

Numerical Simulation of Dynamic Systems: Hw3 - Problem

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[H3.4] RK Order Increase by Blending

Given two separate n^{th} -order accurate RK algorithms in at least $(n + 1)$ stages:

$$f_1(q) = 1 + q + \frac{q^2}{2!} + \cdots + \frac{q^n}{n!} + c_1 \cdot q^{n+1}$$

$$f_2(q) = 1 + q + \frac{q^2}{2!} + \cdots + \frac{q^n}{n!} + c_2 \cdot q^{n+1}$$

where $c_2 \neq c_1$.

Show that it is always possible to use blending:

$$\mathbf{x}^{\text{blended}} = \vartheta \cdot \mathbf{x}^1 + (1 - \vartheta) \cdot \mathbf{x}^2$$

where $\mathbf{x}^1$ is the solution found using method $f_1(q)$ and $\mathbf{x}^2$ is the solution found using method $f_2(q)$, such that $\mathbf{x}^{\text{blended}}$ is of order $(n + 1)$.

Find a formula for ϑ that will make the blended algorithm accurate to the order $(n + 1)$.

[H3.6] Runge-Kutta Integration

Given the following linear time-invariant continuous-time system:

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{pmatrix} 1250 & -25113 & -60050 & -42647 & -23999 \\ 500 & -10068 & -24057 & -17092 & -9613 \\ 250 & -5060 & -12079 & -8586 & -4826 \\ -750 & 15101 & 36086 & 25637 & 14420 \\ 250 & -4963 & -11896 & -8438 & -4756 \end{pmatrix} \cdot \mathbf{x} + \begin{pmatrix} 5 \\ 2 \\ 1 \\ -3 \\ 1 \end{pmatrix} \cdot u \\ \mathbf{y} &= (-1 \quad 26 \quad 59 \quad 43 \quad 23) \cdot \mathbf{x}\end{aligned}$$

with initial conditions:

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \\ 5 \end{pmatrix}$$

[H3.6] Runge-Kutta Integration II

Simulate the system across **10 seconds** of simulated time with step input using the RK4 algorithm with the α -vector and β -matrix:

$$\alpha = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \\ 1 \end{pmatrix}; \quad \beta = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/6 & 1/3 & 1/3 & 1/6 \end{pmatrix}$$

The following fixed step sizes should be tried:

1. $h = 0.32$,
2. $h = 0.032$,
3. $h = 0.0032$.

Plot the three trajectories on top of each other. What can you conclude about the accuracy of the results?

[H3.12] BI4/5_{0.45} Integration for Linear Systems

Given the following linear time-invariant continuous-time system:

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{pmatrix} 1250 & -25113 & -60050 & -42647 & -23999 \\ 500 & -10068 & -24057 & -17092 & -9613 \\ 250 & -5060 & -12079 & -8586 & -4826 \\ -750 & 15101 & 36086 & 25637 & 14420 \\ 250 & -4963 & -11896 & -8438 & -4756 \end{pmatrix} \cdot \mathbf{x} + \begin{pmatrix} 5 \\ 2 \\ 1 \\ -3 \\ 1 \end{pmatrix} \cdot u \\ \mathbf{y} &= (-1 \quad 26 \quad 59 \quad 43 \quad 23) \cdot \mathbf{x}\end{aligned}$$

with initial conditions:

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \\ 5 \end{pmatrix}$$

[H3.12] BI4/5_{0.45} Integration for Linear Systems II

Simulate the system across **10 seconds** of simulated time with step input using BI4/5_{0.45}. The explicit semi-step uses the fourth-order approximation of RKF4/5. There is no need to compute the fifth-order corrector. The implicit semi-step uses the fifth-order corrector. There is no need to compute the fourth-order corrector. Since the system to be simulated is linear, the implicit semi-step can be implemented using matrix inversion. No step-size control is attempted.

The following fixed step sizes should be tried:

1. $h = 0.32$,
2. $h = 0.032$,
3. $h = 0.0032$.

Plot the three trajectories on top of each other.

[H3.14] BI4/5_{0.45} Integration for Non-linear Systems

Repeat Hw.[H3.12]. This time, we want to replace the matrix inversion by Newton iteration. Of course, since the problem is linear and time-invariant, Newton iteration and modified Newton iteration are identical. Iterate until $\delta_{\text{rel}} \leq 10^{-5}$, where:

$$\delta_{\text{rel}} = \frac{\|\mathbf{x}_{k+\frac{1}{2}}^{\text{right}} - \mathbf{x}_{k+\frac{1}{2}}^{\text{left}}\|_{\infty}}{\max(\|\mathbf{x}_{k+\frac{1}{2}}^{\text{left}}\|_2, \|\mathbf{x}_{k+\frac{1}{2}}^{\text{right}}\|_2, \delta)}$$

Compare the results obtained with those found in Hw.[H3.12].