Numerical Simulation of Dynamic Systems: Hw1 - Problem

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[H1.2] Discretization of State Equations

Given the following explicit ODE model:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A} \cdot \mathbf{x} + \mathbf{b} \cdot u \\ y &= \mathbf{c}' \cdot \mathbf{x} + d \cdot u \end{aligned}$$

where:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -4 & -5 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{c}' = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \qquad d = 10$$

Engineers would usually call such a model a *linear single-input, single-output (SISO) continuous-time state–space model*.

[H1.2] Discretization of State Equations II

We wish to simulate this model using the following integration algorithm:

$$\mathbf{x_{k+1}} = \mathbf{x_k} + h \cdot \dot{\mathbf{x}}_\mathbf{k}$$

which is known as the *Forward Euler (FE) integration algorithm*. If $\mathbf{x_k}$ denotes the state vector at time t^*:

$$\mathbf{x_k} = \mathbf{x}(t)\Big|_{t=t^*}$$

then $\mathbf{x_{k+1}}$ represents the state vector one time step later:

$$\mathbf{x_{k+1}} = \mathbf{x}(t)\Big|_{t=t^*+h}$$

[H1.2] Discretization of State Equations III

Obtain an *explicit difference equation (ΔE) model* by substituting the state equations into the integrator equations. You obtain a model of the type:

$$\begin{aligned} \mathbf{x_{k+1}} &= \mathbf{F} \cdot \mathbf{x_k} + \mathbf{g} \cdot u_k \\ y_k &= \mathbf{h}' \cdot \mathbf{x_k} + i \cdot u_k \end{aligned}$$

which engineers would normally call a *linear single-input, single-output (SISO) discrete-time state-space model*.

Let $h = 0.01\ sec$, $t_f = 5\ sec$, $u(t) = 5 \cdot sin(2t)$, $x_0 = \mathrm{ones}(4,1)$, where t_f denotes the final time of the simulation.

Simulate the ΔE model using MATLAB by iterating over the difference equations. Plot the output variable as a function of time.

[H1.4] Van-der-Pol Oscillator and Time Reversal

Given the following non-linear system:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

This system exhibits an oscillatory behavior. It is commonly referred to as the *Van-der-Pol oscillator*. We wish to simulate this system with $\mu = 2.0$ and $x_0 = \dot{x}_0 = 0.1$.

Draw a block diagram of this system. The output variable is x. The system is autonomous, i.e., it doesn't have an input variable.

Derive a state-space description of this system. To this end, choose the outputs of the two integrators as your two state variables.

[H1.4] Van-der-Pol Oscillator and Time Reversal II

Simulate the system across 2 *sec* of simulated time. Since the system is non-linear, you cannot use MATLAB's lsim function. Use function ode45 instead.

At time $t = 2.0$ *sec*, apply the time reversal algorithm, and simulate the system further across another 2 *sec* of simulated time. This is best accomplished by adjusting the model such that it contains a factor c in front of each state equation. $c = +1$ during the first 2 *sec* of simulated time, and $c = -1$ thereafter. You can interpret c as an input variable to the model. Make sure that $t = 2.0$ *sec* defines an output point.

As you simulate the system backward through time for the same time period that you previously used to simulate the system forward through time, the final values of your two state variables ought to be identical to the initial values except for numerical inaccuracies of the simulation. Verify that this is indeed the case. How large is the accumulated error of the final values? The accumulated simulation error is defined as the norm of the difference between final and initial values.

Plot $x(t)$ and $\dot{x}(t)$ on the same graph.

[H1.4] Van-der-Pol Oscillator and Time Reversal III

Repeat the previous experiment, this time simulating the system forward during 20 *sec* of simulated time, then backward through another 20 *sec* of simulated time.

What do you conclude?