Modelreduction of linear, time-invariant Systems by Minimizing the Equation Error

(Ref: Eduard Eitelberg, PhD Thesis, Karlsruhe, FRG 1979)

Problem:

$$\left|\begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} \end{array}\right| \qquad \underline{x} \in R^n$$

Find

$$\left|\begin{array}{l} \dot{\underline{\xi}} = A_r\underline{\xi} + B_r\underline{u} \\ \underline{\eta} = C_r\underline{\xi} \end{array}\right| \qquad \underline{\xi} \in R^\nu$$

$$\nu < n$$

such that $\underline{\eta} \approx \underline{y}$.

We can always find a reduction matrix R such that:

$$C = C_r \cdot R$$

We choose R such that all relevant components of the output are represented.

E.g.
$$\underline{\dot{x}} = \begin{bmatrix} -1 & \mu \\ 0 & a \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = [0 \quad 1] \underline{x}$$

In this case, y is only influenced by x_2, and we can choose:
$$R = [0 \quad 1]$$
$$\Rightarrow \quad C_r = 1 \, .$$

In other situations, we may need a linear combination. The idea is to determine the desired order of the reduced model first (e.g. by looking at the actual output y, and from there conclude what R and C_r have to be.

Approach: Cancel all column vectors that are $= 0$ out of C

$$\Rightarrow \quad \underline{y} = C\,\underline{x} = C_r\,\underline{x}_r = C_r R\,\underline{x}$$

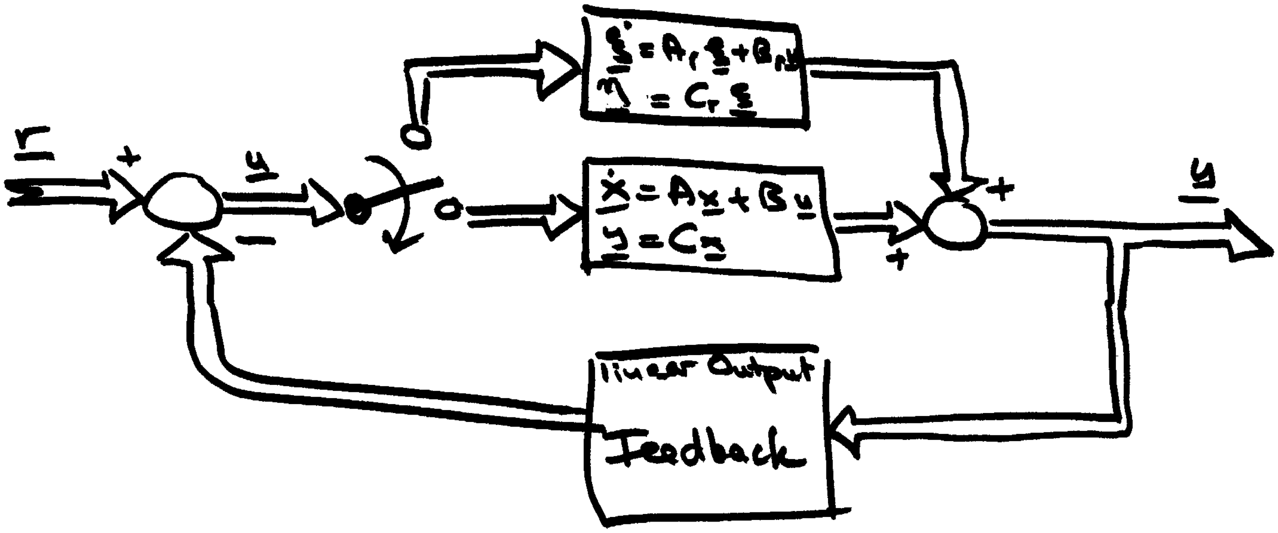
Now: $\quad \underline{\eta} \approx \underline{y} \iff \underline{x}_r \approx \underline{\xi}$

Wishes:

(1) The steady-state error between the original system and the reduced model is to be zero for any control

(2) The dynamic error between the original system and the reduced model is to be minimized for any control.

In practice, we have the following situation:



In steady-state, $\underline{y}$ must not change independent of the switch position for any value of $\underline{r}$ and any linear output feedback.

In transient, the influence of the switch position should be minimal for any input function $\underline{r}$ and any (stable) linear output feedback.

Problem: Find unknown A_r, B_r.

A) Steady-state accuracy

We write the linear output feedback as:

$$\left| \begin{aligned} \dot{\underline{z}} &= E\underline{z} + F\underline{y} \\ \underline{u} &= \underline{r} - K\underline{z} - L\underline{y} \end{aligned} \right|$$

(This is more general than just state feedback. We allow eigendynamics of the controller.)

In steady-state, all derivatives vanish, thus:

$$\left| \begin{aligned} A\underline{x}_{ss} &= -B\underline{u}_{ss} \\ \underline{y}_{ss} &= C\underline{x}_{ss} \end{aligned} \right|$$

and: $$\left| \begin{aligned} A_r\underline{\xi}_{ss} &= -B_r\underline{u}_{ss} \\ \underline{\eta}_{ss} &= C_r\underline{\xi}_{ss} \end{aligned} \right|$$

Finally:

$$\left| \begin{aligned} E\,\underline{z}_{ss} &= -F\,\underline{y}_{ss} \\ \underline{u}_{ss} &= \underline{r} - K\,\underline{z}_{ss} - L\,\underline{y}_{ss} \end{aligned} \right|$$

or using the reduced model:

$$\left| \begin{aligned} E\,\underline{\zeta}_{ss} &= -F\,\underline{\eta}_{ss} \\ \underline{v}_{ss} &= \underline{r} - K\,\underline{\zeta}_{ss} - L\,\underline{\eta}_{ss} \end{aligned} \right|$$

For the moment, let us assume that A, A_r are regular matrices (no eigenvalue at zero).

$$\Rightarrow \left| \begin{aligned} \underline{x}_{ss} &= -A^{-1}B\,\underline{u} \\ \underline{\xi}_{ss} &= -A_r^{-1}B_r\,\underline{u} \end{aligned} \right|$$

but: $\underline{\xi}_{ss} = \underline{x}_{r_{ss}} = R\,\underline{x}_{ss}$

$$\Rightarrow \quad \underline{\xi}_{ss} = -RA^{-1}B\,\underline{u} \equiv -A_r^{-1}B_r\,\underline{u}$$

$$\Rightarrow \quad \boxed{B_r = A_r R A^{-1} B} \qquad (1)$$

It is easy to show that (eq. 1) is a necessary and sufficient condition for error-free steady-state if A, A_r regular.

Let us now assume that A be singular. We introduce a static feedback ($K = 0$)

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{u} = r - LC\underline{x} \end{array} \right|$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = (A - BLC)\underline{x} + B\underline{r} \\ \underline{y} = C\underline{x} \end{array} \right|$$

We call $\quad A - BLC \equiv \hat{A}$

and: $\quad A_r - B_r LC \equiv \hat{A}_r$

Thus, we have a new problem that can easily be made regular. (In fact, we can choose the eigenvalues of $\hat{A}$ freely if we assume that we have decoupled beforehand all noncontrollable and nonobservable modes.)

In these new variables, we can write the necessary and sufficient condition:

$$\boxed{B_r = \hat{A}_r R \hat{A}^{-1} B}$$

But now, we have the same situation as before, and the condition will work for <u>any</u> $\hat{K}, \hat{L}$, in particular also for $\hat{L} = -L$.

Example:

$$\left|\begin{array}{l} \dot{\underline{x}} = \begin{bmatrix} -1 & \mu \\ 0 & a \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\[2ex] y = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x} \end{array}\right|$$

Let : $\quad R = \begin{bmatrix} 0 & 1 \end{bmatrix} \iff c_r = 1$

If $\quad a \neq 0$:

$$B_r = A_r R A^{-1} B =$$

$$A_r \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & \mu/a \\ 0 & 1/a \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A_r \cdot \frac{1}{a}$$

$$\boxed{B_r = \frac{A_r}{a}}$$

Obviously, $a = 0$ is not a meaningful proposition in this formula.

However, using our approach, we introduce a constant feedback ℓ, and get

$$\hat{A} = A - BLC$$

$$\Rightarrow \quad \dot{\hat{\underline{x}}} = \begin{bmatrix} -1 & \mu \\ 0 & -\ell \end{bmatrix} \hat{\underline{x}} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\hat{x}_{2_{ss}} = \frac{u}{\ell}$$

$$B_r = \hat{A}_r R (\hat{A})^{-1} B = \hat{A}_r [0 \;\; 1] \begin{bmatrix} -1 & \frac{\mu}{\ell} \\ 0 & -1/\ell \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \hat{A}_r \left(-\frac{1}{\ell}\right)$$

$$\Rightarrow \boxed{B_r = -\frac{\hat{A}_r}{\ell}}$$

$$\underline{\underline{A_r}} = \hat{A}_r + B_r L C_r = \hat{A}_r - \frac{\hat{A}_r}{\ell} \ell \cdot 1 \equiv \underline{\underline{0}}$$

B) Dynamic Accuracy:

$$\dot{\underline{\xi}} = A_r \underline{\xi} + B_r \underline{u}$$

$$\underline{x}_r = R\underline{x} \qquad\qquad \underline{x}_r \approx \underline{\xi}$$

$$\Rightarrow \quad \dot{\underline{x}}_r \approx A_r \underline{x}_r + B_r \underline{u}$$

$$\Rightarrow \quad \underline{d} = \dot{\underline{x}}_r - A_r \underline{x}_r - B_r \underline{u} \overset{!}{=} \min$$

Proof: $\underline{e} = \underline{x}_r - \underline{\xi}$

$$\Rightarrow \dot{\underline{e}} = A_r \underline{e} + (\dot{\underline{x}}_r - A_r \underline{x}_r - B_r \underline{u})$$

$$= A_r \underline{e} + \underline{d}$$

$$\Rightarrow \quad \underline{d}(t) \overset{!}{=} \min \iff \underline{e}(t) \overset{!}{=} \min$$

$$\underline{d} = \dot{\underline{x}}_r - A_r \underline{x}_r - B_r \underline{u}$$

$$= R\dot{\underline{x}} - A_r R\underline{x} - B_r \underline{u}$$

$$= R(A\underline{x} + B\underline{u}) - A_r R\underline{x} - B_r \underline{u}$$

$$\Rightarrow \underline{d} = (RA - A_r R)\underline{x} + (RB - B_r)\underline{u}$$

(Again, if A is singular, replace A by $\hat{A}$ as before.)

Step response:

$$\underline{x}(t) = (e^{At} - I)A^{-1}B\,\underline{\varepsilon}(t)$$

$$\Rightarrow \underline{d} = (RAe^{At}A^{-1}B - A_r R e^{At}A^{-1}B - RAA^{-1}B + A_r RA^{-1}B + RB - B_r)\,\underline{\varepsilon}(t)$$

$$= \left[(RA - A_r R)e^{At}A^{-1}B\right]\underline{\varepsilon}(t) + \Big[\underbrace{(A_r RA^{-1}B - B_r)}_{\equiv 0}\Big]\underline{\varepsilon}(t)$$

$\equiv 0$ due to steady-state accuracy.

$$\Rightarrow \underline{d} = \left[(RA - A_r R)e^{At}A^{-1}B\right]\underline{\varepsilon}(t)$$

$$\Rightarrow D = (RA - A_r R)e^{At}A^{-1}B \overset{!}{=} \min.$$

In the sense of regression analysis, we can write:

$$PI = \int_0^\infty \mathrm{Spur}\{ D(t) \cdot G(t) \cdot D'(t) \} dt \overset{!}{=} \min_{\forall A_r}$$

for any positive (semi-)definit weighting function $G(t)$

$$\Longrightarrow \quad A_r = RASR' \cdot (RSR')^{-1}$$

where: $$S = \int_0^\infty e^{At} (A^{-1}B) G(t) (A^{-1}B) e^{A't} dt$$

(looks pretty much like a Gramian !)

If (A, B) is controllable, $\Longrightarrow$ S is unique (except for the weighting function.

Example:

$$\dot{\underline{x}} = \begin{bmatrix} -1 & \mu \\ 0 & a \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$R = [0 \quad 1]$$

$$S = \int_0^\infty \exp\left\{ \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix} t \right\} \begin{bmatrix} 0 \\ 1/a \end{bmatrix} [0 \quad 1/a] \exp\left\{ \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix} t \right\} dt$$

$$= \int_0^\infty \begin{bmatrix} 0 & 0 \\ 0 & \frac{e^{2at}}{a^2} \end{bmatrix} dt = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{2a^3} \end{bmatrix}$$

... $\Rightarrow$ $\underline{\underline{A_r = a}}$ ✓

How compute S?

Assume for the moment:

$G(t) = I$.

We know: $$\int_{t_1}^{t_2} U \cdot \frac{dV}{dt} \cdot dt \equiv U \cdot V \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{dU}{dt} \cdot V \cdot dt$$

(matrix form of partial integral)

Set: $$U = e^{At}(A^{-1}B)(A^{-1}B)'$$

$$\frac{dV}{dt} = e^{A't} \quad \Rightarrow V(t) = e^{A't} \cdot (A')^{-1}$$

$$\Rightarrow S = e^{At}(A^{-1}B)(A^{-1}B)' e^{A't}(A')^{-1} \Big|_0^{\infty}$$

$$- \int_0^{\infty} A e^{At}(A^{-1}B)(A^{-1}B)' e^{A't}(A')^{-1} dt$$

$$= -(A^{-1}B)(A^{-1}B)'(A')^{-1} - AS(A')^{-1}$$

$$\Rightarrow \boxed{AS + SA' = -(A^{-1}B)(A^{-1}B)'}$$

$\Rightarrow$ Ljapunov Equation.

We met this before. We use singular value decomposition of $(A^{-1}B)$ to get a good numerical behavior!!!

Problem: If original system is instable $\Longrightarrow$ S does not converge.

Solution: Choose $G(t) = Qe^{2\alpha t}$

$$\Rightarrow S = \int_0^\infty e^{At}(A^{-1}B)Qe^{2\alpha t}(A^{-1}B)'e^{A't}\,dt$$

$$= \int_0^\infty e^{At}e^{\alpha t}(A^{-1}B)Q(A^{-1}B)'e^{\alpha t}e^{A't}\,dt$$

$$= \int_0^\infty e^{(A+\alpha I)t}(A^{-1}B)Q(A^{-1}B)'e^{(A+\alpha I)'t}\,dt$$

Set $\tilde{A} = A + \alpha I$

eig $(\tilde{A})$ $\therefore$ $\quad \tilde{A}\tilde{\underline{v}}_i = \tilde{\lambda}_i \tilde{\underline{v}}_i$

$$\Rightarrow (A + \alpha I)\tilde{\underline{v}}_i = \tilde{\lambda}_i \tilde{\underline{v}}_i$$

$$\Rightarrow A\tilde{\underline{v}}_i = (\tilde{\lambda}_i - \alpha)\tilde{\underline{v}}_i$$

$$\Rightarrow \quad \underline{v}_i \equiv \tilde{\underline{v}}_i \quad ; \quad \lambda_i = \tilde{\lambda}_i - \alpha$$

Thus: λ_i instable, choose α such that $\tilde{\lambda}_i$ stable.

$$\Rightarrow \boxed{(A+\alpha I)S + S(A+\alpha I)' = -(A^{-1}B)Q(A^{-1}B)'}$$

Algorithm:

(1) Given $\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} \end{array} \right|$

(2) If A singular: $\hat{A} = A - BLC$

Choose L

such that $\hat{A}$ regular. [Choose L such that $\hat{A}$ stable.]

(3) Calculate R, C_r

(4) Choose Q, α $\quad (Q > 0)$

α such that system is stable

(5) Solve Ljapunov equation

$$(A + \alpha I)S + S(A + \alpha I)' = -(A^{-1}B)Q(A^{-1}B)'$$

(6) Compute: $A_r = RASR'(RSR')^{-1}$

(7) Compute $B_r = A_r R A^{-1} B$

Simulate reduced system, and compare. If not satisfied, go back and choose a better Q [Q has the same function as in Riccati equation.]