Numerical Simulation of Dynamic Systems: Hw6 - Problem

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[H5.3] Stability Domain of GE4/AB3

The method introduced in earlier chapters for drawing stability domains was geared towards *linear time-invariant homogeneous multi-variable state-space models*:

$$\dot{\mathbf{x}} = \mathbf{A} \cdot \mathbf{x}$$

We generated real-valued **A**-matrices $\in \Re^{2\times 2}$ with their eigenvalues located on the unit circle, at an angle α away from the negative real axis. We then computed the **F**-matrix corresponding to that **A**-matrix for the given algorithm, and found the largest value of the step size h, for which all eigenvalues of **F** remained inside the unit circle. This gave us one point on the stability domain. We repeated this procedure for all suitable values of the angle α.

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The algorithm needs to be modified for dealing with second derivative systems described by the *linear time-invariant homogeneous multi-variable second-derivative model*:

$$\ddot{\mathbf{x}} = \mathbf{A}^2 \cdot \mathbf{x} + \mathbf{B} \cdot \dot{\mathbf{x}}$$

[H5.3] Stability Domain of GE4/AB3 II

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This can be accomplished using the scalar model:

$$\ddot{x} = a^2 \cdot x + b \cdot \dot{x}$$

where:

$$\begin{aligned} a &= \sqrt{a_{21}} \\ b &= a_{22} \end{aligned}$$

of the formerly used **A**-matrix.

[H5.3] Stability Domain of GE4/AB3 III

Write the GE4/AB3 algorithm as follows:

$$
\begin{aligned}
x_{k+1} &= \frac{20}{11} \cdot x_k - \frac{6}{11} \cdot x_{k-1} - \frac{4}{11} \cdot x_{k-2} + \frac{1}{11} \cdot x_{k-3} + \frac{12 \cdot h^2}{11} \cdot \ddot{x}_k \\
h \cdot \dot{x}_{k+1} &= h \cdot \dot{x}_k + \frac{23 \cdot h^2}{12} \cdot \ddot{x}_k - \frac{4 \cdot h^2}{3} \cdot \ddot{x}_{k-1} + \frac{5 \cdot h^2}{12} \cdot \ddot{x}_{k-2} \\
\ddot{x} &= a^2 \cdot x + b \cdot \dot{x}
\end{aligned}
$$

Substitute the model equation into the two solver equations, and rewrite the resulting equations in a state-space form:

$$\xi_{\mathbf{k+1}} = \mathbf{F} \cdot \xi_{\mathbf{k}}$$

[H5.3] Stability Domain of GE4/AB3 IV

whereby the state vector ξ is chosen as:

$$\xi_{\mathbf{k}} = \begin{pmatrix} x_{k-3} \\ h \cdot \dot{x}_{k-3} \\ x_{k-2} \\ h \cdot \dot{x}_{k-2} \\ x_{k-1} \\ h \cdot \dot{x}_{k-1} \\ x_k \\ h \cdot \dot{x}_k \end{pmatrix}$$

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Draw the stability domain of GE4/AB3 using this approach.

[P5.1] Houbolt's Integration Algorithm

John Houbolt proposed already in 1950 a second-derivative integration algorithm that is very similar to the GI3/BDF2 method introduced in this chapter. *Houbolt's algorithm* can be written as follows:

$$\begin{aligned} \mathbf{x_{k+1}} &= \frac{5}{2} \cdot \mathbf{x_k} - 2 \cdot \mathbf{x_{k-1}} + \frac{1}{2} \cdot \mathbf{x_{k-2}} + \frac{h^2}{2} \cdot \ddot{\mathbf{x}}_{\mathbf{k+1}} \\ h \cdot \dot{\mathbf{x}}_{\mathbf{k+1}} &= \frac{11}{6} \cdot \mathbf{x_{k+1}} - 3 \cdot \mathbf{x_k} + \frac{3}{2} \cdot \mathbf{x_{k-1}} - \frac{1}{3} \cdot \mathbf{x_{k-2}} \end{aligned}$$

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The second derivative formula of Houbolt's algorithm can immediately be identified as **GI3**. The formula used for the velocity vector is **BDF3**; however, the formula was used differently from the way, it had been employed by us in the description of the GI3/BDF2 algorithm. Clearly, the Houbolt algorithm is third-order accurate. Although it would have sufficed to use BDF2 for the velocity vector, nothing would have been gained computationally by choosing the reduced-order algorithm.

[P5.1] Houbolt's Integration Algorithm II

We can transform the Houbolt algorithm to the form that we meanwhile got used to by substituting the GI3 solver into the BDF3 solver to eliminate $\mathbf{x_{k+1}}$ from the latter. The so rewritten Houbolt algorithm assumes the form:

$$\begin{aligned}
\mathbf{x_{k+1}} &= \frac{5}{2} \cdot \mathbf{x_k} - 2 \cdot \mathbf{x_{k-1}} + \frac{1}{2} \cdot \mathbf{x_{k-2}} + \frac{h^2}{2} \cdot \ddot{\mathbf{x}}_{\mathbf{k+1}} \\
h \cdot \dot{\mathbf{x}}_{\mathbf{k+1}} &= \frac{19}{12} \cdot \mathbf{x_k} - \frac{13}{6} \cdot \mathbf{x_{k-1}} + \frac{7}{12} \cdot \mathbf{x_{k-2}} + \frac{11 \cdot h^2}{12} \cdot \ddot{\mathbf{x}}_{\mathbf{k+1}}
\end{aligned}$$

Find the stability domain and damping plot of Houbolt's algorithm, and discuss the properties of this algorithm in the same way, as Newmark's algorithm has been discussed in class.