Numerical Simulation of Dynamic Systems: Hw3 - Problem

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[H3.4] RK Order Increase by Blending

Given two separate n^{th}-order accurate RK algorithms in at least $(n+1)$ stages:

$$\begin{aligned} f_1(q) &= 1 + q + \frac{q^2}{2!} + \cdots + \frac{q^n}{n!} + c_1 \cdot q^{n+1} \\ f_2(q) &= 1 + q + \frac{q^2}{2!} + \cdots + \frac{q^n}{n!} + c_2 \cdot q^{n+1} \end{aligned}$$

where $c_2 \neq c_1$.

Show that it is always possible to use blending:

$$\mathbf{x}^{\mathbf{blended}} = \vartheta \cdot \mathbf{x}^{\mathbf{1}} + (1-\vartheta) \cdot \mathbf{x}^{\mathbf{2}}$$

where $\mathbf{x}^{\mathbf{1}}$ is the solution found using method $f_1(q)$ and $\mathbf{x}^{\mathbf{2}}$ is the solution found using method $f_2(q)$, such that $\mathbf{x}^{\mathbf{blended}}$ is of order $(n+1)$.

Find a formula for ϑ that will make the blended algorithm accurate to the order $(n+1)$.

[H3.6] Runge-Kutta Integration

Given the following linear time-invariant continuous-time system:

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{pmatrix} 1250 & -25113 & -60050 & -42647 & -23999 \\ 500 & -10068 & -24057 & -17092 & -9613 \\ 250 & -5060 & -12079 & -8586 & -4826 \\ -750 & 15101 & 36086 & 25637 & 14420 \\ 250 & -4963 & -11896 & -8438 & -4756 \end{pmatrix} \cdot \mathbf{x} + \begin{pmatrix} 5 \\ 2 \\ 1 \\ -3 \\ 1 \end{pmatrix} \cdot u \\ \mathbf{y} &= \begin{pmatrix} -1 & 26 & 59 & 43 & 23 \end{pmatrix} \cdot \mathbf{x} \end{aligned}$$

with initial conditions:

$$\mathbf{x_0} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \\ 5 \end{pmatrix}$$

[H3.6] Runge-Kutta Integration II

Simulate the system across 10 seconds of simulated time with step input using the RK4 algorithm with the α-vector and β-matrix:

$$\alpha = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \\ 1 \end{pmatrix} ; \quad \beta = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/6 & 1/3 & 1/3 & 1/6 \end{pmatrix}$$

The following fixed step sizes should be tried:

1. $h = 0.32$,
2. $h = 0.032$,
3. $h = 0.0032$.

Plot the three trajectories on top of each other. What can you conclude about the accuracy of the results?

[H3.12] BI4/$5_{0.45}$ Integration for Linear Systems

Given the following linear time-invariant continuous-time system:

$$\dot{\mathbf{x}} = \begin{pmatrix} 1250 & -25113 & -60050 & -42647 & -23999 \\ 500 & -10068 & -24057 & -17092 & -9613 \\ 250 & -5060 & -12079 & -8586 & -4826 \\ -750 & 15101 & 36086 & 25637 & 14420 \\ 250 & -4963 & -11896 & -8438 & -4756 \end{pmatrix} \cdot \mathbf{x} + \begin{pmatrix} 5 \\ 2 \\ 1 \\ -3 \\ 1 \end{pmatrix} \cdot u$$

$$\mathbf{y} = \begin{pmatrix} -1 & 26 & 59 & 43 & 23 \end{pmatrix} \cdot \mathbf{x}$$

with initial conditions:

$$\mathbf{x_0} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \\ 5 \end{pmatrix}$$

[H3.12] BI4/$5_{0.45}$ Integration for Linear Systems II

Simulate the system across 10 seconds of simulated time with step input using BI4/$5_{0.45}$. The explicit semi-step uses the fourth-order approximation of RKF4/5. There is no need to compute the fifth-order corrector. The implicit semi-step uses the fifth-order corrector. There is no need to compute the fourth–order corrector. Since the system to be simulated is linear, the implicit semi-step can be implemented using matrix inversion. No step-size control is attempted.

The following fixed step sizes should be tried:

1. $h = 0.32$,
2. $h = 0.032$,
3. $h = 0.0032$.

Plot the three trajectories on top of each other.

[H3.14] BI4/$5_{0.45}$ Integration for Non-linear Systems

Repeat Hw.[H3.12]. This time, we want to replace the matrix inversion by Newton iteration. Of course, since the problem is linear and time-invariant, Newton iteration and modified Newton iteration are identical. Iterate until $\delta_{\mathrm{rel}} \leq 10^{-5}$, where:

$$\delta_{\mathrm{rel}} = \frac{\|\mathbf{x}^{\mathrm{right}}_{\mathbf{k}+\frac{1}{2}} - \mathbf{x}^{\mathrm{left}}_{\mathbf{k}+\frac{1}{2}}\|_\infty}{\max(\|\mathbf{x}^{\mathrm{left}}_{\mathbf{k}+\frac{1}{2}}\|_2, \|\mathbf{x}^{\mathrm{right}}_{\mathbf{k}+\frac{1}{2}}\|_2, \delta)}$$

Compare the results obtained with those found in Hw.[H3.12].