Numerical Simulation of Dynamic Systems: Hw7 - Solution

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[H6.1a] Interpolation of Measurement Data

Given a set of measurement data of surface air temperature:

t [hours]	u °C
0	6
6	16
12	28
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where t denotes the time measured in hours, and u denotes the surface air temperature at that time measured in degrees Centigrade.

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We shall need intermediate values, and therefore, we require an *interpolation routine*.

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Make use of the spline routine to obtain interpolated data points at all multiples of six hours, such that the new "measurement" data are equidistantly spaced over the range of values $t \in \{0\}, \{6\}, \ldots, \{72\}$ *hours*.

[H6.1a] Interpolation of Measurement Data III

We now wish to compare two different interpolation routines:

1. cubic spline interpolation
2. 3^{rd}-order Nordsieck interpolation

To this end, we need to develop a Matlab function nordsieck_intpol that has the same parameters as the spline function.

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We start out with the formula:

$$\begin{pmatrix} x_k \\ h \cdot \dot{x}_k \\ \frac{h^2}{2} \cdot \ddot{x}_k \\ \frac{h^3}{6} \cdot x_k^{(\mathrm{iii})} \end{pmatrix} = \frac{1}{6} \cdot \begin{pmatrix} 6 & 0 & 0 & 0 \\ 11 & -18 & 9 & -2 \\ 6 & -15 & 12 & -3 \\ 1 & -3 & 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ x_{k-3} \end{pmatrix} \tag{1}$$

and move the grid by the minimal amount necessary to make one of the grid points coincide with the desired interpolation point.

[H6.1a] Interpolation of Measurement Data IV

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What do you conclude?

[H6.1a] Interpolation of Measurement Data V

I chose to normalize the support values in the same way as I did for the Newton-Gregory polynomials, i.e., $x = 0$ is the leftmost normalized measurement point, $x = 1$ is the next normalized measurement point to the right, etc.

```
[n, m] = size(xvec);
nm = n * m;
x0 = xvec(1);
dx = xvec(2) - xvec(1);
xvec = 0 : nm - 1;
xvec_new = (xvec_new - x0*ones(size(xvec_new)))/dx;
[n1, m1] = size(xvec_new);
nm1 = n1 * m1;
yvec_new = zeros(n1, m1);
```

Here, $xvec$ is the vector of normalized support values, and $xvec_{new}$ is the vector of normalized values, at which we wish to interpolate.

[H6.1a] Interpolation of Measurement Data VI

Since we wish to use the 3^{rd}-order Nordsieck vector, we always need four neighboring support values for the interpolation. We need to determine, which values to use.

```
x = xvec_new(i);
ix = floor(x);
if ix == 0,
    indx = 4 : -1 : 1;
    c = 1;
elseif ix >= nm - 2,
    indx = nm : -1 : nm - 3;
    c = 3;
else
    indx = ix + 3 : -1 : ix;
    c = 2;
end
```

The *indx* vector computes the indices of the four neighboring support values. We need to treat the leftmost and rightmost segments differently. $c = 1$ denotes the leftmost segment, $c = 3$ marks the rightmost segment, and $c = 2$ handles all of the interior segments in between.

[H6.1a] Interpolation of Measurement Data VII

We need to determine, by how much the support values must be moved, i.e., we must compute the value of the normalized dx_{new}. If the value where we wish to interpolate is in the left half of a segment, we shall shorten dx, otherwise we shall lengthen dx. This rule holds everywhere except in the rightmost segment, where we must always shorten dx.

```
d = x - ix;
if c == 1,
    if d < 0.5,
        dxn = (3 - d)/3;
    else
        dxn = (3 - d)/2;
    end
elseif c == 2,
    if d < 0.5,
        dxn = (2 - d)/2;
    else
        dxn = (2 - d);
    end
else
    dxn = 1 - d;
end
```

[H6.1a] Interpolation of Measurement Data VIII

Now, we can move the support values:

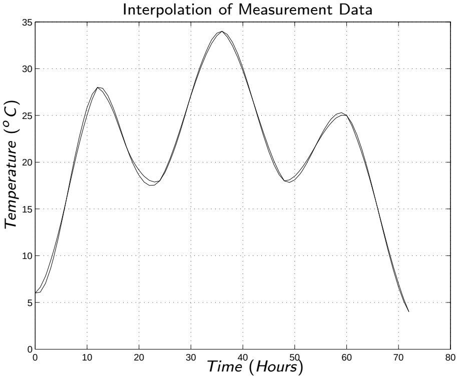
$$
\begin{aligned}
&dxn2 \;=\; dxn * dxn;\\
&dxn3 \;=\; dxn2 * dxn;\\
&H \;= \mathrm{diag}([\;1\;,\;dxn\;,\;dxn2\;,\;dxn3\;]);\\
&v \;=\; yvec(indx);\\
&vnew \;=\; T \backslash H * T * v;
\end{aligned}
$$

[H6.1a] Interpolation of Measurement Data IX

Finally, we need to pick the correct shifted support value:

```
if c == 1,
    if d < 0.5,
        y = vnew(4);
    else
        y = vnew(3);
    end
elseif c == 2,
    if d < 0.5,
        y = vnew(3);
    else
        y = vnew(2);
    end
else
    y = vnew(2);
end
if x == nm - 1,
    y = vnew(1);
end
yvec_new(i) = y;
```

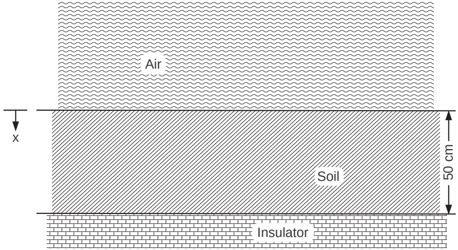
[H6.1a] Interpolation of Measurement Data X



The interpolation worked correctly, but not as smoothly as the cubic spline interpolation.

[H6.1b] Heat Diffusion in the Soil

Agricultural engineers are interested in knowing the temperature distribution in the soil as a function of the surface air temperature. As shown below, we assume that we have a soil layer of 50 cm. Underneath the soil, there is a layer that acts as an ideal heat insulator.



[H6.1b] Heat Diffusion in the Soil II

The heat flow problem can be written as:

$$\frac{\partial u}{\partial t} = \frac{\lambda}{\rho \cdot c} \cdot \frac{\partial^2 u}{\partial x^2}$$

where $\lambda = 0.004 \ \mathrm{cal\ cm^{-1}\ sec^{-1}\ K^{-1}}$ is the specific thermal conductance of soil, $\rho = 1.335 \ \mathrm{g\ cm^{-3}}$ is the density of soil, and $c = 0.2 \ \mathrm{cal\ g^{-1}\ K^{-1}}$ is the specific thermal capacitance of soil.

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Simulate the resulting linear ODE system using MATLAB's built-in stiff system solver (ode15s).

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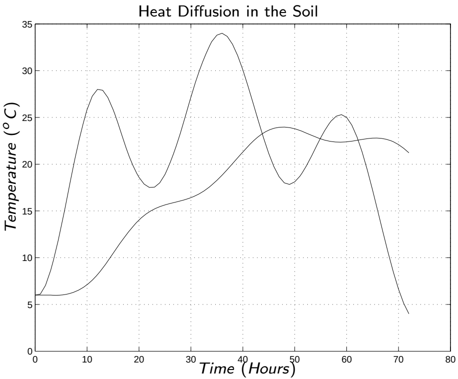
Generate also a three-dimensional plot showing the temperature distribution in the soil as a function of time and space.

[H6.1b] Heat Diffusion in the Soil IV



[H6.1b] Heat Diffusion in the Soil V