Spaces:

A <u>normed Vector space</u> is any vector space (e.g. $\mathbb{R}^n$ or $\mathbb{C}^n$) with an appropriate norm associated with it.

A sequence of values $\{x_n\}$ from the vector space V is called a <u>Cauchy sequence</u>, if

$$\|x_n - x_m\|_\circ \to 0 \quad , \quad n,m \to \infty$$

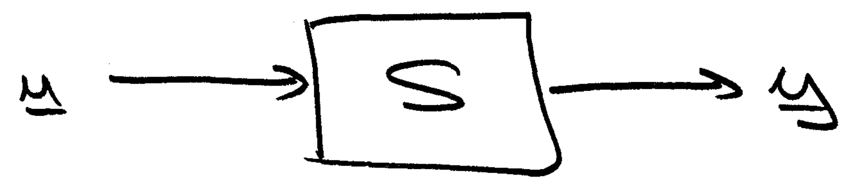
If:

$$\|x_n - x\|_\circ \to 0 \quad , \quad n \to \infty$$

the Cauchy sequence <u>converges</u> on x. If every sequence that belongs to V converges in V, V is called <u>complete</u>.

A complete normed vector space is called a <u>Banach space</u>.

Example:



$\|\underline{u}\|_{\bullet} < \infty \iff \underline{u}$ is bounded

$\| S \|_{\bullet} < \infty \iff S$ is stable

$\Rightarrow \|\underline{y}\|_{\bullet} < \infty \iff \underline{y}$ is bounded

Given a System S, stable
$\Rightarrow$ for all bounded inputs, the outputs form a Banach space.

Examples of Banach spaces:

$l_p[\emptyset, \infty)$: a space of discrete sequences $\{x_n\}$ such that:

$$\sum_{i=\emptyset}^{\infty} |x_i|^p < \infty$$

$$\|\underline{x}\|_{\bullet} = \|\underline{x}\|_p = \left(\sum_{i=\emptyset}^{\infty} |x_i|^p \right)^{1/p}$$

In particular:

$\underline{\ell_2[0,\infty)}$: All stable discrete time series using the Euclidean norm.

$\underline{\ell_\infty[0,\infty)}$: All stable discrete time series using the maximum norm.

$$\|\underline{x}\|_\infty \overset{!}{=} \sup_i |x_i|$$

$\underline{\mathcal{L}_p[0,\infty)}$: All stable continuous time series with

$$\|x\|_p = \left(\int_0^\infty |x(t)|^p dt \right)^{1/p} < \infty$$

Such functions are called <u>Lebesque measurable</u> functions.

In particular:

$\underline{\mathcal{L}_2[0,\infty)}$: All stable continuous time functions using the Euclidean norm.

$\underline{\mathcal{L}_\infty[0,\infty)}$: All stable continuous time functions using the maximum norm:

$$\|x\|_\infty = \sup_t |x(t)|$$

Inner Products:

$$\langle \underline{x}, \underline{y} \rangle \doteq \underline{x}^* \cdot \underline{y}$$

$$\Rightarrow \langle \underline{x}, \underline{x} \rangle = \underline{x}^* \cdot \underline{x} = \left(\|\underline{x}\|_2\right)^2$$

$$\cos \angle(\underline{x}, \underline{y}) = \frac{\langle \underline{x}, \underline{y} \rangle}{\|\underline{x}\|_2 \cdot \|\underline{y}\|_2}$$

$$\langle A\underline{x}, \underline{y} \rangle = \langle \underline{x}, A^*\underline{y} \rangle .$$

Generalization:

We define a generalized inner product $\langle \cdot, \cdot \rangle$ over a vector space V as follows:

(α) $\langle \underline{x}, \alpha \underline{y} + \beta \underline{z} \rangle = \alpha \langle \underline{x}, \underline{y} \rangle + \beta \langle \underline{x}, \underline{z} \rangle$

(β) $\langle \underline{x}, \underline{y} \rangle = \overline{\langle \underline{y}, \underline{x} \rangle}$

(γ) $\langle \underline{x}, \underline{x} \rangle \geq \emptyset$

(δ) $\langle \underline{x}, \underline{x} \rangle = \emptyset \iff \underline{x} = \emptyset$

A vector space with an inner product defined for it is called an <u>inner product space</u>.

Each inner product space <u>induces</u> a norm:

$$\|\underline{x}\|_\circ = \sqrt{\langle \underline{x}, \underline{x} \rangle}$$

A inner product space that uses its induced norm is called a <u>Hilbert space</u>.

<u>Special Hilbert spaces</u>:

<u>$\ell_2[\emptyset, \infty)$</u>: As before with

$$\langle \underline{x}, \underline{y} \rangle \overset{!}{=} \underline{x}^* \cdot \underline{y}$$

and $\|\underline{x}\|_\circ = \|\underline{x}\|_2$

This can be generalized to vectors of functions:

$$\langle X, Y \rangle \stackrel{!}{=} \mathrm{Trace}\,(X^{*} \cdot Y)$$

where:

$$\begin{array}{c|c} & 1 \;\cdots\; n \\ \hline \begin{array}{c} 0 \\ \Delta t \\ 2\Delta t \\ \vdots \\ \downarrow \\ \infty \end{array} & \left[\begin{array}{c} \\ \\ X \\ \\ \\ \end{array} \right] \end{array}$$

is an infinite-dimensional matrix.

$$\mathrm{Trace}\,(M) \stackrel{!}{=} \sum_{i=1}^{n} m_{ii}$$

$$\mathrm{Trace}\,(A^{*}B) = \sum_{i=1}^{n} \sum_{j=1}^{m} \bar{a}_{ij} \cdot b_{ij}$$

$\mathcal{L}_2[0, \infty)$:

$$\langle f, g \rangle = \int_0^\infty f^{*}(t) \cdot g(t)\, dt$$

or:

$$\langle f, g \rangle = \int_0^\infty \mathrm{Trace}\left(\underline{f}^{*}(t) \cdot \underline{g}(t)\right) d$$

(Notice: Hilbert spaces are Banach spaces.)

<u>$\mathcal{L}_2(j\mathbb{R})$:</u> with inner product defined as:

$$\langle \underline{\underline{F}}, \underline{\underline{G}} \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}\left(\underline{\underline{F}}^*(j\omega)\cdot \underline{\underline{G}}(j\omega)\right)d\omega$$

and $\|\underline{\underline{F}}\|_2 = \sqrt{\langle \underline{\underline{F}}, \underline{\underline{F}} \rangle}$

let: $\underline{\underline{F}}(j\omega) = \mathcal{L}\{\underline{\underline{F}}(t)\}\big|_{s=j\omega}$

Because of <u>Parseval theorem</u>:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}\left(\underline{\underline{F}}^*(j\omega)\cdot \underline{\underline{G}}(j\omega)\right)d\omega \equiv \int_0^{\infty} \underset{\text{Trace}}{\left(\underline{\underline{F}}^*(t)\cdot \underline{\underline{F}}(t)\right)}dt$$

$$\Rightarrow \quad \|\underline{\underline{F}}(j\omega)\|_2 \equiv \|\underline{\underline{F}}(t)\|_2$$

$\mathcal{L}_2(j\mathbb{R})$: is defined for all transfer function matrices without poles on imaginary axis.

$\mathcal{H}_2$: Is a Banach space, with the following norm:

$$\|\underline{\underline{F}}\|_2 = \sqrt{\sup_{\sigma > 0} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}\left[\underline{\underline{F}}^*(\sigma+j\omega) \cdot \underline{\underline{F}}(\sigma+j\omega)\right] d\omega \right\}}$$

$\mathcal{H}_2$: is defined for all stable transfer function matrices.

Because of Maximum Modulus Theorem.

$$\|\underline{\underline{F}}\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}\left[\underline{\underline{F}}^*(j\omega) \cdot \underline{\underline{F}}(j\omega)\right] d\omega}$$

$\mathcal{H}_\infty$: Is another Banach space with the following norm:

$$\|\underline{\underline{F}}\|_\infty = \sup_{\omega \in \mathbb{R}} \overline{\sigma}\left[F(j\omega)\right].$$

F bounded for $Re(s) > 0$.

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ are called Hardy spaces.

Power and Spectral Signals:

Given a signal $u(t)$:

We can define its autocorrelation matrix as:

$$R_{uu}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t+\tau)\, u^*(t)\, dt$$

From this, we can define the spectral density matrix as:

$$S_{uu}(j\omega) = \mathcal{F}\{R_{uu}(\tau)\} = \int_{-\infty}^{\infty} R_{uu}(\tau) e^{-j\omega\tau} d\tau$$

Thus:

$$R_{uu}(\tau) = \mathcal{F}^{-1}\{S_{uu}(\tau)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{uu}(j\omega) e^{+j\omega\tau} d\omega$$

$$\Rightarrow \|u\|_{\mathcal{P}} = \sqrt{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|u(t)\|_2^2 dt} = \sqrt{\mathrm{Trace}(R_{uu}(0))}$$

is only a seminorm.

$$\|u\|_s = \sqrt{\|S_{uu}(j\omega)\|_\infty}$$

It makes sense to generalize these norms to vector signals in the following way:

$$\| \underline{u} \|_P \stackrel{!}{=} \| (\| u_i \|_P) \|_1$$

which is the total consumed power.

$$\| \underline{u} \|_S \stackrel{!}{=} \| (\| u_i \|_S) \|_\infty$$

Which is the largest among the individual spectrum norms.

Sometimes, it is useful to define the <u>Hankel norm</u> of a system:

$$\| S \|_H = \sqrt{\max(\mathrm{eig}(G_c \cdot G_o))}$$

is the largest <u>Hankel singular value</u>.

Given: $S = \{A, B; C, D\}$

$$\Rightarrow \sigma_{H_i}(S) \overset{!}{=} \sqrt{eig(G_c \cdot G_o)}$$
$$= \sqrt{eig(G_o \cdot G_c)}$$

Usually, system norms are defined in the frequency domain to guarantee invariance. However, the Hankel singular values are invariant to similarity transformations, thus:

$$\|S\|_H = \sigma_H(S)_{max}$$

is okay!