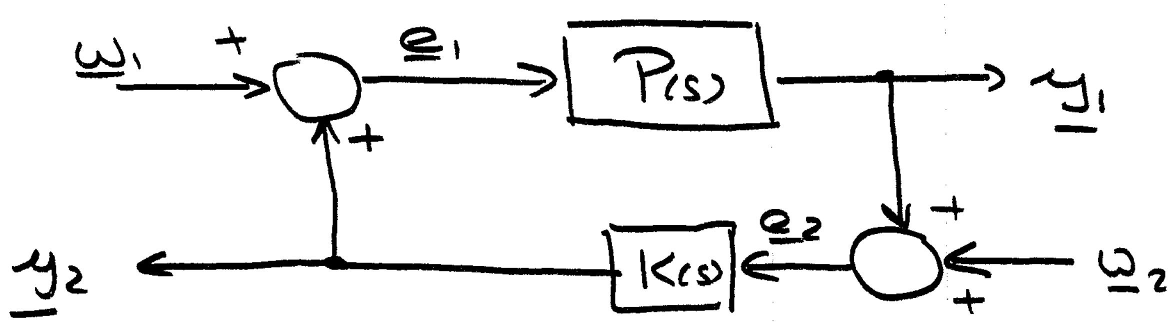
Characterization of ALL Stabilizing Controllers

Given:



Question:

Can we describe the set of <u>ALL</u> controllers $K(s)$, such that the closed-loop system is stable for a given plant ?

(I) Given <u>any</u> sets of stable controller poles, P_c, and observer poles, P_o, find feedback matrices:

$$F = -\,\text{place}(A_p, B_p, P_c)$$
$$L = -\,\text{place}(A_p^*, C_p^*, P_o)^*$$

Then:

$$K(s) = \left[\begin{array}{c|c} A_p + B_p \cdot F + L \cdot C_p + L \cdot D_p \cdot F & -L \\ \hline F & \phi \end{array}\right]$$

The sets of stable $\{P_c, P_o\}$ characterize (parametrize) ALL stable $K(s)$.

Disadvantages:

(i) It takes 2 parameters rather than 1 to characterize all stable controllers.

(ii) The algorithm to compute $K(s)$ given P_c and P_o is complicated.

(iii) The algorithm cannot be inverted, i.e., given a stabilizing $K(s)$, it is not possible to say what the f_c and f_o were that produce $K(s)$.

(II) Similarly: Given any sets of $Q \geq \emptyset$, $R_i > \emptyset$, $R_o > \emptyset$

Compute:

$$F = -\mathrm{lqr}(A_p, B_p, Q, R_i)$$
$$L = -\mathrm{lqr}(A_p^*, C_p^*, Q, R_o)^*$$

Then:

$K(s)$ in the same way as before.

Again: The sets of all positive semidefinite Q and positive definite R_i, R_o characterize **ALL** stabilizing controllers $K(s)$.

[Non-trivial to prove !!].

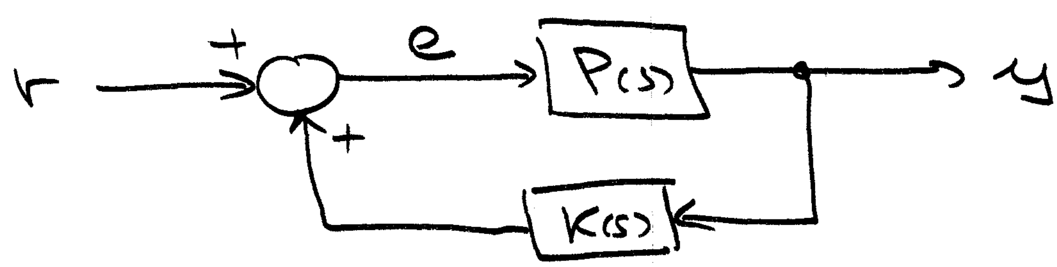
$\Rightarrow$ Same disadvantages!

(III) We compute:

$$\begin{bmatrix} I & -K(s) \\ -P(s) & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-KP)^{-1} & K(I-PK)^{-1} \\ P(I-KP)^{-1} & (I-PK)^{-1} \end{bmatrix}$$

All $K(s)$ that make the above matrix stable are stabilizing controllers. $\Rightarrow$ <u>Still too complicated!</u>

(IV) Let us assume for now, $P(s)$ <u>is</u> stable and SISO.



Since $P(s)$ is stable, $y(s)$ is bounded iff $e(s)$ is bounded.

$$e(s) = \frac{1}{1 - K(s)P(s)}$$

$$\Rightarrow \quad e(s) - K(s)P(s)e(s) = 1$$

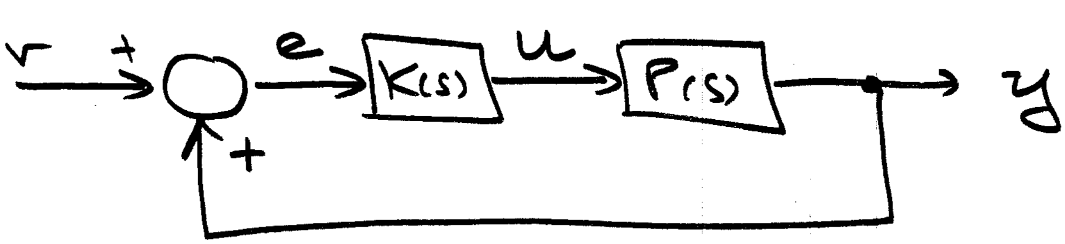
$$\Rightarrow \quad K(s)P(s)\,e(s) = e(s) - 1$$

$$\Rightarrow \quad K(s) = \frac{e(s) - 1}{P(s) \cdot e(s)}$$

All bounded $e(s)$ that are used to compute $K(s)$ using the above formula characterize the set of stabilizing $K(s)$.

(V) For stability, it doesn't really matter where the input is.

$\Rightarrow$ Equivalent problem:



Since $P(s)$ is stable, $y(s)$ is bounded if $u(s)$ is bounded.

$$U(s) = \frac{K(s)}{1 - K(s)P(s)}$$

$\Rightarrow$ Closed-loop system is stable iff

$$Q = \frac{K}{1-KP}$$

is stable.

$$\Rightarrow \quad \underline{\underline{K = \frac{Q}{1+PQ}}}$$

(VI) Generalization: $P(s)$ is no longer assumed stable and SISO.

Let: $\left.\begin{array}{l} P = N \cdot M^{-1} \\ K_{nom} = U \cdot V^{-1} \end{array}\right\}$ right coprime factorization

$\uparrow$ nominal controller

$$\Rightarrow K_{all} = (U + MQ)(V+NQ)^{-1} \quad \forall Q \begin{cases}\text{stab} \\ \text{prop}\end{cases}$$

is called Youla-Kucera parametrization.

Ref: Youla, Jabri, Bongiorno, 1976
Kucera, 1975
Desoer, Liu, Murray, Saeks, 1980

Special case: $P(s)$ is stable.

$\Rightarrow$ We don't need feedback to stabilize!

$$P = N \cdot M^{-1} \Rightarrow \begin{cases} P = N \\ M = I \end{cases}$$

$$K_{nom} = U \cdot V^{-1} = \phi \Rightarrow \begin{cases} U = \phi \\ V = I \end{cases}$$

$$\Rightarrow K_{all} = (\phi + I \cdot Q)(I + PQ)^{-1}$$

$$\Rightarrow K_{all} = Q(I + PQ)^{-1}$$

as before.

(VII) Similarly:

$$\left.\begin{array}{l} P = \tilde{M}^{-1} \cdot \tilde{N} \\ K_{nom} = \tilde{V}^{-1} \cdot \tilde{U} \end{array}\right\} \text{left coprime factorization}$$

$$\Rightarrow K_{all} = \left(\tilde{V} + \tilde{Q} \cdot \tilde{N}\right)^{-1} \cdot \left(\tilde{U} + \tilde{Q} \cdot \tilde{M}\right)$$

$$\forall\ \tilde{Q} = \begin{cases} \text{stable} \\ \text{proper} \end{cases}$$

also characterize the set of stabilizing controllers.

Notice: K_{all} is still a coprime factorization:

$$K_{all} = U_{all} \cdot V_{all}^{-1}$$

where:
$$U_{all} = U + MQ$$
$$V_{all} = V + NQ$$

etc.

Corollary: Because of symmetry, we can also turn the question around:

Given a $K(s)$, what is the set of plants $P(s)$ that are stabilized by $K(s)$.

$\Rightarrow$ The answer must be the symmetric one:

$$P_{nom} = N \cdot M^{-1}$$
$$K = U \cdot V^{-1}$$

$$\Rightarrow P_{all} = (N + VQ)(M + UQ)^{-1}$$
$$= N_{all} \cdot M_{all}^{-1}$$

with: $N_{all} = N + VQ$
$M_{all} = M + UQ$

is the set of all plants stabilized by controller K. etc.