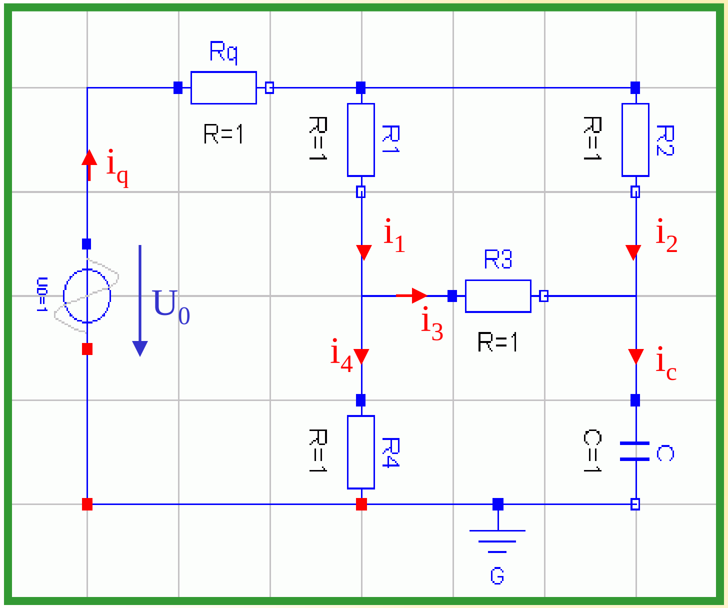
4th Homework Solution

- In this homework problem, we wish to exercise the tearing and relaxation methods by means of a slightly larger problem than that presented in the lecture.
- We also wish to compare the computational efficiency of the simulation codes obtained by the two methods.

- Tearing Algorithm
- Relaxation Algorithm
- Tearing *vs.* Relaxation

Tearing Algorithm I



Given the electrical circuit of the figure on the left, determine a complete set of equations in currents and Voltages (by use of both node and mesh equations).

Make the equation system causal, while trying to get by with two tearing variables (and residual equations).

$$
\begin{aligned}
U_0 &= f(t) \\
u_q &= R_q \cdot i_q \\
u_1 &= R_1 \cdot i_1 \\
u_2 &= R_2 \cdot i_2 \\
u_3 &= R_3 \cdot i_3 \\
u_4 &= R_4 \cdot i_4 \\
i_c &= C \cdot du_c/dt \\
i_q &= i_1 + i_2 \\
i_1 &= i_3 + i_4 \\
i_c &= i_2 + i_3 \\
U_0 &= u_q + u_1 + u_4 \\
u_2 &= u_1 + u_3 \\
u_4 &= u_3 + u_c
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
U_0 &= f(t) \\
u_q &= R_q \cdot i_q \\
u_1 &= R_1 \cdot i_1 \\
u_2 &= R_2 \cdot i_2 \\
u_3 &= R_3 \cdot i_3 \\
u_4 &= R_4 \cdot i_4 \\
i_c &= C \cdot du_c/dt \\
i_q &= i_1 + i_2 \\
i_1 &= \boxed{i_3} + i_4 \\
i_c &= i_2 + i_3 \\
U_0 &= u_q + u_1 + u_4 \\
u_2 &= u_1 + u_3 \\
u_4 &= u_3 + u_c
\end{aligned}
$$

selected tearing variable

better choice (this choice would have allowed us to get by with a single tearing variable) – however, I prefer to demonstrate the algorithm with 2 tearing variables.

$$\begin{aligned}
U_0 &= f(t)\\
u_q &= R_q \cdot i_q\\
u_1 &= R_1 \cdot i_1\\
u_2 &= R_2 \cdot i_2\\
u_3 &= R_3 \cdot i_3\\
u_4 &= R_4 \cdot i_4\\
i_c &= C \cdot du_c/dt\\
\boxed{i_q} &= i_1 + i_2\\
i_1 &= i_3 + i_4\\
i_c &= i_2 + i_3\\
U_0 &= u_q + u_1 + u_4\\
u_2 &= u_1 + u_3\\
u_4 &= u_3 + u_c
\end{aligned}$$

2nd selected tearing variable

$\Rightarrow$

$$\begin{aligned}
U_0 &= f(t)\\
u_q &= R_q \cdot i_q\\
u_1 &= R_1 \cdot i_1\\
u_2 &= R_2 \cdot i_2\\
u_3 &= R_3 \cdot i_3\\
u_4 &= R_4 \cdot i_4\\
i_c &= C \cdot du_c/dt\\
i_q &= i_1 + i_2\\
i_1 &= i_3 + i_4\\
i_c &= i_2 + i_3\\
U_0 &= u_q + u_1 + u_4\\
u_2 &= u_1 + u_3\\
u_4 &= u_3 + u_c
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
U_0 &= f(t)\\
u_q &= R_q \cdot i_q\\
i_1 &= u_1 / R_1\\
i_2 &= u_2 / R_2\\
u_3 &= R_3 \cdot i_3\\
i_4 &= u_4 / R_4\\
du_c/dt &= i_c / C\\
i_q &= i_1 + i_2\\
i_3 &= i_1 - i_4\\
i_c &= i_2 + i_3\\
u_1 &= U_0 - u_q - u_4\\
u_2 &= u_1 + u_3\\
u_4 &= u_3 + u_c
\end{aligned}$$

Tearing Algorithm II

- Solve symbolically for the tearing variables, and find replacement equations for the two residual equations, which permit to make the entire set of equations causal.
- Find an explicit DAE system by completely sorting the equations both horizontally and vertically.

$U_0 = f(t)$
$u_q = R_q \cdot i_q$
$i_1 = u_1 / R_1$
$i_2 = u_2 / R_2$
$u_3 = R_3 \cdot i_3$
$i_4 = u_4 / R_4$
$du_c / dt = i_c / C$
$i_q = i_1 + i_2$
$i_3 = i_1 - i_4$
$i_c = i_2 + i_3$
$u_1 = U_0 - u_q - u_4$
$u_2 = u_1 + u_3$
$u_4 = u_3 + u_c$

$$
\begin{aligned}
i_3 &= i_1 - i_4 \\
&= u_1 / R_1 - u_4 / R_4 \\
&= (U_0 - u_q - u_4) / R_1 - (u_3 + u_c) / R_4 \\
&= (U_0 - R_q \cdot i_q - u_3 - u_c) / R_1 - (u_3 + u_c) / R_4 \\
&= (U_0 - R_q \cdot i_q - R_3 \cdot i_3 - u_c) / R_1 - (R_3 \cdot i_3 + u_c) / R_4
\end{aligned}
$$

$$(1 + R_3/R_1 + R_3/R_4) \cdot i_3 + (R_q/R_1) \cdot i_q = U_0/R_1 - u_c/R_1 - u_c/R_4$$

$$\boxed{(R_1 R_3 + R_1 R_4 + R_3 R_4) \cdot i_3 + R_4 R_q \cdot i_q = R_4 \cdot U_0 - (R_1 + R_4) \cdot u_c}$$

Expressions in the tearing variables

Expressions in known variables

$$U_0 = f(t)$$
$$u_q = R_q \cdot i_q$$
$$i_1 = u_1 / R_1$$
$$i_2 = u_2 / R_2$$
$$u_3 = R_3 \cdot i_3$$
$$i_4 = u_4 / R_4$$
$$du_c / dt = i_c / C$$
$$i_q = i_1 + i_2$$
$$i_3 = i_1 - i_4$$
$$i_c = i_2 + i_3$$
$$u_1 = U_0 - u_q - u_4$$
$$u_2 = u_1 + u_3$$
$$u_4 = u_3 + u_c$$

$$\begin{aligned}
i_q &= i_1 + i_2 \\
&= u_1 / R_1 + u_2 / R_2 \\
&= u_1 / R_1 + (u_1 + u_3) / R_2 \\
&= (R_1+R_2) /(R_1 R_2) \cdot u_1 + u_3 /R_2 \\
&= (R_1+R_2) /(R_1 R_2) \cdot (U_0 - R_q \cdot i_q - R_3 \cdot i_3 - u_c) + R_3/R_2 \cdot i_3
\end{aligned}$$

$$\boxed{R_2 R_3 \cdot i_3 + (R_1 R_2 + R_1 R_q + R_2 R_q) \cdot i_q = (R_1+R_2) \cdot (U_0 - u_c)}$$

$$\begin{bmatrix} (R_1R_3 + R_1R_4 + R_3R_4) & R_4R_q \\ R_2R_3 & (R_1R_2 + R_1R_q + R_2R_q) \end{bmatrix} \cdot \begin{bmatrix} i_3 \\ i_q \end{bmatrix} = \begin{bmatrix} R_4 \cdot U_0 - (R_1+R_4) \cdot u_c \\ (R_1+R_2) \cdot (U_0 - u_c) \end{bmatrix}$$

$\Rightarrow$

$$a_{11} = R_1 \cdot R_3 + R_1 \cdot R_4 + R_3 \cdot R_4$$
$$a_{12} = R_4 \cdot R_q$$
$$a_{21} = R_2 \cdot R_3$$
$$a_{22} = R_1 \cdot R_2 + R_1 \cdot R_q + R_2 \cdot R_q$$
$$b_1 = R_4 \cdot U_0 - (R_1+R_4) \cdot u_c$$
$$b_2 = (R_1+R_2) \cdot (U_0 - u_c)$$
$$d = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$
$$i_3 = (a_{22} \cdot b_1 - a_{12} \cdot b_2)/d$$
$$i_q = (a_{11} \cdot b_2 - a_{21} \cdot b_1)/d$$

Tearing Algorithm III

- Solve symbolically for the tearing variables, and find replacement equations for the two residual equations, which permit to make the entire set of equations causal.
- Find an explicit DAE system by completely sorting the equations both horizontally and vertically.

$U_0 = f(t)$	$u_3 = R_3 \cdot i_3$
$a_{11} = R_1 \cdot R_3 + R_1 \cdot R_4 + R_3 \cdot R_4$	$u_q = R_q \cdot i_q$
$a_{12} = R_4 \cdot R_q$	$u_4 = u_3 + u_c$
$a_{21} = R_2 \cdot R_3$	$u_1 = U_0 - u_q - u_4$
$a_{22} = R_1 \cdot R_2 + R_1 \cdot R_q + R_2 \cdot R_q$	$u_2 = u_1 + u_3$
$b_1 = R_4 \cdot U_0 - (R_1 + R_4) \cdot u_c$	$i_1 = u_1 / R_1$
$b_2 = (R_1 + R_2) \cdot (U_0 - u_c)$	$i_2 = u_2 / R_2$
$d = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$	$i_4 = u_4 / R_4$
$i_3 = (a_{22} \cdot b_1 - a_{12} \cdot b_2)/d$	$du_c/dt = i_c / C$
$i_q = (a_{11} \cdot b_2 - a_{21} \cdot b_1)/d$	$i_c = i_2 + i_3$

20 equations

9 additions

7 subtractions

19 multiplications

6 divisions

$\Rightarrow$ ***41 operations***

Relaxation Algorithm I

- Apply the relaxation algorithm to the same electrical circuit.
- For determining the sequence of equations and variables, make use of the following heuristics:
  - Make the equations causal in the same way as for ***Problem 4.1***.
  - Start with the first residual equation. It is being placed as the ***last equation***, whereby the corresponding tearing variable is the ***last variable***.
  - Number the equations, which can be made causal on the basis of the assumption that the tearing variable is already known, starting with ***equation #1***, and set the variables for which these equations are being solved also at the beginning of the list of variables, starting with ***variable #1***. In this way, the diagonal elements can be normalized to ***1***.

Relaxation Algorithm II

- Set the second residual equation as the second to last equation, whereby the corresponding tearing variable is also the second to last variable.
- Number the equations, which can be causalized based on the assumption that the second tearing variable is already known, in increasing order following the first set of equations (with the exception of the first residual equation, which comes at the very end).
- The resulting equation system in matrix form has diagonal elements that are all normalized to 1, and contains exactly two non-zero elements above the diagonal. These are located in the columns of the tearing variables and in the rows of the first equation of the causalized equation system.

- Consequently, the problem of minimizing the number of non-zero elements above the diagonal in the case of the relaxation algorithm is indeed identical with the search for suitable tearing variables.

$u_q = R_q \cdot i_q$	4
$i_1 = u_1 / R_1$	7
$i_2 = u_2 / R_2$	8
$u_3 = R_3 \cdot i_3$	1
$i_4 = u_4 / R_4$	3
$i_q = i_1 + i_2$	9
$i_3 = i_1 - i_4$	10
$u_1 = U_0 - u_q - u_4$	5
$u_2 = u_1 + u_3$	6
$u_4 = u_3 + u_c$	2

$\Rightarrow$

$$u_3 - R_3 \cdot i_3 = 0$$
$$u_4 - u_3 = u_c$$
$$i_4 - u_4 / R_4 = 0$$
$$u_q - R_q \cdot i_q = 0$$
$$u_1 + u_q + u_4 = U_0$$
$$u_2 - u_1 - u_3 = 0$$
$$i_1 - u_1 / R_1 = 0$$
$$i_2 - u_2 / R_2 = 0$$
$$i_q - i_1 - i_2 = 0$$
$$i_3 - i_1 + i_4 = 0$$

$$u_3 - R_3 \cdot i_3 = 0$$
$$u_4 - u_3 = u_c$$
$$i_4 - u_4 / R_4 = 0$$
$$u_q - R_q \cdot i_q = 0$$
$$u_1 + u_q + u_4 = U_0$$
$$u_2 - u_1 - u_3 = 0$$
$$i_1 - u_1 / R_1 = 0$$
$$i_2 - u_2 / R_2 = 0$$
$$i_q - i_1 - i_2 = 0$$
$$i_3 - i_1 + i_4 = 0$$

$\Rightarrow$

u3	u4	i4	uq	u1	u2	i1	i2	iq	i3	
1	0	0	0	0	0	0	0	0	-R3	0
-1	1	0	0	0	0	0	0	0	0	uC
0	-1/R4	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	-Rq	0	0
0	1	0	1	1	0	0	0	0	0	U0
-1	0	0	0	-1	1	0	0	0	0	0
0	0	0	0	-1/R1	0	1	0	0	0	0
0	0	0	0	0	-1/R2	0	1	0	0	0
0	0	0	0	0	0	-1	-1	1	0	0
0	0	1	0	0	0	-1	0	0	1	0

u3	u4	i4	uq	u1	u2	i1	i2	iq	i3	
1	0	0	0	0	0	0	0	0	-R3	0
-1	1	0	0	0	0	0	0	0	0	uC
0	-1/R4	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	-Rq	0	0
0	1	0	1	1	0	0	0	0	0	U0
-1	0	0	0	-1	1	0	0	0	0	0
0	0	0	0	-1/R1	0	1	0	0	0	0
0	0	0	0	0	-1/R2	0	1	0	0	0
0	0	0	0	0	0	-1	-1	1	0	0
0	0	1	0	0	0	-1	0	0	1	0

$$u_3 = R_3 \cdot i_3$$

u4	i4	uq	u1	u2	i1	i2	iq	i3	
1	0	0	0	0	0	0	0	c1	uC
-1/R4	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	-Rq	0	0
1	0	1	1	0	0	0	0	0	U0
0	0	0	-1	1	0	0	0	c1	0
0	0	0	-1/R1	0	1	0	0	0	0
0	0	0	0	-1/R2	0	1	0	0	0
0	0	0	0	0	-1	-1	1	0	0
0	1	0	0	0	-1	0	0	1	0

$$c_1 = -R_3$$

$$u_4 = u_C - c_1 \cdot i_3$$

i4	uq	u1	u2	i1	i2	iq	i3	
1	0	0	0	0	0	0	c2	c3
0	1	0	0	0	0	-Rq	0	0
0	1	1	0	0	0	0	c4	c5
0	0	-1	1	0	0	0	c1	0
0	0	-1/R1	0	1	0	0	0	0
0	0	0	-1/R2	0	1	0	0	0
0	0	0	0	-1	-1	1	0	0
1	0	0	0	-1	0	0	1	0

$$c_2 = c_1 / R_4$$
$$c_3 = u_C / R_4$$
$$c_4 = -c_1$$
$$c_5 = U_0 - u_C$$

$$i_4 = c_3 - c_2 \cdot i_3$$

uq	u1	u2	i1	i2	iq	i3	
1	0	0	0	0	-Rq	0	0
1	1	0	0	0	0	c4	c5
0	-1	1	0	0	0	c1	0
0	-1/R1	0	1	0	0	0	0
0	0	-1/R2	0	1	0	0	0
0	0	0	-1	-1	1	0	0
0	0	0	-1	0	0	c6	c7

$$c_6 = 1 - c_2$$
$$c_7 = -c_3$$

$$u_q = R_q \cdot i_q$$

u1	u2	i1	i2	iq	i3	
1	0	0	0	c8	c4	c5
-1	1	0	0	0	c1	0
-1/R1	0	1	0	0	0	0
0	-1/R2	0	1	0	0	0
0	0	-1	-1	1	0	0
0	0	-1	0	0	c6	c7

$$c_8 = R_q$$

$$u_1 = c_5 - c_8 \cdot i_q - c_4 \cdot i_3$$

u2	i1	i2	iq	i3	
1	0	0	c9	c10	c11
0	1	0	c12	c13	c14
-1/R2	0	1	0	0	0
0	-1	-1	1	0	0
0	-1	0	0	c6	c7

$$c_9 = c_8$$
$$c_{10} = c_1 + c_4$$
$$c_{11} = c_5$$
$$c_{12} = c_8 / R_1$$
$$c_{13} = c_4 / R_1$$
$$c_{14} = c_5 / R_1$$

$$u_2 = c_{11} - c_9 \cdot i_q - c_{10} \cdot i_3$$

i1	i2	iq	i3	
1	0	c12	c13	c14
0	1	c15	c16	c17
-1	-1	1	0	0
-1	0	0	c6	c7

$$c_{15} = c_9 / R_2$$
$$c_{16} = c_{10} / R_2$$
$$c_{17} = c_{11} / R_2$$

$$i_1 = c_{14} - c_{12} \cdot i_q - c_{13} \cdot i_3$$

i2	iq	i3	
1	c15	c16	c17
-1	c18	c19	c20
0	c21	c22	c23

$$c_{18} = 1 + c_{12}$$
$$c_{19} = c_{13}$$
$$c_{20} = c_{14}$$
$$c_{21} = c_{12}$$
$$c_{22} = c_6 + c_{13}$$
$$c_{23} = c_7 + c_{14}$$

$$i_2 = c_{17} - c_{15} \cdot i_q - c_{16} \cdot i_3$$

iq	i3	
c24	c25	c26
c21	c22	c23

$$c_{24} = c_{18} + c_{15}$$
$$c_{25} = c_{19} + c_{16}$$
$$c_{26} = c_{20} + c_{17}$$

$$i_q = (c_{26} - c_{25} \cdot i_3) / c_{24}$$

i3	
c27	c28

$$c_{27} = c_{22} - c_{21} \cdot c_{25} / c_{24}$$
$$c_{28} = c_{23} - c_{21} \cdot c_{26} / c_{24}$$

$$i_3 = c_{28} / c_{27}$$

Tearing *vs.* Relaxation

- Apply the relaxation algorithm to the resulting set of equations, and determine an explicit DAE system in this way.
- Count the number of additions, subtractions, multiplications, and divisions of the two sets of equations that result from the two algorithms, and determine, which of the two algorithms is more economical in the case of the given example.

$U_0 = f(t)$	$c_{14} = c_5 / R_1$	$c_{28} = c_{23} - c_{21} \cdot c_{26} / c_{24}$
$c_1 = -R_3$	$c_{15} = c_9 / R_2$	$i_3 = c_{28} / c_{27}$
$c_2 = c_1 / R_4$	$c_{16} = c_{10} / R_2$	$i_q = (c_{26} - c_{25} \cdot i_3) / c_{24}$
$c_3 = u_C / R_4$	$c_{17} = c_{11} / R_2$	$i_2 = c_{17} - c_{15} \cdot i_q - c_{16} \cdot i_3$
$c_4 = -c_1$	$c_{18} = 1 + c_{12}$	$i_1 = c_{14} - c_{12} \cdot i_q - c_{13} \cdot i_3$
$c_5 = U_0 - u_C$	$c_{19} = c_{13}$	$u_2 = c_{11} - c_9 \cdot i_q - c_{10} \cdot i_3$
$c_6 = 1 - c_2$	$c_{20} = c_{14}$	$u_1 = c_5 - c_8 \cdot i_q - c_4 \cdot i_3$
$c_7 = -c_3$	$c_{21} = c_{12}$	$u_q = R_q \cdot i_q$
$c_8 = R_q$	$c_{22} = c_6 + c_{13}$	$i_4 = c_3 - c_2 \cdot i_3$
$c_9 = c_8$	$c_{23} = c_7 + c_{14}$	$u_4 = u_C - c_1 \cdot i_3$
$c_{10} = c_1 + c_4$	$c_{24} = c_{18} + c_{15}$	$u_3 = R_3 \cdot i_3$
$c_{11} = c_5$	$c_{25} = c_{19} + c_{16}$	$du_c / dt = i_c / C$
$c_{12} = c_8 / R_1$	$c_{26} = c_{20} + c_{17}$	$i_c = i_2 + i_3$
$c_{13} = c_4 / R_1$	$c_{27} = c_{22} - c_{21} \cdot c_{25} / c_{24}$	

41 equations

8 additions

18 subtractions

15 multiplications

13 divisions

$\Rightarrow$ ***54 operations***

It is also possible to mix the relaxation algorithm and the tearing algorithm.

$U_0 = f(t)$	$c_{14} = c_5 / R_1$	$c_{28} = c_{23} - c_{21} \cdot c_{26} / c_{24}$
$c_1 = -R_3$	$c_{15} = c_9 / R_2$	$i_3 = c_{28} / c_{27}$
$c_2 = c_1 / R_4$	$c_{16} = c_{10} / R_2$	$i_q = (c_{26} - c_{25} \cdot i_3) / c_{24}$
$c_3 = u_C / R_4$	$c_{17} = c_{11} / R_2$	$u_3 = R_3 \cdot i_3$
$c_4 = -c_1$	$c_{18} = 1 + c_{12}$	$u_q = R_q \cdot i_q$
$c_5 = U_0 - u_C$	$c_{19} = c_{13}$	$u_4 = u_3 + u_c$
$c_6 = 1 - c_2$	$c_{20} = c_{14}$	$u_1 = U_0 - u_q - u_4$
$c_7 = -c_3$	$c_{21} = c_{12}$	$u_2 = u_1 + u_3$
$c_8 = R_q$	$c_{22} = c_6 + c_{13}$	$i_1 = u_1 / R_1$
$c_9 = c_8$	$c_{23} = c_7 + c_{14}$	$i_2 = u_2 / R_2$
$c_{10} = c_1 + c_4$	$c_{24} = c_{18} + c_{15}$	$i_4 = u_4 / R_4$
$c_{11} = c_5$	$c_{25} = c_{19} + c_{16}$	$du_c / dt = i_c / C$
$c_{12} = c_8 / R_1$	$c_{26} = c_{20} + c_{17}$	$i_c = i_2 + i_3$
$c_{13} = c_4 / R_1$	$c_{27} = c_{22} - c_{21} \cdot c_{25} / c_{24}$	

41 equations

10 additions

10 subtractions

5 multiplications

16 divisions

$\Rightarrow$ ***41 operations***