdot C_3) \cdot du_{C2} \\
&\quad + (i_{R1} - i_{R2})/C_1 - i_{R4}/C_3
\end{aligned}$$

$$\Rightarrow \quad du_{C2} = \frac{C_3 \cdot (i_{R1} - i_{R2}) - C_1 \cdot i_{R4}}{C_1 \cdot C_2 + C_1 \cdot C_3 + C_2 \cdot C_3}$$

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05: $v_4 = u_{C3} - v_0$
06: $u_{C2} = v_2 - v_4$
07: $dv_0 = 0$
08: $u_{R1} = v_1 - v_2$
09: $i_{R1} = u_{R1}/R_1$
10: $i_{R2} = \dfrac{(R_3 + R_4) \cdot v_2 - R_4 \cdot v_0 - R_3 \cdot v_4}{R_2 \cdot R_3 + R_2 \cdot R_4 + R_3 \cdot R_4}$
11: $u_{R2} = R_2 \cdot i_{R2}$
12: $v_3 = v_2 - u_{R2}$
13: $u_{R3} = v_3 - v_0$
14: $u_{R4} = v_3 - v_4$
15: $i_{R3} = u_{R3}/R_3$
16: $i_{R4} = u_{R4}/R_4$
17: $du_{C2} = \dfrac{C_3 \cdot (i_{R1} - i_{R2}) - C_1 \cdot i_{R4}}{C_1 \cdot C_2 + C_1 \cdot C_3 + C_2 \cdot C_3}$
18: $i_{C2} = C_2 \cdot du_{C2}$
19: $i_{C1} = i_{R1} - i_{C2} - i_{R2}$
20: $i_{C3} = i_{R4} + i_{C2}$
21: $du_{C1}/dt = i_{C1}/C_1$
22: $du_{C3}/dt = i_{C3}/C_3$
23: $dv_2 = du_{C1}/dt - dv_0$
24: $dv_4 = du_{C3}/dt - dv_0$
25: $i_0 = i_{R1}$

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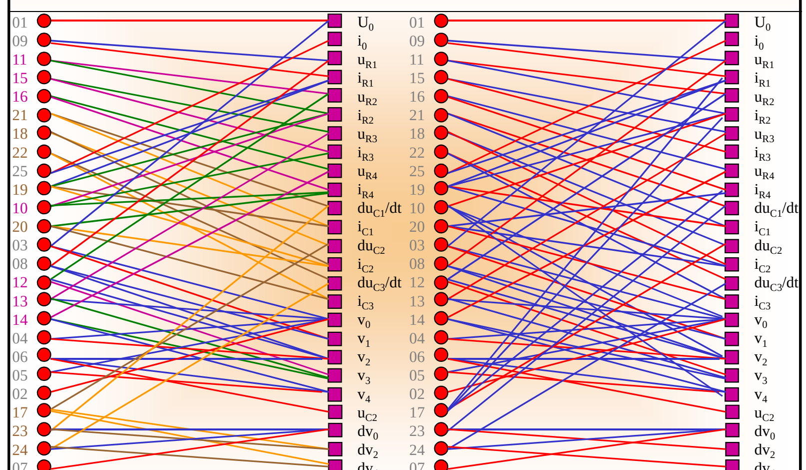

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S =

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2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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6	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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9	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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11	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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19	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
23	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0
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25	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1