# The Structural Singularity Removal Algorithm by Pantelides

- This lecture deals with a procedure that can be used to remove structural singularities from a model in a systematic and algorithmic fashion. It is called the ***Algorithm of Pantelides***.
- The algorithm of Pantelides is a symbolic index-reduction algorithm.





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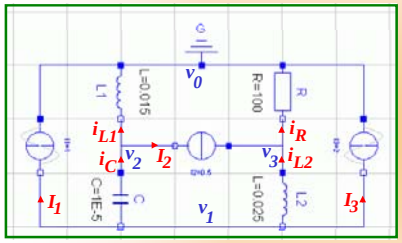
# Structural Singularities: An Example I




***We compose a model using the currents, the Voltages and the potentials. Consequently, the mesh equations are being ignored.***

***We have 7 circuit components plus the ground, therefore 2×7 + 1 = 15 equations. In addition, there are 4 nodes giving rise to another 3 equations. Therefore, we expect 18 equations in 18 unknowns.***

For passive components, it is customary to normalize the Voltages in the same direction as the currents. For active components (sources), the reverse is true.

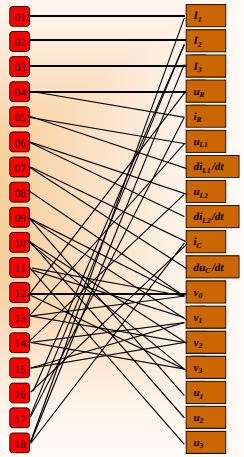




# Structural Singularities: An Example II

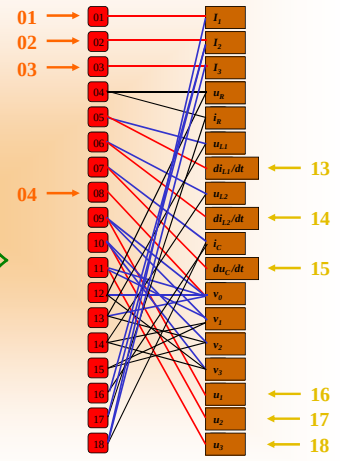
| | |
|---|---|
| 1: $I_1 = f_1(t)$ | 9: $u_1 = v_0 - v_1$ |
| 2: $I_2 = f_2(t)$ | 10: $u_2 = v_3 - v_2$ |
| 3: $I_3 = f_3(t)$ | 11: $u_3 = v_0 - v_1$ |
| 4: $u_R = R \cdot i_R$ | 12: $u_R = v_3 - v_0$ |
| 5: $u_{L1} = L_1 \cdot di_{L1}/dt$ | 13: $u_{L1} = v_2 - v_0$ |
| 6: $u_{L2} = L_2 \cdot di_{L2}/dt$ | 14: $u_{L2} = v_1 - v_3$ |
| 7: $i_C = C \cdot du_C/dt$ | 15: $u_C = v_1 - v_2$ |
| 8: $v_0 = 0$ | |
| 16: $i_C = i_{L1} + I_2$ | |
| 17: $i_R = i_{L2} + I_2$ | |
| 18: $I_1 + i_C + i_{L2} + I_3 = 0$ | |

⇒



## Structural Singularities: An Example III

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}/dt$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$
9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$
16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$



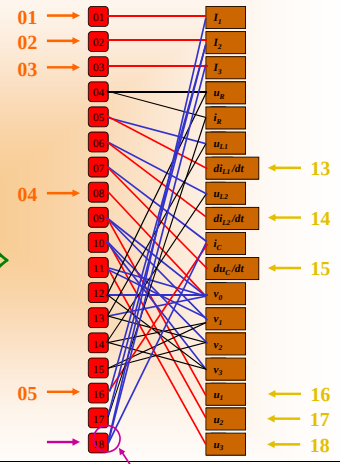





## Structural Singularities: An Example IV

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}/dt$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$
9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$
16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$

Constraint Equation




All connections are *blue*





## Coloring of the Structure Digraph

- The algorithm for ***coloring the structure digraph*** is completely analogous to the previously used method for making the equations causal.
- An implementation of the method by means of a computer program probably prefers the digraph, since this algorithm can directly be mapped onto data structures of conventional programming languages.
- For the human eye, the coloring of the equations may be more readable. For this reason, we shall continue, in the lecture, to color equations rather than digraphs.
- The vertical sorting can happen simultaneously by renumbering of the equations.

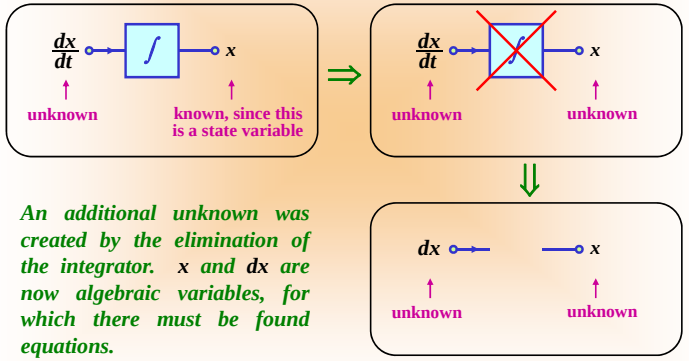




## The Algorithm by Pantelides I

- As soon as a ***constraint equation*** has been found, this equation must be ***differentiated***.
- In the algorithm of Pantelides, the differentiated constraint equation is ***added*** to the set of equations.
- Consequently, the set of equations has now one equation too many.
- In order to re-equalize the number of equations and unknowns, one of the integrators that is associated with the constraint equation is being eliminated.

# The Algorithm by Pantelides II




*An additional unknown was created by the elimination of the integrator. x and dx are now algebraic variables, for which there must be found equations.*





# The Algorithm by Pantelides III

- When differentiating constraint equations, it can happen that additional new variables are being created, e.g. $v \rightarrow dv$, where $v$ is an algebraic variable.
- Since $v$ was already ***blue*** (otherwise, this would not have been a constraint equation), there already exists another equation to compute $v$.
- This equation must also be differentiated.
- The differentiation of additional equations continues until no additional variables are being created.





# The Algorithm by Pantelides : An Example I

eliminated integrator

$dI_1 + di_C + di_{L2} + dI_3 = 0$

newly introduced variables

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}/dt$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$
9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$
16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$





# The Algorithm by Pantelides : An Example II

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$
9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$
16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$
19: $dI_1 + di_C + di_{L2} + dI_3 = 0$

# The Algorithm by Pantelides : An Example III

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$
9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$
16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$
19: $dI_1 + di_C + di_{L2} + dI_3 = 0$
20: $dI_1 = df_1(t)/dt$
21: $dI_3 = df_3(t)/dt$
22: $di_C = di_{L1}/dt + dI_2$
23: $dI_2 = df_2(t)/dt$

newly introduced variable





# The Algorithm by Pantelides : An Example IV

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$
9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$
16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$
19: $dI_1 + di_C + di_{L2} + dI_3 = 0$
20: $dI_1 = df_1(t)/dt$
21: $dI_3 = df_3(t)/dt$
22: $di_C = di_{L1}/dt + dI_2$
23: $dI_2 = df_2(t)/dt$





# The Algorithm by Pantelides : An Example V

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$
9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$
16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$
19: $dI_1 + di_C + di_{L2} + dI_3 = 0$
20: $dI_1 = df_1(t)/dt$
21: $dI_3 = df_3(t)/dt$
22: $di_C = di_{L1}/dt + dI_2$
23: $dI_2 = df_2(t)/dt$





# The Algorithm by Pantelides : An Example VI

1: $I_1 = f_1(t)$
2: $I_2 = f_2(t)$
3: $I_3 = f_3(t)$
4: $u_R = R \cdot i_R$
5: $u_{L1} = L_1 \cdot di_{L1}/dt$
6: $u_{L2} = L_2 \cdot di_{L2}$
7: $i_C = C \cdot du_C/dt$
8: $v_0 = 0$
9: $u_1 = v_0 - v_1$
10: $u_2 = v_3 - v_2$
11: $u_3 = v_0 - v_1$
12: $u_R = v_3 - v_0$
13: $u_{L1} = v_2 - v_0$
14: $u_{L2} = v_1 - v_3$
15: $u_C = v_1 - v_2$
16: $i_C = i_{L1} + I_2$
17: $i_R = i_{L2} + I_2$
18: $I_1 + i_C + i_{L2} + I_3 = 0$
19: $dI_1 + di_C + di_{L2} + dI_3 = 0$
20: $dI_1 = df_1(t)/dt$
21: $dI_3 = df_3(t)/dt$
22: $di_C = di_{L1}/dt + dI_2$
23: $dI_2 = df_2(t)/dt$

# The Algorithm by Pantelides : An Example VII

| | |
|---|---|
| 1: $I_1 = f_1(t)$ | 9: $u_1 = v_0 - v_1$ |
| 2: $I_2 = f_2(t)$ | 10: $u_2 = v_3 - v_2$ |
| 3: $I_3 = f_3(t)$ | 11: $u_3 = v_0 - v_1$ |
| 4: $u_R = R \cdot i_R$ | 12: $u_R = v_3 - v_0$ |
| 5: $u_{L1} = L_1 \cdot di_{L1}/dt$ | 13: $u_{L1} = v_2 - v_0$ |
| 6: $u_{L2} = L_2 \cdot di_{L2}$ | 14: $u_{L2} = v_1 - v_3$ |
| 7: $i_C = C \cdot du_C/dt$ | 15: $u_C = v_1 - v_2$ |
| 8: $v_0 = 0$ | |
| 16: $i_C = i_{L1} + I_2$ | 20: $dI_1 = df_1(t)/dt$ |
| 17: $i_R = i_{L2} + I_2$ | 21: $dI_3 = df_3(t)/dt$ |
| 18: $I_1 + i_C + i_{L2} + I_3 = 0$ | 22: $di_C = di_{L1}/dt + dI_2$ |
| 19: $dI_1 + di_C + di_{L2} + dI_3 = 0$ | 23: $dI_2 = df_2(t)/dt$ |





# The Algorithm by Pantelides : An Example VIII

| | |
|---|---|
| 1: $I_1 = f_1(t)$ | 9: $u_1 = v_0 - v_1$ |
| 2: $I_2 = f_2(t)$ | 10: $u_2 = v_3 - v_2$ |
| 3: $I_3 = f_3(t)$ | 11: $u_3 = v_0 - v_1$ |
| 4: $u_R = R \cdot i_R$ | 12: $u_R = v_3 - v_0$ |
| 5: $u_{L1} = L_1 \cdot di_{L1}/dt$ | 13: $u_{L1} = v_2 - v_0$ |
| 6: $u_{L2} = L_2 \cdot di_{L2}$ | 14: $u_{L2} = v_1 - v_3$ |
| 7: $i_C = C \cdot du_C/dt$ | 15: $u_C = v_1 - v_2$ |
| 8: $v_0 = 0$ | |
| 16: $i_C = i_{L1} + I_2$ | 20: $dI_1 = df_1(t)/dt$ |
| 17: $i_R = i_{L2} + I_2$ | 21: $dI_3 = df_3(t)/dt$ |
| 18: $I_1 + i_C + i_{L2} + I_3 = 0$ | 22: $di_C = di_{L1}/dt + dI_2$ |
| 19: $dI_1 + di_C + \boxed{di_{L2}} + dI_3 = 0$ | 23: $dI_2 = df_2(t)/dt$ |

*Choice*

*There now exists an algebraically coupled system with 7 equations in 7 unknowns.*





# The Algorithm by Pantelides : An Example IX

| | |
|---|---|
| 1: $I_1 = f_1(t)$ | 9: $u_1 = v_0 - v_1$ |
| 2: $I_2 = f_2(t)$ | 10: $u_2 = v_3 - v_2$ |
| 3: $I_3 = f_3(t)$ | 11: $u_3 = v_0 - v_1$ |
| 4: $u_R = R \cdot i_R$ | 12: $u_R = v_3 - v_0$ |
| 5: $u_{L1} = L_1 \cdot di_{L1}/dt$ | 13: $u_{L1} = v_2 - v_0$ |
| 6: $u_{L2} = L_2 \cdot di_{L2}$ | 14: $u_{L2} = v_1 - v_3$ |
| 7: $i_C = C \cdot du_C/dt$ | 15: $u_C = v_1 - v_2$ |
| 8: $v_0 = 0$ | |
| 16: $i_C = i_{L1} + I_2$ | 20: $dI_1 = df_1(t)/dt$ |
| 17: $i_R = i_{L2} + I_2$ | 21: $dI_3 = df_3(t)/dt$ |
| 18: $I_1 + i_C + i_{L2} + I_3 = 0$ | 22: $di_C = di_{L1}/dt + dI_2$ |
| 19: $dI_1 + di_C + di_{L2} + dI_3 = 0$ | 23: $dI_2 = df_2(t)/dt$ |





# Summary I

- First, we find a complete set of a-causal DAEs.
- The ***graph coloring algorithm by Tarjan*** is then applied to this set of DAEs.
- If an equation is found that is colored entirely in ***blue***, then the system is structurally singular.
- The structurally singular system is made non-singular by means of the ***algorithm by Pantelides***.
- It may be necessary to apply the Pantelides algorithm multiple times.

## Summary II

- The ***graph coloring algoriths by Tarjan*** is now applied to the modified non-singular set of DAEs.
- If the algorithm stalls, the modified system now contains one or several ***algebraic loops***. The occurrence of algebraic loops after application of the Pantelides algorithm to a structurally singular system is quite common.
- The system can now be further processed. The ***tearing algorithm***, which has already been presented, is one possible approach to deal with algebraically coupled systems.



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Mathematical Modeling of Physical Systems

## References


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- Pantelides, C.C. (1988), "The consistent initialization of differential-algebraic systems," *SIAM Journal Scientific Statistical Computation*, **9**(2), pp. 213-231.