

1D Mechanical Systems

- In this lecture, we shall deal with 1D mechanical systems that can either translate or rotate in a one-dimensional space.
- We shall demonstrate the similarities between the mathematical descriptions of these systems and the electrical circuits discussed in the previous lecture.
- In particular, it will be shown that the symbolic formulae manipulation algorithms (sorting algorithms) that were introduced in the previous lecture can be applied to these systems just as easily and without any modification.

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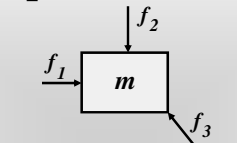
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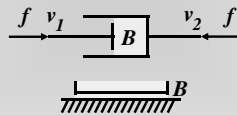
Linear Components of Translation

- Mass



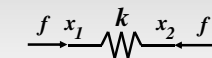
$$\begin{aligned} m \cdot a &= \sum_i (f_i) \\ \frac{dv}{dt} &= a \\ \frac{dx}{dt} &= v \end{aligned}$$

- Friction



$$f = \mathbf{B} \cdot (\mathbf{v}_1 - \mathbf{v}_2)$$

- Spring



$$f = k \cdot (x_1 - x_2)$$

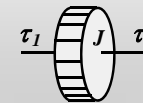
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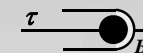
Linear Components of Rotation

- Inertia



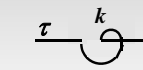
$$\begin{aligned} J \cdot \alpha &= \sum_i (\tau_i) \\ \frac{d\omega}{dt} &= \alpha \\ \frac{d\theta}{dt} &= \omega \end{aligned}$$

- Friction



$$\tau = B \cdot (\omega_1 - \omega_2)$$

- Spring



$$\tau = k \cdot (\theta_1 - \theta_2)$$

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Mathematical Modeling of Physical Systems

Joints without Degrees of Freedom

- Node (Translation)

$x_a = x_b = x_c$
 $v_a = v_b = v_c$
 $a_a = a_b = a_c$
 $f_a + f_b + f_c = 0$
- Node (Rotation)

$\theta_a = \theta_b = \theta_c$
 $\omega_a = \omega_b = \omega_c$
 $\alpha_a = \alpha_b = \alpha_c$
 $\tau_a + \tau_b + \tau_c = 0$

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Mathematical Modeling of Physical Systems

Joints with One Degree of Freedom

- Prismatic

$x_1 \neq x_2$
 $y_1 = y_2$
 $\theta_1 = \theta_2$
- Cylindrical

$x_1 = x_2$
 $y_1 = y_2$
 $\theta_1 \neq \theta_2$
- Scissors

$\theta_1 \neq \theta_2$

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Mathematical Modeling of Physical Systems

The D'Alembert Principle

- By introduction of an *inertial force*:

$$f_m = -m \cdot a$$
 the second law of Newton:

$$m \cdot a = \sum_i (f_i)$$
 can be converted to a law of the form:

$$\sum_i (f_i) = 0$$

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Mathematical Modeling of Physical Systems

Sign Conventions

$$f_m = + \frac{d(m \cdot v)}{dt}$$

$$f_k = + k \cdot (x - x_{Neighbor})$$

$$f_B = + B \cdot (v - v_{Neighbor})$$

$$f_m = - \frac{d(m \cdot v)}{dt}$$

$$f_k = - k \cdot (x - x_{Neighbor})$$

$$f_B = - B \cdot (v - v_{Neighbor})$$

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Mathematical Modeling of Physical Systems

1. Example (Translation)

Network View

Topological View

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Mathematical Modeling of Physical Systems

1. Example (Translation) II

The system is being cut open between the individual masses, and cutting forces are introduced.

The D'Alembert principle can now be applied to each body separately.

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Mathematical Modeling of Physical Systems

1. Example (continued)

$$F(t) = F_{13} + F_{Ba} + F_{Bb}$$

$$F_{Ba} = F_{12} + F_{Bc} + F_{B2} + F_{k2}$$

$$F_{Bb} + F_{B2} = F_{11} + F_{Bd} + F_{k1}$$

$$F_{11} = m_1 \cdot \frac{dv_1}{dt}$$

$$\frac{dx_1}{dt} = v_1$$

$$F_{12} = m_2 \cdot \frac{dv_2}{dt}$$

$$\frac{dx_2}{dt} = v_2$$

$$F_{13} = m_3 \cdot \frac{dv_3}{dt}$$

$$\frac{dx_3}{dt} = v_3$$

$$F_{Ba} = B_1 \cdot (v_3 - v_2)$$

$$F_{Bb} = B_1 \cdot (v_3 - v_1)$$

$$F_{Bc} = B_1 \cdot v_2$$

$$F_{Bd} = B_1 \cdot v_1$$

$$F_{B2} = B_2 \cdot (v_2 - v_1)$$

$$F_{k1} = k_1 \cdot x_1$$

$$F_{k2} = k_2 \cdot x_2$$

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Mathematical Modeling of Physical Systems

Horizontal Sorting I

$$F(t) = F_{13} + F_{Ba} + F_{Bb}$$

$$F_{Ba} = F_{12} + F_{Bc} + F_{B2} + F_{k2}$$

$$F_{Bb} + F_{B2} = F_{11} + F_{Bd} + F_{k1}$$

$$F_{11} = m_1 \cdot \frac{dv_1}{dt}$$

$$\frac{dx_1}{dt} = v_1$$

$$F_{12} = m_2 \cdot \frac{dv_2}{dt}$$

$$\frac{dx_2}{dt} = v_2$$

$$F_{13} = m_3 \cdot \frac{dv_3}{dt}$$

$$\frac{dx_3}{dt} = v_3$$

$$F_{Ba} = B_1 \cdot (v_3 - v_2)$$

$$F_{Bb} = B_1 \cdot (v_3 - v_1)$$

$$F_{Bc} = B_1 \cdot v_2$$

$$F_{Bd} = B_1 \cdot v_1$$

$$F_{B2} = B_2 \cdot (v_2 - v_1)$$


$$F_{k1} = k_1 \cdot x_1$$

$$F_{k2} = k_2 \cdot x_2$$

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Mathematical Modeling of Physical Systems

Horizontal Sorting II

$$F(t) = F_{I3} + F_{Ba} + F_{Bb}$$

$$F_{Ba} = F_{I2} + F_{Bc} + F_{B2} + F_{k2}$$

$$F_{Bb} + F_{B2} = F_{I1} + F_{Bd} + F_{k1}$$

$$F_{I1} = m_1 \cdot \frac{dv_1}{dt}$$

$$\frac{dx_1}{dt} = v_1$$

$$F_{I2} = m_2 \cdot \frac{dv_2}{dt}$$

$$\frac{dx_2}{dt} = v_2$$

$$F_{I3} = m_3 \cdot \frac{dv_3}{dt}$$

$$\frac{dx_3}{dt} = v_3$$

$$F_{Ba} = B_1 \cdot (v_3 - v_2)$$

$$F_{Bb} = B_1 \cdot (v_3 - v_1)$$

$$F_{Bc} = B_1 \cdot v_2$$

$$F_{Bd} = B_1 \cdot v_1$$

$$F_{B2} = B_2 \cdot (v_2 - v_1)$$

$$F_{k1} = k_1 \cdot x_1$$

$$F_{k2} = k_2 \cdot x_2$$

⇒

$$F(t) = F_{I3} + F_{Ba} + F_{Bb}$$

$$F_{Ba} = F_{I2} + F_{Bc} + F_{B2} + F_{k2}$$

$$F_{Bb} + F_{B2} = F_{I1} + F_{Bd} + F_{k1}$$

$$F_{I1} = m_1 \cdot \frac{dv_1}{dt}$$

$$\frac{dx_1}{dt} = v_1$$

$$F_{I2} = m_2 \cdot \frac{dv_2}{dt}$$

$$\frac{dx_2}{dt} = v_2$$

$$F_{I3} = m_3 \cdot \frac{dv_3}{dt}$$

$$\frac{dx_3}{dt} = v_3$$

$$F_{Ba} = B_1 \cdot (v_3 - v_2)$$

$$F_{Bb} = B_1 \cdot (v_3 - v_1)$$


$$F_{Bc} = B_1 \cdot v_2$$


$$F_{Bd} = B_1 \cdot v_1$$

$$F_{B2} = B_2 \cdot (v_2 - v_1)$$

$$F_{k1} = k_1 \cdot x_1$$

$$F_{k2} = k_2 \cdot x_2$$

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Mathematical Modeling of Physical Systems

Horizontal Sorting III

$$F(t) = F_{I3} + F_{Ba} + F_{Bb}$$

$$F_{Ba} = F_{I2} + F_{Bc} + F_{B2} + F_{k2}$$

$$F_{Bb} + F_{B2} = F_{I1} + F_{Bd} + F_{k1}$$

$$F_{I1} = m_1 \cdot \frac{dv_1}{dt}$$

$$\frac{dx_1}{dt} = v_1$$

$$F_{I2} = m_2 \cdot \frac{dv_2}{dt}$$

$$\frac{dx_2}{dt} = v_2$$

$$F_{I3} = m_3 \cdot \frac{dv_3}{dt}$$

$$\frac{dx_3}{dt} = v_3$$

$$F_{Ba} = B_1 \cdot (v_3 - v_2)$$

$$F_{Bb} = B_1 \cdot (v_3 - v_1)$$

$$F_{Bc} = B_1 \cdot v_2$$

$$F_{Bd} = B_1 \cdot v_1$$

$$F_{B2} = B_2 \cdot (v_2 - v_1)$$

$$F_{k1} = k_1 \cdot x_1$$

$$F_{k2} = k_2 \cdot x_2$$

⇒

$$F(t) = F_{I3} + F_{Ba} + F_{Bb}$$

$$F_{Ba} = F_{I2} + F_{Bc} + F_{B2} + F_{k2}$$

$$F_{Bb} + F_{B2} = F_{I1} + F_{Bd} + F_{k1}$$

$$F_{I1} = m_1 \cdot \frac{dv_1}{dt}$$

$$\frac{dx_1}{dt} = v_1$$

$$F_{I2} = m_2 \cdot \frac{dv_2}{dt}$$

$$\frac{dx_2}{dt} = v_2$$

$$F_{I3} = m_3 \cdot \frac{dv_3}{dt}$$

$$\frac{dx_3}{dt} = v_3$$

$$F_{Ba} = B_1 \cdot (v_3 - v_2)$$

$$F_{Bb} = B_1 \cdot (v_3 - v_1)$$


$$F_{Bc} = B_1 \cdot v_2$$


$$F_{Bd} = B_1 \cdot v_1$$

$$F_{B2} = B_2 \cdot (v_2 - v_1)$$

$$F_{k1} = k_1 \cdot x_1$$

$$F_{k2} = k_2 \cdot x_2$$

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Mathematical Modeling of Physical Systems

Horizontal Sorting IV

$$F(t) = F_{I3} + F_{Ba} + F_{Bb}$$

$$F_{Ba} = F_{I2} + F_{Bc} + F_{B2} + F_{k2}$$

$$F_{Bb} + F_{B2} = F_{I1} + F_{Bd} + F_{k1}$$

$$F_{I1} = m_1 \cdot \frac{dv_1}{dt}$$

$$\frac{dx_1}{dt} = v_1$$

$$F_{I2} = m_2 \cdot \frac{dv_2}{dt}$$

$$\frac{dx_2}{dt} = v_2$$

$$F_{I3} = m_3 \cdot \frac{dv_3}{dt}$$

$$\frac{dx_3}{dt} = v_3$$

$$F_{Ba} = B_1 \cdot (v_3 - v_2)$$

$$F_{Bb} = B_1 \cdot (v_3 - v_1)$$

$$F_{Bc} = B_1 \cdot v_2$$

$$F_{Bd} = B_1 \cdot v_1$$

$$F_{B2} = B_2 \cdot (v_2 - v_1)$$

$$F_{k1} = k_1 \cdot x_1$$

$$F_{k2} = k_2 \cdot x_2$$

⇒

$$F_{I3} = F(t) - F_{Ba} - F_{Bb}$$

$$F_{I2} = F_{Ba} - F_{Bc} - F_{B2} - F_{k2}$$

$$F_{I1} = F_{Bb} + F_{B2} - F_{Bd} - F_{k1}$$

$$\frac{dv_1}{dt} = F_{I1} / m_1$$

$$\frac{dx_1}{dt} = v_1$$

$$\frac{dv_2}{dt} = F_{I2} / m_2$$

$$\frac{dx_2}{dt} = v_2$$

$$\frac{dv_3}{dt} = F_{I3} / m_3$$

$$\frac{dx_3}{dt} = v_3$$

$$F_{Ba} = B_1 \cdot (v_3 - v_2)$$

$$F_{Bb} = B_1 \cdot (v_3 - v_1)$$


$$F_{Bc} = B_1 \cdot v_2$$


$$F_{Bd} = B_1 \cdot v_1$$

$$F_{B2} = B_2 \cdot (v_2 - v_1)$$

$$F_{k1} = k_1 \cdot x_1$$

$$F_{k2} = k_2 \cdot x_2$$

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


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Mathematical Modeling of Physical Systems

Sorting Algorithm

- The sorting algorithm operates exactly in the same way as for electrical circuits. It is totally independent of the application domain.

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References

- Cellier, F.E. (1991), *Continuous System Modeling*, Springer-Verlag, New York, Chapter 4.