The Effects of Model Uncertainty

The plant to be controlled is <u>never</u> totally known.

$P(s) :=$ nominal plant

$\Pi(s) := \left\{ \begin{array}{l} \text{all plants that} \\ \text{are possible} \\ \text{given the existing} \\ \text{knowledge} \end{array} \right.$

Additive uncertainty:

$$\Pi(s) = P(s) + W_1(s)\,\Delta(s)\,W_2(s)$$

$W_1(s), W_2(s)$ are known transfer function

$\Delta(s)$ is the uncertainty, normalized to $\|\Delta(s)\|_\infty = 1$

Multiplicative Uncertainty:

$$\Pi(s) = [I + W_1(s)\Delta(s)W_2(s)]P(s)$$

$$\|\Delta(s)\|_\infty = 1$$

<u>Example</u>:

a_k is known to 10%

$$\Rightarrow \quad \alpha_k = (1 + 0.1 \cdot \Delta)\, a_k$$

<u>Definitions</u>:

<u>Nominal Stability</u>:

K(s) internally stabilizes P(s)

<u>Robust Stability</u>:

K(s) internally stabilizes $\Pi(s)$

<u>Nominal Performance</u>:

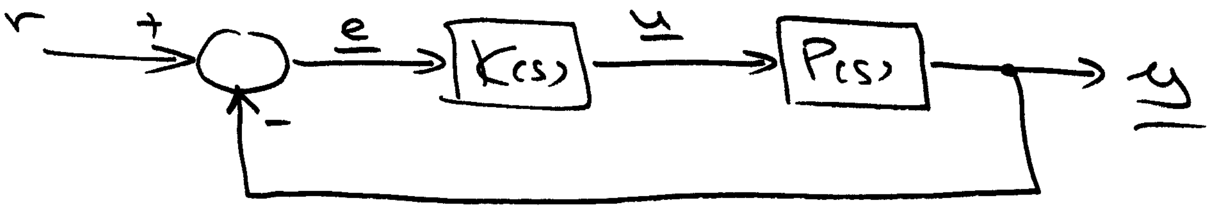
Performance criteria satisfied for $\langle P(s), K(s) \rangle$

<u>Robust Performance</u>:

Performance criteria satisfied for $\langle \Pi(s), K(s) \rangle$.

Robust Stability under unstructured additive model uncertainty

Remember:



is stable and well-posed iff

$$\begin{bmatrix} I & K \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I+KP)^{-1} & K(I+PK)^{-1} \\ P(I+KP)^{-1} & (I+PK)^{-1} \end{bmatrix}$$

is stable.

For robust stability, we replace $P \to \Pi$.

$$\begin{bmatrix} I & K \\ -\Pi & I \end{bmatrix}^{-1} = \begin{bmatrix} (I+KP+KW_1\Delta W_2)^{-1} & K(I+PK+W_1\Delta W_2 \\ P(I+KP+KW_1\Delta W_2)^{-1} & (I+PK+W_1\Delta W_2K) \end{bmatrix}$$

Let us look at the first term:

$$(I + KP + KW_1 \Delta W_2)^{-1}$$

$$\equiv \left[(I+KP)\left[I + (I+KP)^{-1} K W_1 \Delta W_2 \right] \right]^{-1}$$

$$\equiv \left[(I+KP)\left[I + K(I+PK)^{-1} W_1 \Delta W_2 \right] \right]^{-1}$$

$$\equiv \left[I + K(I+PK)^{-1} W_1 \Delta W_2 \right]^{-1} \cdot \left[I + KP \right]^{-1}$$

$$\equiv \underbrace{\left[I + K S_o W_1 \Delta W_2 \right]^{-1}}_{\text{must be stable for robust stability}} \cdot \underbrace{S_i}_{\text{stable, because nominal stability is already assumed}}$$

$\Downarrow$

guaranteed if

$$\| K S_o W_1 \Delta W_2 \|_\infty < 1$$

$$\Rightarrow \| \Delta W_2 K S_o W_1 \|_\infty < 1$$

$$\Rightarrow \underline{\underline{\| W_2 K S_o W_1 \|_\infty < 1}}$$

Now, we can return to the problem of <u>nominal stability</u>.

Among all the stabilizing controllers $\hat{K}(s)$ stabilizing the nominal plant $P(s)$, we pick one that satisfies:

$$\| W_2 K S_o W_1 \|_\infty < 1$$

$\Longrightarrow$ we guarantee that $K(s)$ not only stabilizes $P(s)$, but even $\Pi(s)$.

$\Longrightarrow$ <u>robust stability</u>!

<u>Robust stability under unstructured multiplicative model uncertainty.</u>

$$\Rightarrow (I+K\Pi)^{-1} = (I+K(I+W_1\Delta W_2)P)^{-1}$$

$$= (I+KP+KW_1\Delta W_2 P)^{-1}$$

$$\equiv \left[(I+KP)\left[I + (I+KP)^{-1} K W_1 \Delta W_2 P \right] \right]^{-1}$$

$$= \left[I + (I+KP)^{-1} K W_1 \Delta W_2 P \right]^{-1} \cdot \left[I+KP \right]^{-1}$$

$$= \underbrace{\left[I + K(I+PK)^{-1} W_1 \Delta W_2 P \right]^{-1}}_{\text{must be stable}} \cdot \underbrace{S_i}_{\substack{\text{stability}\\ \text{assumed}}}$$

$$\Downarrow$$

$$\| K(I+PK)^{-1} W_1 \Delta W_2 P \|_\infty < 1$$

$$\Rightarrow \quad \| \Delta W_2 P K (I+PK)^{-1} W_1 \|_\infty < 1$$

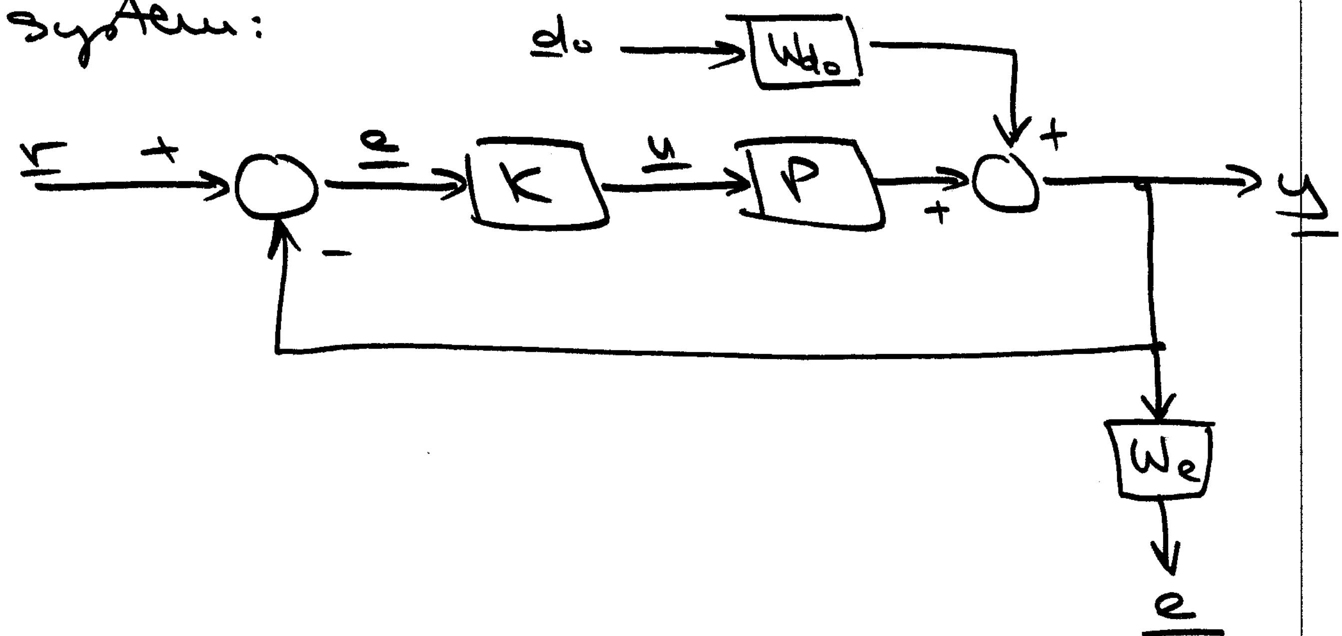
$$\Rightarrow \quad \| W_2 P K (I+PK)^{-1} W_1 \|_\infty < 1$$

$$\Rightarrow \quad \| W_2 T_0 W_1 \|_\infty < 1$$

Corollary:

Remember (p. 43)

For nominal performance of the system:



We required:

$$\underline{\underline{\| W_e S_o W_{do} \|_\infty < 1}}$$

We can reformulate the robust stability problem as a nominal performance problem, e.g. with:

$$\left.\begin{array}{l} W_e \equiv W_2 K \\ W_{do} \equiv W_1 \end{array}\right\} \text{ in case of additive uncertainty}$$

or: $\left.\begin{array}{l} W_e \equiv W_2 P K \\ W_{d_o} \equiv W_1 \end{array}\right\}$ in case of multiplicative uncertainty.

etc.

$\Rightarrow$ Every robust stability problem can be reformulated as a nominal performance problem.

Robust Performance:

Earlier, we had seen that nominal performance requires:

$$\| W_e S_o W_{do} \|_\infty < 1$$

$$\Rightarrow \| W_e (I + PK)^{-1} W_{do} \|_\infty < 1$$

For robust performance, we replace $P \rightarrow \Pi$

$$\Rightarrow \sup_{\forall \Delta} \| W_e (I + \Pi K)^{-1} W_{do} \|_\infty < 1$$

Let us assume multiplicative uncertainty:

$$\Pi = (I + W_1 \Delta W_2) P$$

$$\Rightarrow \sup_{\substack{\forall \Delta \\ \|\Delta\|_\infty < 1}} \left\| W_e \left(I + (I + W_1 \Delta W_2) P K\right)^{-1} W_{do} \right\| <$$

Let us look at:

$$\left[I + (I + W_1 \Delta W_2) P K\right]^{-1}$$

$$\equiv \left[I + PK + W_1 \Delta W_2 P K\right]^{-1}$$

$$\equiv \left[\left(I + W_1 \Delta W_2 P K (I + PK)^{-1}\right)(I + PK)\right]^{-}$$

$$\equiv S_o \cdot \left[I + W_1 \Delta W_2 T_o\right]^{-1}$$

$$\Rightarrow \left\| W_e S_o (I + W_1 \Delta W_2 T_o)^{-1} W_{do} \right\|_\infty < 1$$

$$\| W_e S_o W_{do} \|_\infty < 1$$

already satisfied because of nominal performance

$$\Rightarrow \quad \| W_2 T_o W_1 \|_\infty < 1$$

guarantees robust performance.

<u>Recipe</u>:

- We start with the <u>nominal</u> plant $P(s)$. We design a nominal controller that gives us good stability/performance.
- We then parameterize the set of <u>all</u> controllers that give us good stability/performance for the nominal plant.

- Among those, we pick one that gives us also robust stability.
- Robust stability together with nominal performance implies robust performance.