Solution of Non-linear Equation Systems

- In this lecture, we shall look at the mixed symbolic and numerical solution of algebraically coupled non-linear equation systems.
- The tearing method lends itself also to the efficient treatment of non-linear equation systems.
- The numerical iteration of the non-linear equation system can be limited to the tearing variables.

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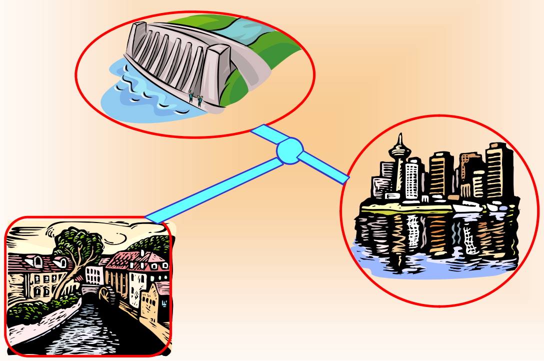
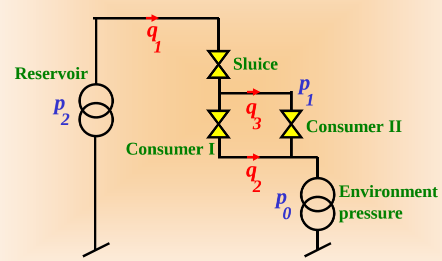
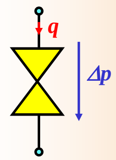
Non-linear Equation System: An Example I



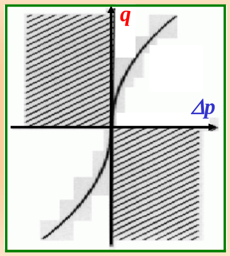
Non-linear Equation System: An Example II



Non-linear Equation System: An Example III

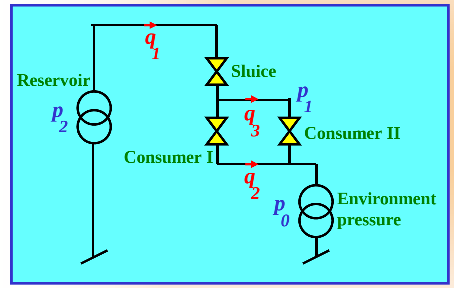


q: Flow rate
Δp: Pressure reduction



$$q = k \cdot sign(\Delta p) \cdot \sqrt{|\Delta p|}$$
$$\Rightarrow \Delta p = sign(q) \cdot q^2 / k^2$$

Non-linear Equation System: An Example IV



$p_2 = 100$
$p_0 = 1$
$f_S(q_1, p_1, p_2) = 0$
$f_I(q_2, p_0, p_1) = 0$
$f_{II}(q_3, p_0, p_1) = 0$
$q_1 = q_2 + q_3$

Non-linear Equation System: An Example V

$p_2 = 100$
$p_0 = 1$
$f_S(q_1, p_1, p_2) = 0$
$f_I(q_2, p_0, p_1) = 0$
$f_{II}(q_3, p_0, p_1) = 0$
$q_1 - q_2 - q_3 = 0$

$\Rightarrow$

$p_2 = 100$
$p_0 = 1$
$f_S(q_1, p_1, p_2) = 0$
$f_I(q_2, p_0, p_1) = 0$
$f_{II}(q_3, p_0, p_1) = 0$
$q_1 - q_2 - q_3 = 0$

Non-linear equation system in 4 unknowns

Newton Iteration I

Non-linear equation system:	$f(x) = 0$	$x \in \Re^n$, $f \in \Re^n$
Initial guess:	x^0	
Iteration formula:	$x^{i+1} = x^i - \Delta x^i$	$\Delta x \in \Re^n$
Increment:	$\Delta x^i = H(x^i)^{-1} \cdot f(x^i)$	$H \in \Re^{n \times n}$
Hessian matrix:	$H(x) = \dfrac{\partial f(x)}{\partial x}$	

Newton Iteration : Example I

$$x = \begin{bmatrix} p_1 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \qquad f(x) = \begin{bmatrix} p_2 - p_1 - sign(q_1) \cdot q_1^2/k_1^2 \\ p_1 - p_0 - sign(q_2) \cdot q_2^2/k_2^2 \\ p_1 - p_0 - sign(q_3) \cdot q_3^2/k_3^2 \\ q_1 - q_2 - q_3 \end{bmatrix} = 0$$

$$H(x) = \begin{bmatrix} -1 & -2|q_1|/k_1^2 & 0 & 0 \\ 1 & 0 & -2|q_2|/k_2^2 & 0 \\ 1 & 0 & 0 & -2|q_3|/k_3^2 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

Newton Iteration II

Computation of increment:

$$\Delta x^i = H(x^i)^{-1} \cdot f(x^i)$$
$$\Rightarrow H(x^i) \cdot \Delta x^i = f(x^i)$$

$\Rightarrow$ *Linear equation system in the unknowns* Δx

$\Rightarrow \Delta x \in \Re^n$

Newton Iteration with Tearing I

$p_2 = 100$
$p_0 = 1$
$f_S(q_1, p_1, p_2) = 0$
$f_I(q_2, p_0, p_1) = 0$
$f_{II}(q_3, p_0, p_1) = 0$
$q_1 - q_2 - q_3 = 0$

Choice (q_1)

$\Rightarrow$

$p_2 = 100$
$p_0 = 1$
$f_S(q_1, p_1, p_2) = 0$
$f_I(q_2, p_0, p_1) = 0$
$f_{II}(q_3, p_0, p_1) = 0$
$q_1 - q_2 - q_3 = 0$

$\Rightarrow$

$p_2 = 100$
$p_0 = 1$
$f_S(q_1, p_1, p_2) = 0$
$f_I(q_2, p_0, p_1) = 0$
$f_{II}(q_3, p_0, p_1) = 0$
$q_1 - q_2 - q_3 = 0$

Newton Iteration with Tearing II

$p_2 = 100$
$p_0 = 1$
$f_S(q_1, p_1, p_2) = 0$
$f_I(q_2, p_0, p_1) = 0$
$f_{II}(q_3, p_0, p_1) = 0$
$q_1 - q_2 - q_3 = 0$

$\Rightarrow$

$p_2 = 100$
$p_0 = 1$
$q_1 = q_2 + q_3$
$p_1 = f_1(q_1, p_2)$
$q_2 = f_2(p_0, p_1)$
$q_3 = f_3(p_0, p_1)$

$$q_1 = f_2(p_0, p_1) + f_3(p_0, p_1)$$
$$= f_2(p_0, f_1(q_1, p_2)) + f_3(p_0, f_1(q_1, p_2))$$

Newton Iteration with Tearing III

$q_1 = f_2(p_0, p_1) + f_3(p_0, p_1)$
$= f_2(p_0, f_1(q_1, p_2)) + f_3(p_0, f_1(q_1, p_2))$

$x = q_1$

$f(x) = q_1 - f_2(p_0, f_1(q_1, p_2)) - f_3(p_0, f_1(q_1, p_2)) = 0$

$\Rightarrow$ $H(x^i) \cdot \Delta x^i = f(x^i)$

$\Rightarrow$ *Linear equation system in the unknown Δx*

$\Rightarrow$ $\Delta x \in \Re^1$

Newton Iteration : Example II

$p_2 = 100$
$p_0 = 1$
$q_1 = q_2 + q_3$
$p_1 = p_2 - sign(q_1) \cdot q_1^2 / k_1^2$
$q_2 = k_2 \cdot sign(p_1 - p_0) \cdot \sqrt{|p_1 - p_0|}$
$q_3 = k_3 \cdot sign(p_1 - p_0) \cdot \sqrt{|p_1 - p_0|}$
$pq_1q_1 = 1$
$pp_1q_1 = -2|q_1| / k_1^2$
$pq_2q_1 = k_2 / (2 \cdot \sqrt{|p_1 - p_0|}) \cdot pp_1q_1$
$pq_3q_1 = k_3 / (2 \cdot \sqrt{|p_1 - p_0|}) \cdot pp_1q_1$
$f = q_1 - q_2 - q_3$
$h = pq_1q_1 - pq_2q_1 - pq_3q_1$

$\Rightarrow$ *The symbolic substitution of expressions is hardly ever worthwhile. It is much better to iterate over all equations and to differentiate every equation separately in the determination of the partial derivatives.*

Newton Iteration : Example III

q_1 = *Initial guess*
$dx = 1$
while $dx > dxmin$
  $p_1 = p_2 - sign(q_1) \cdot q_1^2 / k_1^2$
  $q_2 = k_2 \cdot sign(p_1 - p_0) \cdot \sqrt{|p_1 - p_0|}$
  $q_3 = k_3 \cdot sign(p_1 - p_0) \cdot \sqrt{|p_1 - p_0|}$
  $pp_1 = -2|q_1| / k_1^2$
  $pq_2 = k_2 / (2 \cdot \sqrt{|p_1 - p_0|}) \cdot pp_1$
  $pq_3 = k_3 / (2 \cdot \sqrt{|p_1 - p_0|}) \cdot pp_1$
  $f = q_1 - q_2 - q_3$
  $h = 1 - pq_2 - pq_3$
  $dx = h \backslash f$
  $q_1 = q_1 - dx$
end

$\Rightarrow$ *The iteration is carried out over all equations. However, the internal linear equation system is only solved for the tearing variables.*

Newton Iteration for Linear Systems

Linear system: $A \cdot x = b$

$\Rightarrow f(x) = A \cdot x - b = 0$

$\Rightarrow H(x) = \partial f(x) / \partial x = A$

$\Rightarrow A \cdot \Delta x = A \cdot x - b$

$\Rightarrow \Delta x = x - A^{-1} \cdot b$

$\Rightarrow x^1 = x^0 - (x^0 - A^{-1} \cdot b) = A^{-1} \cdot b$

$\Rightarrow$ *The Newton iteration converges here in a single iteration step*

Summary

- The tearing method is equally suitable for use in non-linear as in linear systems.
- The Newton iteration of a non-linear equation system leads internally to the solution of a linear equation system. The Hessian matrix of this equation system needs only to be determined for the tearing variables.
- The Newton iteration can also be used very efficiently for the solution of linear systems in many variables, since it converges (with correct computation of the *H(x)* matrix) in a single step.
- In practice, the *H(x)* matrix is often numerically approximated rather than analytically computed.
- Yet, symbolic formula manipulation techniques can be used to come up with symbolic expressions for the elements of the Hessian matrix.