Numerical Simulation of Dynamic Systems: Hw6 - Solution

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April 3, 2012

[H5.3] Stability Domain of GE4/AB3

The method introduced in earlier chapters for drawing stability domains was geared towards *linear time-invariant homogeneous multi-variable state-space models*:

$$\dot{\mathbf{x}} = \mathbf{A} \cdot \mathbf{x}$$

We generated real-valued **A**-matrices $\in \Re^{2\times 2}$ with their eigenvalues located on the unit circle, at an angle α away from the negative real axis. We then computed the **F**-matrix corresponding to that **A**-matrix for the given algorithm, and found the largest value of the step size *h*, for which all eigenvalues of **F** remained inside the unit circle. This gave us one point on the stability domain. We repeated this procedure for all suitable values of the angle α.

The algorithm needs to be modified for dealing with second derivative systems described by the *linear time-invariant homogeneous multi-variable second-derivative model*:

$$\ddot{\mathbf{x}} = \mathbf{A}^2 \cdot \mathbf{x} + \mathbf{B} \cdot \dot{\mathbf{x}}$$

[H5.3] Stability Domain of GE4/AB3 II

We need to find real-valued **A**- and **B**-matrices such that the second derivative model has its eigenvalues located on the unit circle.

This can be accomplished using the scalar model:

$$\ddot{x} = a^2 \cdot x + b \cdot \dot{x}$$

where:

$$\begin{aligned} a &= \sqrt{a_{21}} \\ b &= a_{22} \end{aligned}$$

of the formerly used **A**-matrix.

[H5.3] Stability Domain of GE4/AB3 III

Write the GE4/AB3 algorithm as follows:

$$
\begin{aligned}
x_{k+1} &= \frac{20}{11} \cdot x_k - \frac{6}{11} \cdot x_{k-1} - \frac{4}{11} \cdot x_{k-2} + \frac{1}{11} \cdot x_{k-3} + \frac{12 \cdot h^2}{11} \cdot \ddot{x}_k \\
h \cdot \dot{x}_{k+1} &= h \cdot \dot{x}_k + \frac{23 \cdot h^2}{12} \cdot \ddot{x}_k - \frac{4 \cdot h^2}{3} \cdot \ddot{x}_{k-1} + \frac{5 \cdot h^2}{12} \cdot \ddot{x}_{k-2} \\
\ddot{x} &= a^2 \cdot x + b \cdot \dot{x}
\end{aligned}
$$

Substitute the model equation into the two solver equations, and rewrite the resulting equations in a state-space form:

$$\xi_{\mathbf{k+1}} = \mathbf{F} \cdot \xi_{\mathbf{k}}$$

[H5.3] Stability Domain of GE4/AB3 IV

whereby the state vector ξ is chosen as:

$$\xi_{\mathbf{k}} = \begin{pmatrix} x_{k-3} \\ h \cdot \dot{x}_{k-3} \\ x_{k-2} \\ h \cdot \dot{x}_{k-2} \\ x_{k-1} \\ h \cdot \dot{x}_{k-1} \\ x_k \\ h \cdot \dot{x}_k \end{pmatrix}$$

The **F**-matrix turns out to be a function of $(a \cdot h)^2$ and of $b \cdot h$.

The remainder of the algorithm remains the same as before.

Draw the stability domain of GE4/AB3 using this approach.

[H5.3] Stability Domain of GE4/AB3 V

The **F**-matrix of **GE4/AB3** is:

$$\mathbf{F} = \begin{pmatrix} \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{I}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} \\ \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{I}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} \\ \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{I}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} \\ \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{I}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} \\ \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{I}^{(n)} & \mathbf{Z}^{(n)} \\ \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{I}^{(n)} \\ -\frac{1}{11}\mathbf{I}^{(n)} & \mathbf{Z}^{(n)} & -\frac{4}{11}\mathbf{I}^{(n)} & \mathbf{Z}^{(n)} & -\frac{6}{11}\mathbf{I}^{(n)} & \mathbf{Z}^{(n)} & \left[\frac{20}{11}\mathbf{I}^{(n)} + \frac{12}{11}(\mathbf{A}h)^2\right] & \frac{12}{11}(\mathbf{B}h) \\ \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \frac{5}{12}(\mathbf{A}h)^2 & \frac{5}{12}(\mathbf{B}h) & -\frac{4}{12}(\mathbf{A}h)^2 & -\frac{4}{12}(\mathbf{B}h) & \frac{23}{12}(\mathbf{A}h)^2 & \left[\mathbf{I}^{(n)} + \frac{23}{12}(\mathbf{B}h)\right] \end{pmatrix}$$

We generate the **A**-matrix as always, then extract:

$$\begin{aligned} a &= a_{21} \cdot h^2 \\ b &= a_{22} \cdot h \end{aligned}$$

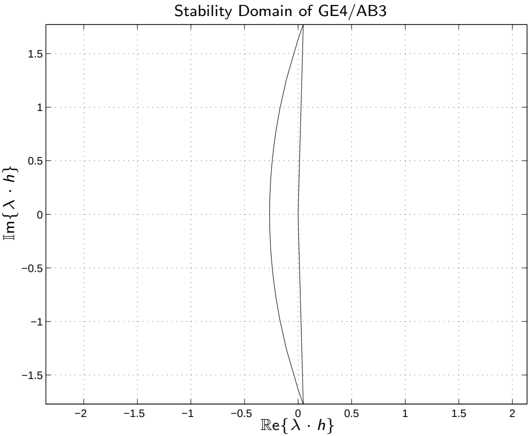
[H5.3] Stability Domain of GE4/AB3 VI

We then rewrite the **F**-matrix as follows:

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{11} & 0 & -\frac{4}{11} & 0 & -\frac{6}{11} & 0 & \left[\frac{20}{11} + \frac{12}{11}a\right] & \frac{12}{11}b \\ 0 & 0 & \frac{5}{12}a & \frac{5}{12}b & -\frac{4}{12}a & -\frac{4}{12}b & \frac{23}{12}a & \left[1 + \frac{23}{12}b\right] \end{pmatrix}$$

[H5.3] Stability Domain of GE4/AB3 VII

We can now apply our standard stability domain plotting routine.



[P5.1] Houbolt's Integration Algorithm

John Houbolt proposed already in 1950 a second-derivative integration algorithm that is very similar to the GI3/BDF2 method introduced in this chapter. *Houbolt's algorithm* can be written as follows:

$$
\begin{aligned}
\mathbf{x_{k+1}} &= \frac{5}{2} \cdot \mathbf{x_k} - 2 \cdot \mathbf{x_{k-1}} + \frac{1}{2} \cdot \mathbf{x_{k-2}} + \frac{h^2}{2} \cdot \ddot{\mathbf{x}}_{\mathbf{k+1}} \\
h \cdot \dot{\mathbf{x}}_{\mathbf{k+1}} &= \frac{11}{6} \cdot \mathbf{x_{k+1}} - 3 \cdot \mathbf{x_k} + \frac{3}{2} \cdot \mathbf{x_{k-1}} - \frac{1}{3} \cdot \mathbf{x_{k-2}}
\end{aligned}
$$

The second derivative formula of Houbolt's algorithm can immediately be identified as **GI3**. The formula used for the velocity vector is **BDF3**; however, the formula was used differently from the way, it had been employed by us in the description of the GI3/BDF2 algorithm. Clearly, the Houbolt algorithm is third-order accurate. Although it would have sufficed to use BDF2 for the velocity vector, nothing would have been gained computationally by choosing the reduced-order algorithm.

[P5.1] Houbolt's Integration Algorithm II

We can transform the Houbolt algorithm to the form that we meanwhile got used to by substituting the GI3 solver into the BDF3 solver to eliminate $\mathbf{x_{k+1}}$ from the latter. The so rewritten Houbolt algorithm assumes the form:

$$\begin{aligned} \mathbf{x_{k+1}} &= \frac{5}{2} \cdot \mathbf{x_k} - 2 \cdot \mathbf{x_{k-1}} + \frac{1}{2} \cdot \mathbf{x_{k-2}} + \frac{h^2}{2} \cdot \ddot{\mathbf{x}}_{\mathbf{k+1}} \\ h \cdot \dot{\mathbf{x}}_{\mathbf{k+1}} &= \frac{19}{12} \cdot \mathbf{x_k} - \frac{13}{6} \cdot \mathbf{x_{k-1}} + \frac{7}{12} \cdot \mathbf{x_{k-2}} + \frac{11 \cdot h^2}{12} \cdot \ddot{\mathbf{x}}_{\mathbf{k+1}} \end{aligned}$$

Find the stability domain and damping plot of Houbolt's algorithm, and discuss the properties of this algorithm in the same way, as Newmark's algorithm has been discussed in class.

[P5.1] Houbolt's Integration Algorithm III

The **F**-matrix of the *Houbolt algorithm* is:

$$\mathbf{F} = \begin{pmatrix} \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{I}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} \\ \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{I}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} \\ \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{I}^{(n)} & \mathbf{Z}^{(n)} \\ \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{Z}^{(n)} & \mathbf{I}^{(n)} \\ \frac{1}{2}\mathbf{I}^{(n)} & \mathbf{Z}^{(n)} & -2\mathbf{I}^{(n)} & \mathbf{Z}^{(n)} & \left[\frac{5}{2}\mathbf{I}^{(n)} + \frac{1}{2}(\mathbf{A}h)^2\right] & \frac{1}{2}(\mathbf{B}h) \\ \frac{7}{12}\mathbf{I}^{(n)} & \mathbf{Z}^{(n)} & -\frac{13}{6}\mathbf{I}^{(n)} & \mathbf{Z}^{(n)} & \left[\frac{19}{12}\mathbf{I}^{(n)} + \frac{11}{12}(\mathbf{A}h)^2\right] & \frac{11}{12}(\mathbf{B}h) \end{pmatrix}$$

We generate the **A**-matrix as always, then extract:

$$\begin{aligned} a &= a_{21} \cdot h^2 \\ b &= a_{22} \cdot h \end{aligned}$$

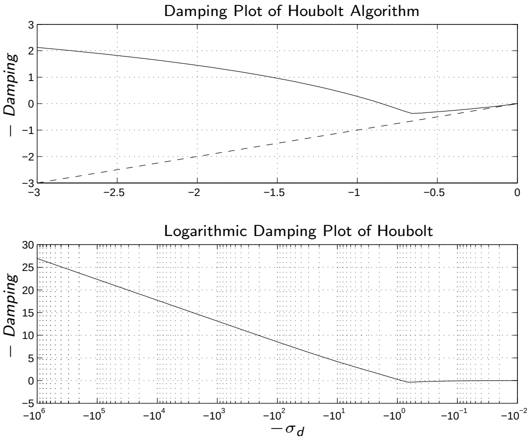
[P5.1] Houbolt's Integration Algorithm IV

We then rewrite the **F**-matrix as follows:

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & -2 & 0 & \left[\frac{5}{2} + \frac{1}{2}a\right] & \frac{1}{2}b \\ \frac{7}{12} & 0 & -\frac{13}{6} & 0 & \left[\frac{19}{12} + \frac{11}{12}a\right] & \frac{11}{12}b \end{pmatrix}$$

[P5.1] Houbolt's Integration Algorithm V

Let us start by drawing the *damping plot* of the Houbolt algorithm:



[P5.1] Houbolt's Integration Algorithm VI

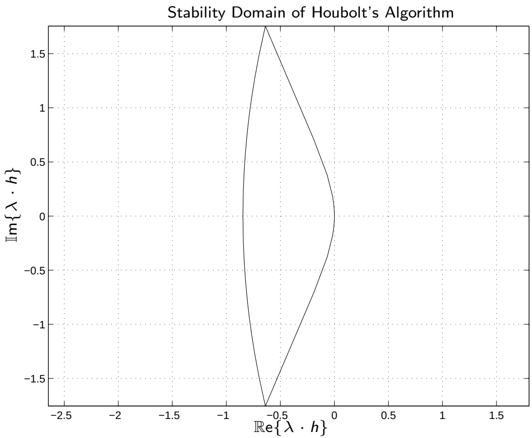
- Houbolt's algorithm exhibits negative damping farther out along the negative real axis. Thus, the algorithm is *not stiffly stable*, and the stability domain will loop in the left-half complex plane.

- Houbolt's algorithm has an asymptotic region around the origin, which, unfortunately, is rather small. Along the negative real axis, the asymptotic region ends for $\sigma_d \approx 0.1$.

- Since Houbolt's algorithm is an *implicit second-derivative ODE solver*, we had hoped for stiff stability. Without that property, the algorithm will perform poorly in comparison with the explicit GE3/AB2 algorithm.

[P5.1] Houbolt's Integration Algorithm VII

Let us now plot the stability domain of Houbolt's algorithm:



[P5.1] Houbolt's Integration Algorithm VIII

- The flying-saucer stability domain of Houbolt's algorithm looks beautiful, but the method is hardly convincing.
- Unfortunately, we still haven't discovered an *L-stable second-derivative ODE solver*.
- Such an algorithm does probably exist, but we haven't found it yet.