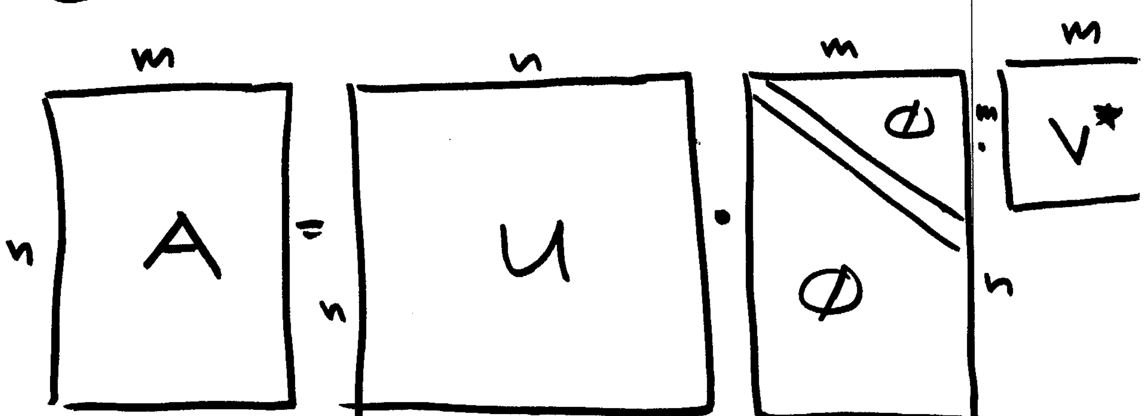
The Singular Value Decomposition

Def: Any (even non-square) matrix can be decomposed into:

$$\boxed{A = U \cdot \Sigma \cdot V^*}$$

where U and V are two unitary matrices, and Σ is a diagonal matrix.



CTRL-C: $s = \mathrm{SVD}(A) \Rightarrow$ vector with σ_i

$[U, S, V] = \mathrm{SVD}(A) \Rightarrow$ 3 matrices

Proof:

- $A \cdot A^* = (U \cdot \Sigma \cdot V^*) \cdot (V \cdot \Sigma^* \cdot U^*)$

$$= U \cdot \Sigma \cdot \underbrace{V^* \cdot V}_{I} \cdot \underbrace{\Sigma^*}_{\Sigma} \cdot U^*$$

$$\Rightarrow \boxed{A \cdot A^* = U \cdot \Sigma^2 \cdot U^*}$$

spectral decomposition of $(A \cdot A^*)$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \phi \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_m \\ & \phi & & \\ & & & \end{bmatrix} \Rightarrow \underline{\underline{\{\sigma_i\} = \sqrt{Eig(A \cdot A^*)}}}$$

U is the right modal matrix of $A \cdot A^*$.

- $A^* \cdot A = (V \cdot \Sigma^* \cdot U^*) \cdot (U \cdot \Sigma \cdot V^*)$

$$= V \cdot \underbrace{\Sigma^*}_{\Sigma} \cdot \underbrace{U^* \cdot U}_{I} \cdot \Sigma \cdot V^*$$

$$\Rightarrow \boxed{A^* A = V \cdot \Sigma^2 \cdot V^*}$$

$$\Rightarrow \quad \sigma_i = \sqrt{Eig(A \cdot A^*)} \equiv \sqrt{Eig(A^* \cdot A)}$$

σ_i are positive real.

V is the right modal matrix of $A^* \cdot A$.

Def. the σ_i are called the singular values of A. They are usually ordered such that:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$$

Algorithm:

```
[U,S2]= EIG(A*A');
S2 = DIAG(S2);
S= SQRT(S2);
[V,S2]= EIG(A'*A);
SS= U'*A*V
FOR I= 1:N,...
    IF SS(I,I) < 0 ,...
        V(:,I) = -V(:,I); ...
    END,...
END
```

$$\boxed{A = U \cdot \Sigma \cdot V^*}$$

As U, V are unitary

$\Rightarrow$ $\text{Rank}(\Sigma) \equiv \text{Rank}(A)$

$\Rightarrow$ The Rank of A equals the # of nonzero singular values of A.

$\Rightarrow$ The Nullity of A equals the # of zero singular values of A.

- As A^*A and AA^* are Hermitian, the two eigenvalue problems are very well conditioned $\Longrightarrow$ the SVD-algorithm is numerically benign.

- This is the algorithm that CTRL-C uses to determine the RANK of a matrix.

$$\boxed{A} = \left[\begin{array}{c:c} U_1 & U_2 \end{array}\right] \cdot \left[\begin{array}{c:c} \overbrace{\Sigma}^{r} & \emptyset \\ \hdashline \emptyset & \emptyset \end{array}\right] \cdot \left[\begin{array}{c} V_1^* \\ \hdashline V_2^* \end{array}\right]$$

$$\text{Rank}(A) = r < n$$

$$\Rightarrow A = U \cdot (\Sigma \cdot V_1^*) = U_1 \cdot (\Sigma \cdot V_1^*) + U_2 \cdot \emptyset$$

As U is unitary

$\Rightarrow$ U_1 is a column-image of A
U_2 is a column-nullspace of A

$$A^* = (U \Sigma V^*)^* = V \Sigma^* U^* = V \Sigma U^*$$

$$\boxed{A^*} = \left[\begin{array}{c:c} V_1 & V_2 \end{array}\right] \cdot \left[\begin{array}{c:c} \overbrace{\Sigma}^{r} & \\ \hdashline & \end{array}\right] \cdot \left[\begin{array}{c} U_1^* \\ \hdashline U_2^* \end{array}\right]$$

$$= \left[\begin{array}{c:c} V_1 & V_2 \end{array}\right] \cdot \left[\begin{array}{c} \Sigma U_1^* \\ \hdashline \emptyset \end{array}\right]$$

As V is unitary

$\Rightarrow$ V_1^* is a row-image of A

V_2^* is a row-nullspace of A.

$\Rightarrow$ One SVD gives all Images and Nullspaces at once. (This is even better, though more expensive, than QR.)