Treatment of Discontinuities

- Today, we shall look at the problem of dealing with discontinuities in models.
- Models from engineering often exhibit discontinuities that describe situations such as switching, limiters, dry friction, impulses, or similar phenomena.
- The modeling environment must deal with these problems in special ways, since they influence strongly the numerical behavior of the underlying differential equation solver.

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Numerical Differential Equation Solvers

- Most of the ***differential equation solvers*** that are currently on the market operate on ***polynomial extrapolation***.
- The value of a state variable ***x*** at time ***t+h***, where ***h*** is the current ***integration step size***, is approximated by fitting a ***polynomial of n^{th} order*** through known supporting values of ***x*** and ***dx/dt*** at the current time ***t*** as well as at past instances of time.
- The value of the extrapolation polynomial at time ***t+h*** represents the approximated solution of the differential equation.
- In the case of ***implicit integration algorithms***, the state derivative at time ***t+h*** is also used as a supporting value.

Examples

Explicit Euler Integration Algorithm of 1st Order:

$$x(t+h) \approx x(t) + h \cdot \dot{x}(t)$$

Implicit Euler Integration Algorithm of 1st Order:

$$x(t+h) \approx x(t) + h \cdot \dot{x}(t+h)$$

Discontinuities in State Equations

- ***Polynomials*** are always ***continuous and continuously differentiable functions***.
- Therefore, when the ***state equations*** of the system:

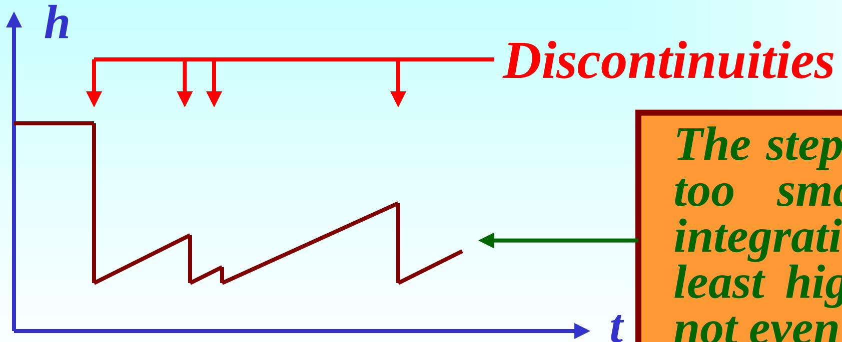
$$\dot{x}(t) = f(x(t),t)$$

exhibit a discontinuity, the polynomial extrapolation is a very poor approximation of reality.

- Consequently, ***integration algorithms*** with a fixed step size exhibit a large ***integration error***, whereas integration algorithms with a variable step size reduce the step size dramatically in the vicinity of a discontinuity.

Integration Across Discontinuities

- An integration algorithm of variable step size reduces the step size at every discontinuity.
- After passing the discontinuity, the step size is only slowly enlarged again, as the integration algorithm cannot distinguish between a ***discontinuity*** on one hand and a ***point of large local stiffness*** (with a large absolute value of the derivative) on the other.

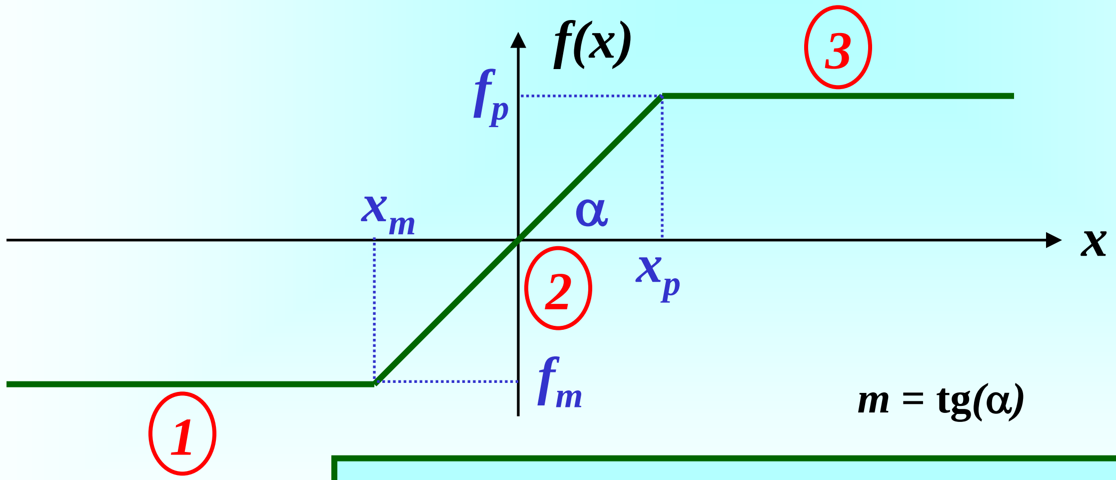


The step size is constantly too small. Thus, the integration algorithm is at least highly inefficient, if not even inaccurate.

The State Event

- These problems can be avoided by telling the integration algorithm explicitly, when and where discontinuities are contained in the model description.

Example: Limiter Function



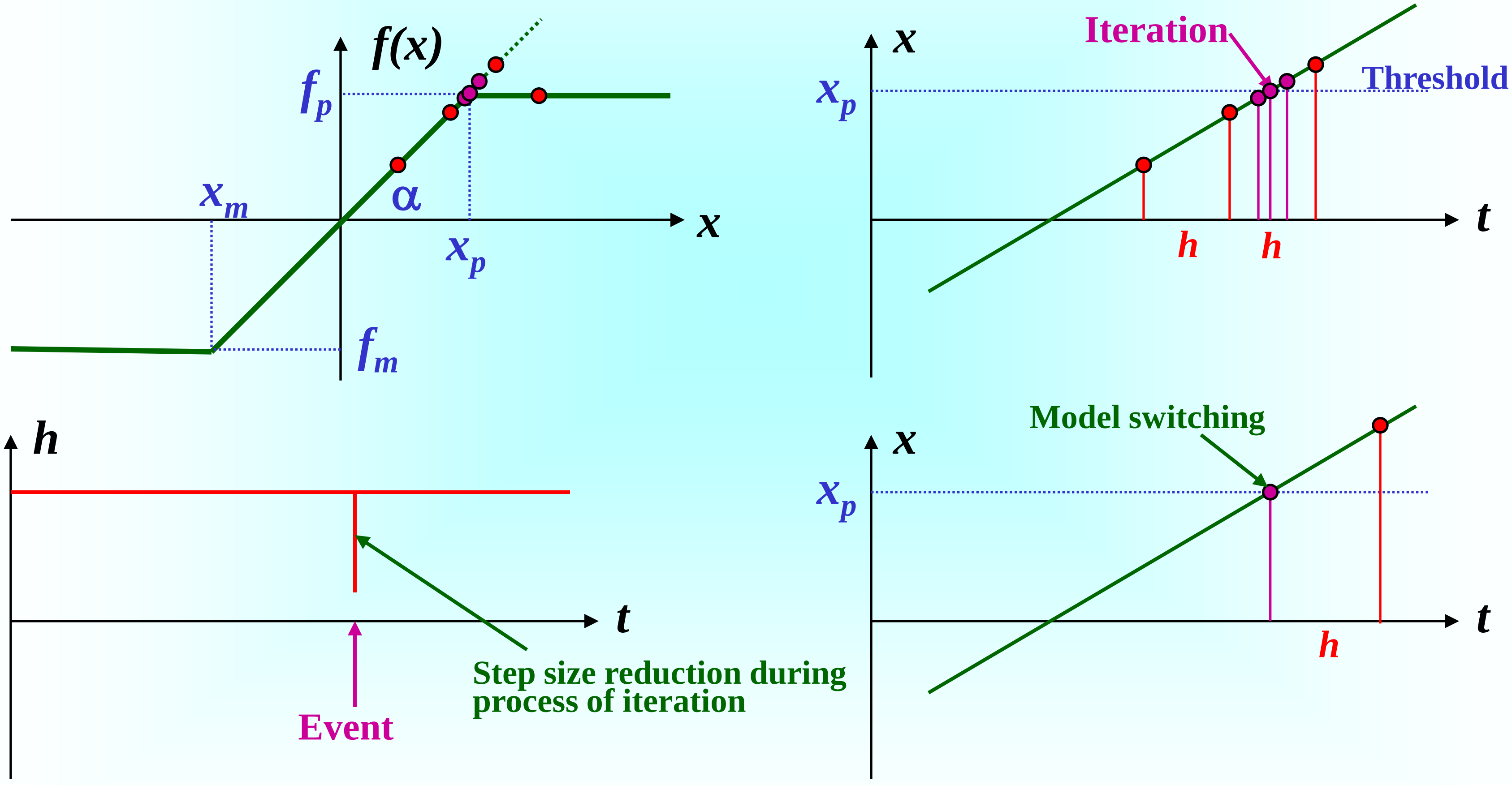
$f = f_m$

$f = m \cdot x$

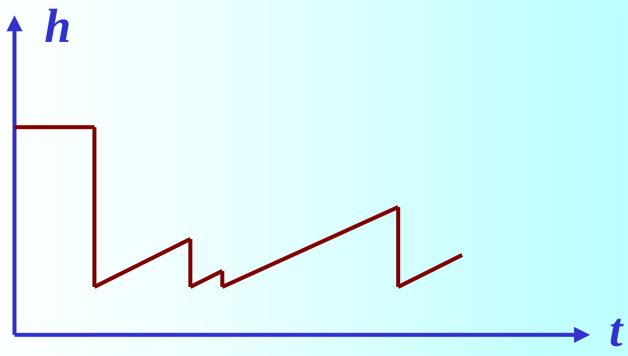
$f = f_p$

f = if x < xm then fm else if x < xp then m*x else fp ;

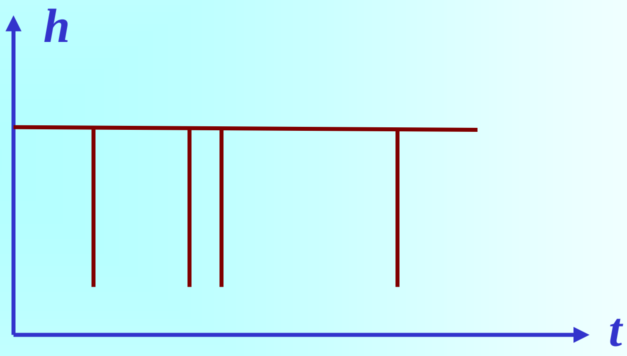
Event Handling I



Event Handling II



Step size as function of time without event handling



Step size as function of time with event handling

Representation of Discontinuities

f = **if** *x* < *xm* **then** *fm* **else if** *x* < *xp* **then** *m**x* **else** *fp* ;

- In ***Modelica***, discontinuities are represented as ***if-statements***.
- In the process of translation, these statements are transformed into correct ***event descriptions*** (sets of ***models with switching conditions***).
- The modeler does not need to concern him- or herself with the mechanisms of event descriptions. These are hidden behind the ***if-statements***.

Problems

- The modeler needs to take into account that the discontinuous solution is temporarily left during iteration.

$$q = \sqrt{|\Delta p|}$$

```
Δp = p1 – p2 ;
absΔp = if Δp > 0 then Δp else –Δp ;
q = sqrt(absΔp) ;
```

may be dangerous, since ***absΔp*** can become temporarily negative.

⇒

```
Δp = p1 – p2 ;
absΔp = noEvent( if Δp > 0 then Δp else –Δp ) ;
q = sqrt(absΔp) ;
```

solves this problem.

The "noEvent" Construct

$\Delta p = p_1 - p_2$;

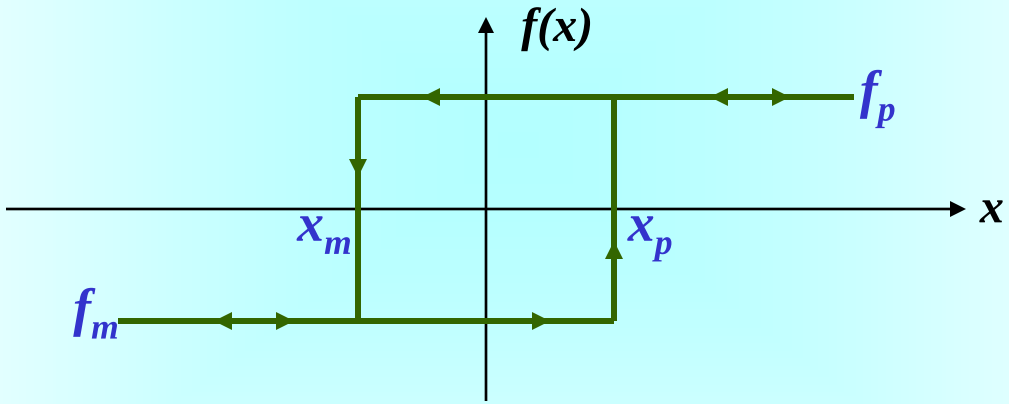
absΔp = **noEvent(if** *Δp* > *0* **then** *Δp* **else** –*Δp* **)** ;

q = sqrt(*absΔp*) ;

- The ***noEvent construct*** has the effect that ***if-statements*** or ***Boolean expressions***, which normally would be translated into simulation code containing correct event handling instructions, are handed over to the integration algorithm untouched.
- Thereby, management of the simulation across these discontinuities is left to the step size control of the numerical Integration algorithm.

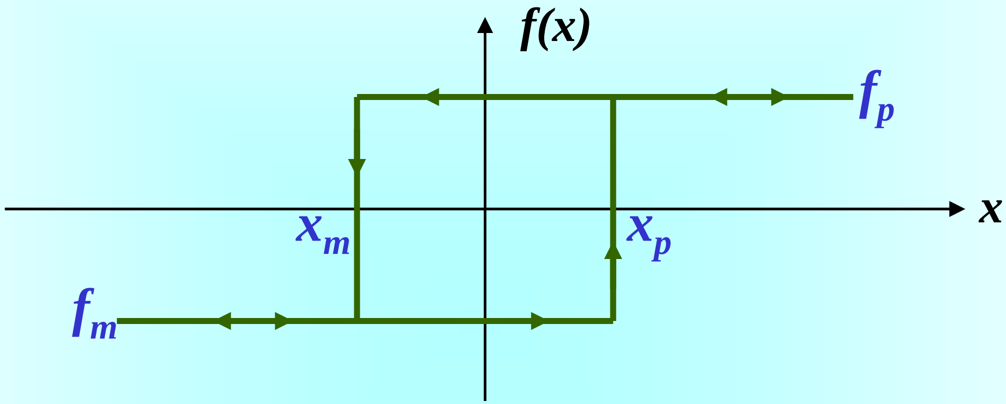
Multi-valued Functions I

- The language constructs that have been introduced so far don't suffice to describe ***multi-valued functions***, such as the ***dry hysteresis function*** shown below.



- When x ***becomes greater*** than x_p, f must be switched from f_m to f_p.
- When x ***becomes smaller*** than x_m, f must be switched from f_p to f_m.

Multi-valued Functions II




```
when initial() then
  reinit(f , fp);
end when;
when  x > xp  or  x < xm  then
  f  =  if  x > 0  then  fp  else  fm;
end when;
```

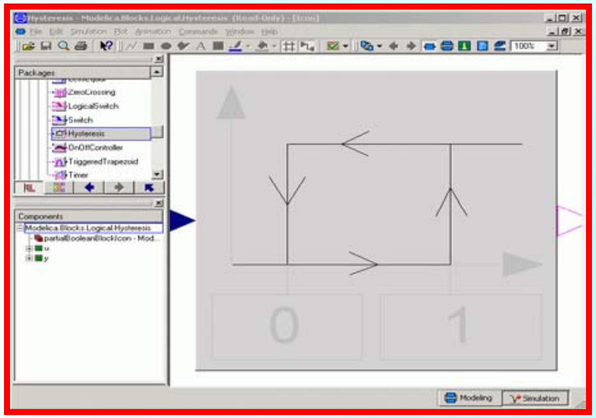
becomes larger (annotation on $x > xp$)

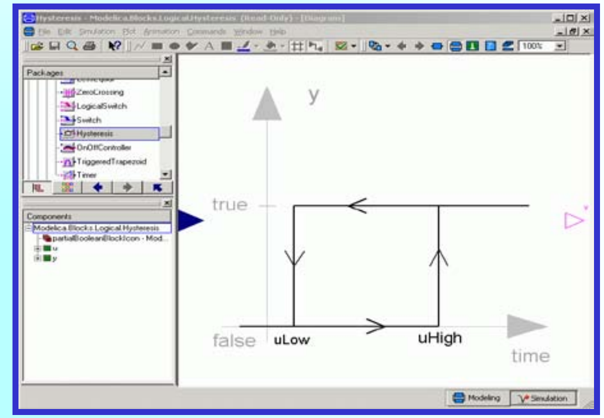
is larger (annotation on $x > 0$)

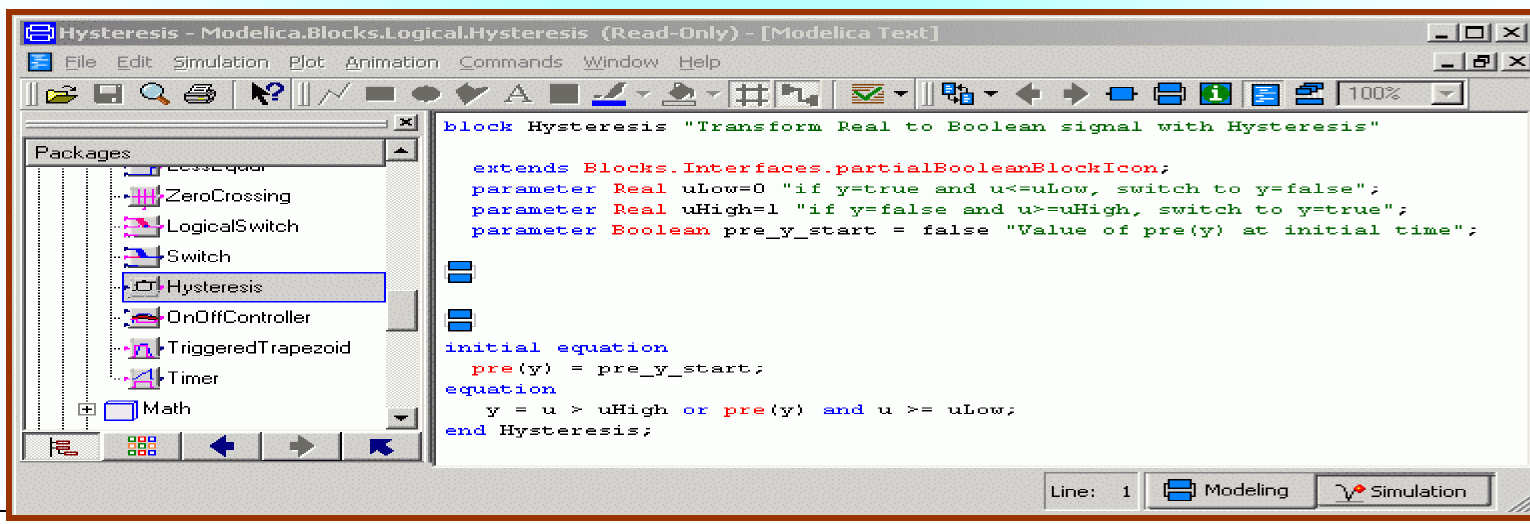
← Executed *at the beginning* of the simulation.

} These statements are only executed, when either x ***becomes larger than*** x_p, or when x ***becomes smaller than*** x_m.

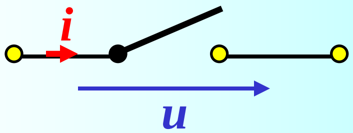
Multi-valued Functions III







The Electrical Switch I



When the switch is ***open***, the current is ***i=0***.
When the switch is ***closed***, the voltage is ***u=0***.

***0* = if *open* then *i* else *u* ;**

The ***if-statement*** in ***Modelica*** is ***a-causal***. It is being sorted together with all other statements.

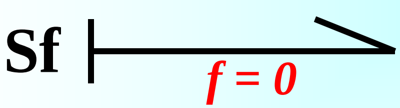
The Electrical Switch II

Possible Implementation:

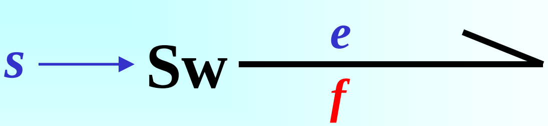
Switch open: $s = 1$
Switch closed: $s = 0$

$$\Rightarrow \quad 0 = s \cdot i + (1 - s) \cdot u$$

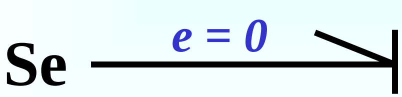
Switch open:



$\Rightarrow$

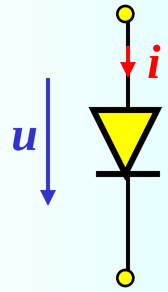


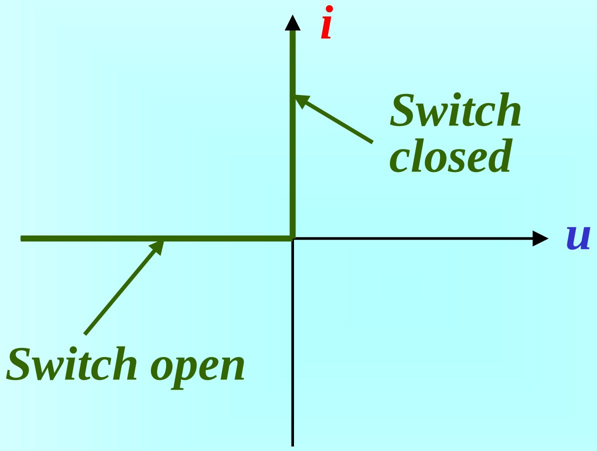
Switch closed:



The causality of the switch element is a function of the value of the control signal s.

The Ideal Diode I

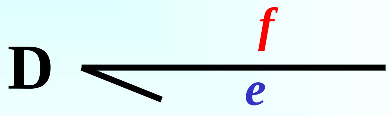




When $u < 0$, the switch is open. No current flows through.

When $u > 0$, the switch is closed. Current may flow. The ideal diode behaves like a short circuit.

open = *u* < *0* ;
0 = **if** *open* **then** *i* **else** *u* ;



The Ideal Diode II

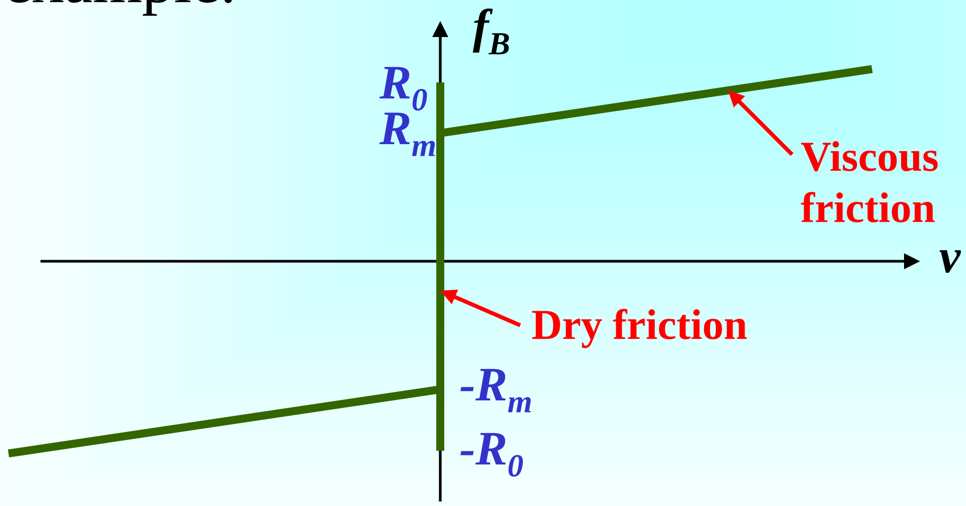
- Since current flowing through a diode cannot simply be interrupted, it is necessary to slightly modify the diode model.

```
open  =  u <= 0 and not i > 0 ;
0  =  if open then i else u ;
```

- The variable ***open*** must be declared as ***Boolean***. The value to the right of the Boolean expression is assigned to it.

The Friction Characteristic I

- More complex phenomena, such as friction characteristics, must be carefully analyzed case by case.
- The approach is discussed here by means of the friction example.



When $v \neq 0$, the friction force is a function of the velocity.

When $v = 0$, the friction force is computed such that the velocity remains 0.

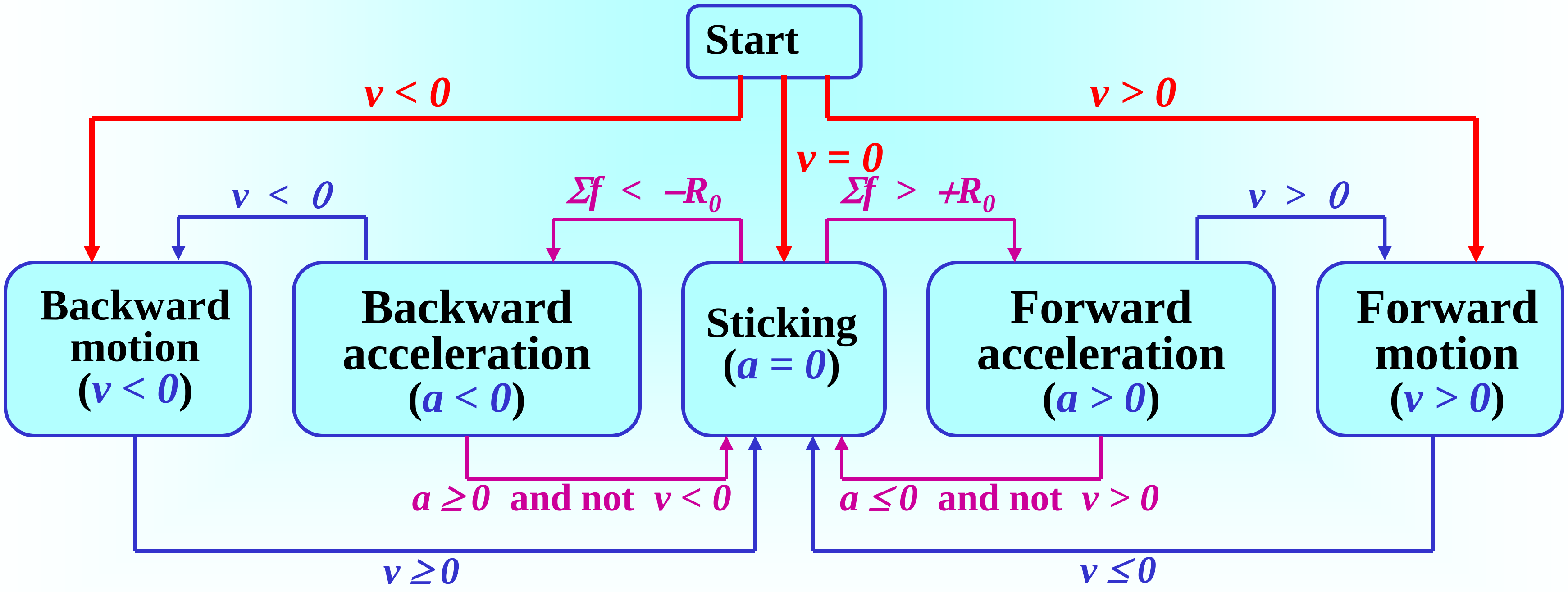
The Friction Characteristic II

- We distinguish between five situations:

$v = 0$ $a = 0$	***Sticking:*** The friction force compensates the sum of all forces attached, except if $\lvert \Sigma f \rvert > R_0$.
$v > 0$	***Moving forward:*** The friction force is computed as: $f_B = R_v \cdot v + R_m$.
$v < 0$	***Moving backward:*** The friction force is computed as: $f_B = R_v \cdot v - R_m$.
$v = 0$ $a > 0$	***Beginning of forward motion:*** The friction force is computed as: $f_B = R_m$.
$v = 0$ $a < 0$	***Beginning of backward motion:*** The friction force is computed as: $f_B = -R_m$.

The State Transition Diagram

- The set of events can be described by a ***state transition diagram***.



The Friction Model I

```
model Friction;
   parameter Real R0, Rm, Rv;
   parameter Boolean ic=false;
   Real fB, fc;
   Boolean Sticking (final start = ic);
   Boolean Forward (final start = ic), Backward (final start = ic);
   Boolean StartFor (final start = ic), StartBack (final start = ic);

   fB = if Forward   then Rv*v + Rm else
          if Backward then Rv*v - Rm  else
          if StartFor  then Rm         else
          if StartBack then -Rm        else fc;
   0 = if Sticking or initial() then a else fc;
```

The Friction Model II

```
when Sticking and not initial() then
   reinit(v,0);
end when;

Forward =  initial()          and v > 0 or
           pre(StartFor)      and v > 0 or
           pre(Forward)       and not v <= 0;
Backward = initial()          and v < 0 or
           pre(StartBack)     and v < 0 or
           pre(Backward)      and not v >= 0;
```

The Friction Model III

```
   StartFor   = pre(Sticking)    and fc > R0 or
                pre(StartFor)    and not (v > 0 or a <= 0 and not v > 0);
   StartBack = pre(Sticking)     and fc < -R0 or
                pre(StartBack)  and not (v < 0 or a >= 0 and not v < 0);
   Sticking = not (Forward or Backward or StartFor or StartBack);

end Friction;
```

References I

- Cellier, F.E. (1979), *Combined Continuous/Discrete System Simulation by Use of Digital Computers: Techniques and Tools*, PhD Dissertation, Swiss Federal Institute of Technology, ETH Zürich, Switzerland.

- Elmqvist, H., F.E. Cellier, and M. Otter (1993), "Object-oriented modeling of hybrid systems," *Proc. ESS'93, SCS European Simulation Symposium*, Delft, The Netherlands, pp.xxxi-xli.

- Cellier, F.E., M. Otter, and H. Elmqvist (1995), "Bond graph modeling of variable structure systems," *Proc. ICBGM'95, 2nd SCS Intl. Conf. on Bond Graph Modeling and Simulation*, Las Vegas, NV, pp. 49-55.

References II

- Elmqvist, H., F.E. Cellier, and M. Otter (1994), "Object-oriented modeling of power-electronic circuits using Dymola," *Proc. CISS'94, First Joint Conference of International Simulation Societies*, Zurich, Switzerland, pp. 156-161.
- Glaser, J.S., F.E. Cellier, and A.F. Witulski (1995), "Object-oriented switching power converter modeling using Dymola with event-handling," *Proc. OOS'95, SCS Object-Oriented Simulation Conference*, Las Vegas, NV, pp. 141-146.