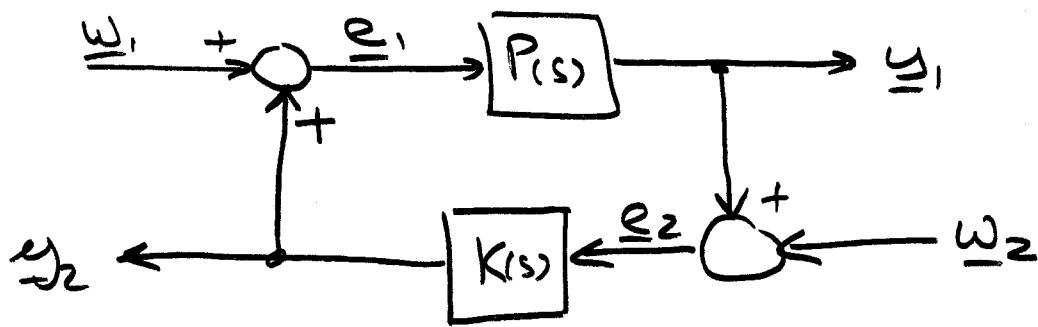


Stability and Performance

Given a plant $P(s)$ with a stabilizing controller $K(s)$:



The controller is well-posed if there are no "short circuits" in the feedback loop:

$$\begin{cases} \underline{e}_1 = \underline{\omega}_1 + K \underline{e}_2 \\ \underline{e}_2 = \underline{\omega}_2 + P \underline{e}_1 \end{cases}$$

$$\Rightarrow \underline{e}_1 = \underline{\omega}_1 + K \underline{\omega}_2 + K P \underline{e}_1,$$

$$\Rightarrow (I - K P) \underline{e}_1 = \underline{\omega}_1 + K \underline{\omega}_2$$

can be solved for \underline{e}_1 iff

$(I - K P)$ is invertible, or

$(I - K(\infty) \cdot P(\infty))$ is invertible.

∴

$$\begin{cases} \underline{\epsilon}_1 - K \underline{\epsilon}_2 = \underline{\Sigma}_1 \\ \underline{\epsilon}_2 - P \underline{\epsilon}_1 = \underline{\omega}_2 \end{cases}$$

$$\begin{bmatrix} H & -K \\ -P & H \end{bmatrix} \begin{bmatrix} \underline{\epsilon}_1 \\ \underline{\epsilon}_2 \end{bmatrix} = \begin{bmatrix} \underline{\Sigma}_1 \\ \underline{\omega}_2 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} H & -K \\ -P & H \end{bmatrix}$ must be invertible,

or (simpler to compute):

$$\begin{bmatrix} H & -K(\infty) \\ -P(\infty) & H \end{bmatrix} \text{ must have an inverse.}$$

In the time domain:

$$P = \begin{bmatrix} A_p & B_p \\ C_p & D_p \end{bmatrix}; \quad K = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$$

$$\begin{cases} \dot{\underline{x}}_p = A_p \underline{x}_p + B_p \underline{\epsilon}_1 \\ \underline{\epsilon}_2 = C_p \underline{x}_p + D_p \underline{\epsilon}_1 + \underline{\omega}_2 \\ \dot{\underline{x}}_k = A_k \underline{x}_k + B_k \underline{\epsilon}_2 \\ \underline{\epsilon}_1 = C_k \underline{x}_k + D_k \underline{\epsilon}_2 + \underline{\Sigma}_1 \end{cases}$$

$$\Rightarrow \begin{cases} \underline{e}_1 - D_K \underline{e}_2 = C_K \underline{x}_K + \underline{\omega}_1 \\ -D_P \underline{e}_1 + \underline{e}_2 = C_P \underline{x}_P + \underline{\omega}_2 \end{cases}$$

$$\Rightarrow \begin{bmatrix} I & -D_K \\ -D_P & I \end{bmatrix} \cdot \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} = \begin{bmatrix} \emptyset & C_K \\ C_P & \emptyset \end{bmatrix} \cdot \begin{bmatrix} \underline{x}_P \\ \underline{x}_K \end{bmatrix} + \begin{bmatrix} \underline{\omega}_1 \\ \underline{\omega}_2 \end{bmatrix}$$

is well-posed, iff

$$\begin{bmatrix} I & -D_K \\ -D_P & I \end{bmatrix} \text{ is invertible.}$$

We shall now assume the well-posedness of a system.

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} = \begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} \begin{bmatrix} \underline{\omega}_1 \\ \underline{\omega}_2 \end{bmatrix}$$

is stable if the transfer function matrix:

$$\begin{bmatrix} I & -K(s) \\ -P(s) & I \end{bmatrix}^{-1}$$

has all poles in the LHP, i.e., belongs to \mathcal{J}_{∞} .

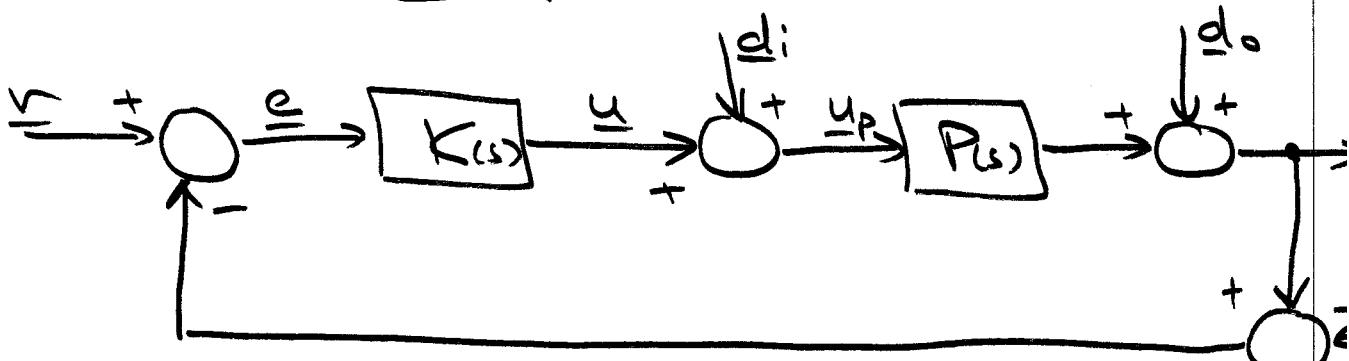
500 SHEETS FOLDED 5x5 SQUARE
50 SHEETS EASY EASY 5x5 SQUARE
100 SHEETS EASY EASY 5x5 SQUARE
42-3886 200 SHEETS EASY EASY 5x5 SQUARE
42-3925 100 RECYCLED WHITE 5x5 SQUARE
42-3999 200 RECYCLED WHITE 5x5 SQUARE
Model No. 5 A



For reference:

$$\begin{aligned} \begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} &= \begin{bmatrix} (I-KP)^{-1} & K(I-PK)^{-1} \\ P(I-KP)^{-1} & (I-PK)^{-1} \end{bmatrix} \\ &\equiv \begin{bmatrix} I + K(I-PK)^{-1}P & K(I-PK)^{-1} \\ (I-PK)^{-1}P & (I-PK)^{-1} \end{bmatrix} \\ &\equiv \begin{bmatrix} (I-KP)^{-1} & (I-KP)^{-1}K \\ P(I-KP)^{-1} & I + P(I-KP)^{-1}K \end{bmatrix} \end{aligned}$$

Given the system:



We define:

$$L_i = KP$$

: input loop transfer matrix

$$L_o = PK$$

: output loop transfer matrix

$$S_i = (I + L_i)^{-1}$$

: input sensitivity matrix

$$S_o = (I + L_o)^{-1}$$

: output sensitivity matrix

$$T_i = I - S_i = L_i (I + L_i)^{-1} : \text{input complement sensitivity mat}$$

$$T_o = I - S_o = L_o (I + L_o)^{-1} : \text{output complement sensitivity mat}$$

$$\underline{y} = T_o (\underline{I} - \underline{n}) + S_o P \underline{d}_i + S_o \underline{d}_o$$

$$(\underline{r} - \underline{y}) = S_o (\underline{I} - \underline{d}) + T_o \underline{n} - S_o P \underline{d}_i$$

$$\underline{u} = K S_o (\underline{I} - \underline{n}) - K S_o \underline{d}_o - T_i \underline{d}_i$$

$$\underline{u}_p = K S_o (\underline{I} - \underline{n}) - K S_o \underline{d}_o - S_i \underline{d}_i$$

\Rightarrow All internal signals can be computed from transfer function that either belong to :

$$P, K, L_i, L_o, S_i, S_o, T_i, T_o$$

or are products of combinations of the above .

\Rightarrow These transfer functions characterize the system completely.

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42-385 100 SHEETS REVERSE 5 SQUARE
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There is some redundancy,
Because:

$$\begin{aligned} P(I+KP)^{-1} &= P(I+L_i)^{-1} = P \cdot S_i \\ \equiv (I+PK)^{-1}P &= (I+L_o)^{-1}P = S_o \cdot P \\ &\quad \underline{\text{etc.}} \end{aligned}$$

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$$\underline{y} = \bar{T}_o(\underline{e}-\underline{d}) + S_o P \underline{d}_i + S_o \underline{d}_o$$

→ Influence of output disturbance
on system output:

$$\underline{y} = S_o \underline{d}_o$$

$$\Rightarrow \|\underline{y}\|_2 \leq \|S_o\|_\infty \cdot \|\underline{d}_o\|_2$$

In order to keep the influence
small, we must make the
infinity norm of S_o small,
e.g.

$$\|S_o\|_\infty = \overline{\sigma}(S_o) < 1$$

Usually, we are interested to suppress disturbances in a certain frequency range:

$$\bar{\sigma}(S_0) < 1 ; \forall \omega \in [\omega_{\min}, \omega_{\max}]$$

Similarly, to suppress the influence of the input disturbance on the plant:

$$u_p = -S_i d_i$$

$$\Rightarrow \|u_p\|_2 < \|S_i\|_\infty \cdot \|d_i\|_2$$

$$\rightarrow \bar{\sigma}(S_i) < 1 ; \forall \omega \in [\omega_{\min}, \omega_{\max}]$$

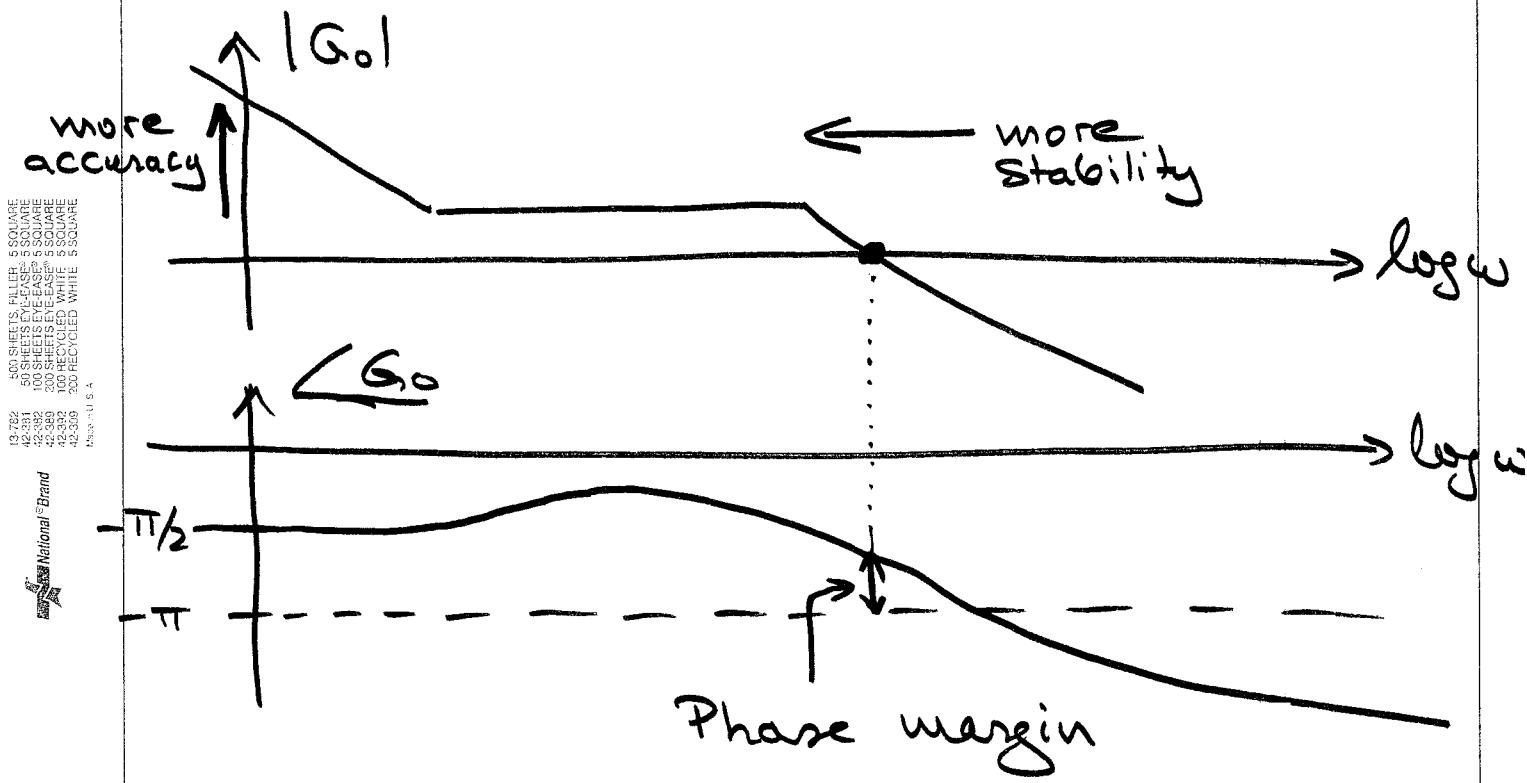
$$\bar{\sigma}(S_0) = \bar{\sigma}((I + PK)^{-1}) = \frac{1}{\sigma(I + PK)}$$

$$\underline{\sigma}(PK) - 1 \leq \sigma(I + PK) \leq \bar{\sigma}(PK) + 1$$

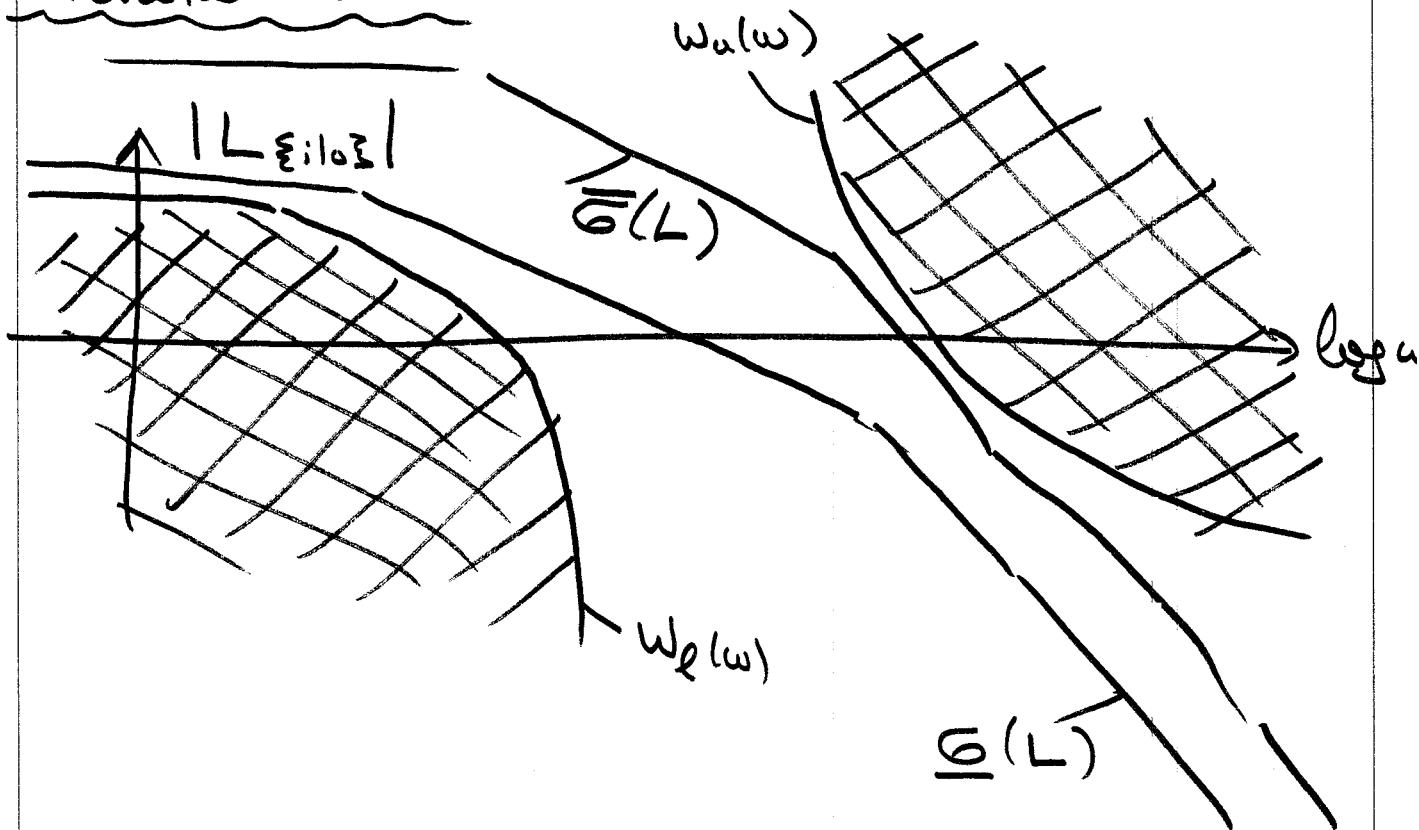
$$\Rightarrow \frac{1}{\bar{\sigma}(PK) + 1} \leq \bar{\sigma}(S_0) \leq \frac{1}{\underline{\sigma}(PK) - 1}$$

$$\Rightarrow \bar{\sigma}(S_0) < 1 \Leftrightarrow \underline{\sigma}(PK) \gg 1$$

From ECE 441:



Generalization:



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13-2626 100 RECYCLED WHITE 5 SQUARE
13-2627 200 RECYCLED WHITE 5 SQUARE

Performance (accuracy), attenuation of disturbances.

$$\underline{\sigma}(L_o) > w_e(\omega)$$

$$\underline{\sigma}(L_i) > w_e(\omega)$$

Stability:

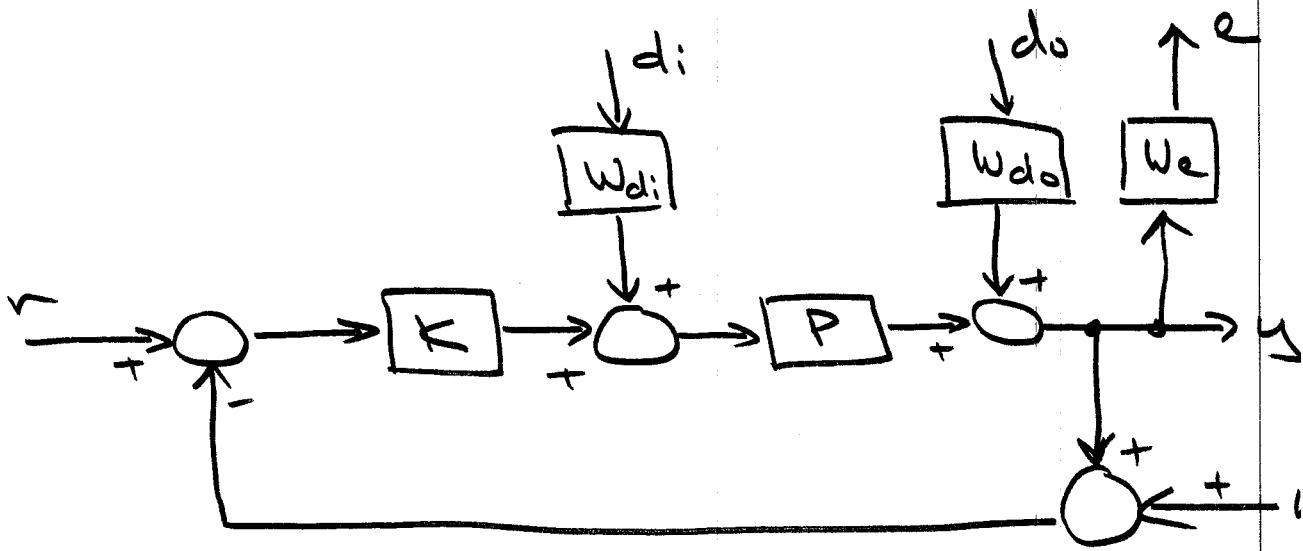
$$\bar{\sigma}(L_o) < w_u(\omega)$$

$$\bar{\sigma}(L_i) < w_{ue}(\omega)$$

Sometimes, more knowledge is available about a disturbance, which can be described by a transfer function $W_d(s)$, or not all frequencies are equally important. Also, the signal to be kept small may be another signal than the output, usually an error signal, e , that can be related to the output by $W_e(s)$.

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42-5352 100 RECYCLED WHITE 5x5 SQUARE
42-5353 200 RECYCLED WHITE 5x5 SQUARE
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42-352 100 RECYCLED WHITE 5 SQUARE
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In those cases, replace

$$\|S_0\|_\infty \rightarrow \|W_e S_0 W_{d0}\|_\infty$$

$$\|S_i\|_\infty \rightarrow \|W_e S_i W_{di}\|_\infty$$

etc.

Coprime Factorization:

We need a mathematical tool that will allow us to characterize sets of controllers, sets of plants, or sets of disturbances.

Two polynomials $m(s)$ and $n(s)$ are relative coprime, if there exist two other polynomials $x(s)$ and $y(s)$, such that :

$$x(s) \cdot m(s) + y(s) \cdot n(s) \equiv 1$$

This is called Bezout identity.

Proof: (trivial)

Assume: $m(s)$ and $n(s)$ have a common factor $t(s)$:

$$m(s) = \hat{m}(s) \cdot t(s)$$

$$n(s) = \hat{n}(s) \cdot t(s)$$

$$\Rightarrow x(s) \cdot m(s) + y(s) \cdot n(s)$$

$$= x(s) \cdot \hat{m}(s) \cdot t(s) + y(s) \cdot \hat{n}(s) \cdot t(s)$$

$$= [x(s) \cdot \hat{m}(s) + y(s) \cdot \hat{n}(s)] \cdot t(s) \neq 1$$

Generalization:

$M(s)$ and $N(s)$ are relative right coprime, if there exist

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matrices $X_r(s)$ and $Y_r(s)$, such that:

$$X_r(s) \cdot M(s) + Y_r(s) \cdot N(s) = I^{(n)}$$

or:

$$\begin{bmatrix} X_r(s) & Y_r(s) \end{bmatrix} \cdot \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = I^{(n)}$$

Similarly, $M(s)$ and $N(s)$ are relative left coprime, if there exist $X_e(s)$ and $Y_e(s)$, such that:

$$M(s) \cdot X_e(s) + N(s) \cdot Y_e(s) = I^{(n)}$$

or:

$$\begin{bmatrix} M(s) & N(s) \end{bmatrix} \cdot \begin{bmatrix} X_e(s) \\ Y_e(s) \end{bmatrix} = I^{(n)}$$

Generalization:

Given $P(s)$ a proper real-rational matrix. We call a right-coprime factorization of $P(s)$ a set of matrices $M(s)$ and $N(s)$, such that

$$P(s) = N(s) \cdot M^{-1}(s).$$

Among all those factorizations, we are particularly interested in those where:

$N(s), M(s)$ are proper real-rational
and: $N(s), M(s)$ are stable.

Example:

$$P(s) = \frac{1}{s-1} \quad \text{is } \left\{ \begin{array}{l} \text{proper} \\ \text{real-rational} \\ \text{unstable.} \end{array} \right.$$

$$\Rightarrow m(s) = \frac{1}{s+1} \quad n(s) = \frac{s-1}{s+1} \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

proper
real-rational
stable

$$m(s) \cdot n^{-1}(s) = \frac{1}{s+1} \cdot \frac{s+1}{s-1} = \frac{1}{s-1} = p(s)$$

✓

$$x(s) = 2$$

$$y(s) = 1$$

$$\Rightarrow x(s)m(s) + y(s)n(s) =$$

$$2 \cdot \frac{1}{s+1} + 1 \cdot \frac{s-1}{s+1} =$$

$$\frac{2}{s+1} + \frac{s-1}{s+1} = \frac{2+s-1}{s+1} = \frac{s+1}{s+1} = 1$$

$\Rightarrow m(s), n(s)$ are coprime.

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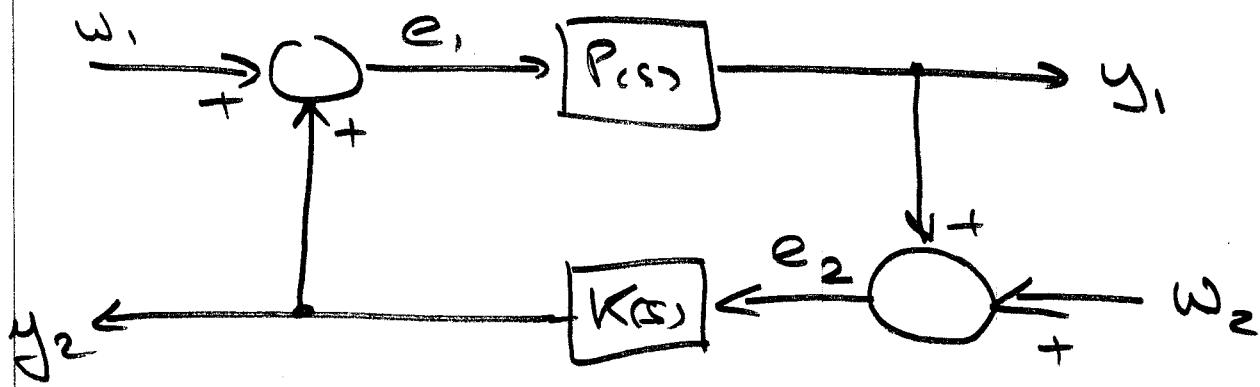


Similarly :

$$P(s) = \tilde{M}^{-1}(s) \cdot \tilde{N}(s)$$

is a left coprime factorization.

Given :



Find : $P(s) = N(s) \cdot M^{-1}(s) = \tilde{M}^{-1}(s) \cdot \tilde{N}(s)$

$$K(s) = U(s) \cdot V^{-1}(s) = \tilde{V}^{-1}(s) \cdot \tilde{U}(s)$$

We want :

- All pairs are coprime
- All individual decomposition matrices are stable
- K stabilizing P

and :