Numerical Simulation of Dynamic Systems: Hw8 - Problem

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[H6.3] Wave Equation

The wave equation has been written as:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}$$

Let us rewrite $u(x,t)$ as $\tilde{u}(v,w)$, where:

$$v = x + c \cdot t$$
$$w = x - c \cdot t$$

What happens?

[H6.9a] Poiseuille Flow Through a Pipe

The following equations describe the stationary flow of an incompressible fluid through a pipe:

$$\frac{d\hat{v}}{d\rho} = \frac{-\sqrt{2\Gamma}}{(\tau_M + 1)^2} \cdot \rho \cdot \tau^2$$

$$\frac{d}{d\rho}\left(\frac{\rho}{T} \cdot \frac{d\tau}{d\rho}\right) = \frac{-\Gamma}{(\tau_M + 1)^3} \cdot \rho^3 \cdot \tau^2$$

where:

$$\rho = \frac{r}{R}$$

$$\tau = \frac{T(r)}{T_W}$$

are two normalized coordinates. r is the distance from the center of the pipe, and R is the radius of the pipe. $T(r)$ is the temperature of the fluid at a distance r from the center, and T_W is the temperature of the pipe wall. T_W is assumed constant.
$\hat{v} = k_1 * v$ is the normalized flow velocity, where k_1 is a constant that depends on the viscosity, the thermal conductivity, and the average temperature of the fluid.

[H6.9a] Poiseuille Flow Through a Pipe II

The boundary conditions are:

$$\frac{d\hat{v}}{d\rho}(\rho = 0.0) = 0.0$$
$$\frac{d\tau}{d\rho}(\rho = 0.0) = 0.0$$
$$\hat{v}(\rho = 1.0) = 0.0$$
$$\tau(\rho = 1.0) = 1.0$$

Thus, this is a *boundary value problem*.

The equations contain two yet unknown parameters. Γ is a constant that depends on the fluid. Let us assume that $\Gamma = 10.0$. τ_M is the value of the normalized temperature at the center of the pipe. We shall introduce the momentary value of that temperature into the equation, and adjust that value as the simulation proceeds.

[H6.9a] Poiseuille Flow Through a Pipe III

We wish to simulate this problem across ρ in the range $\rho = [0.0, 1.0]$ with unknown initial conditions $\hat{v}(\rho = 0.0) = \hat{v}_M$ and $\tau(\rho = 0.0) = \tau_M$.

We begin by introducing an additional variable:

$$w = \frac{\rho}{T} \cdot \frac{d\tau}{d\rho}$$

We thus obtain three first-order ODEs in the variables $\hat{v}$, w, and τ.

You can verify easily that all three derivatives are negative everywhere in the pipe except at the center. All three ODEs are analytically unstable when simulating from $\rho = 0$ to $\rho = 1$.

It thus makes sense to substitute:

$$\sigma = 1 - \rho$$

and simulate from $\rho = 1$ to $\rho = 0$, i.e., from $\sigma = 0$ to $\sigma = 1$.

[H6.9a] Poiseuille Flow Through a Pipe IV

We end up with three differential equations of the form:

$$\frac{d\hat{v}}{d\sigma} = f_{\hat{v}}(\tau, \sigma)$$
$$\frac{dw}{d\sigma} = f_w(\tau, \sigma)$$
$$\frac{d\tau}{d\sigma} = f_\tau(\tau, w, \sigma)$$

You may need to do something about the third equation for $\rho = 0$ to avoid a division by zero.

The boundary conditions are:

$$\hat{v}(\sigma = 0) = 0$$
$$\tau(\sigma = 0) = 1$$
$$w(\sigma = 1) = 0$$

[H6.9a] Poiseuille Flow Through a Pipe V

Simulate the system using any guess for the unknown initial value of w using Matlab's built-in non-stiff ODE solver, ode45, and repeat the simulation multiple times using *fixed-point iteration* on this value, until the final value of $w(\sigma = 1) = 0$ is hit.

This method of solving boundary value problems is sometimes referred to as the *artillery method*.

Plot the three variables, $\hat{v}$, w, and τ as functions of ρ on three subplots of a single graph.

[H6.9b] Poiseuille Flow Through a Pipe

We now wish to solve this same problem in a different way using *invariant embedding*.

We rewrite the three ODEs in DAE form:

$$\frac{d\hat{v}}{d\sigma} - f_{\hat{v}}(\tau, \sigma) = 0$$
$$\frac{dw}{d\sigma} - f_w(\tau, \sigma) = 0$$
$$\frac{d\tau}{d\sigma} - f_\tau(\tau, w, \sigma) = 0$$

and embed the problem in a set of PDEs:

$$\frac{\partial\hat{v}}{\partial\sigma} - f_{\hat{v}}(\tau, \sigma) = \pm\frac{\partial\hat{v}}{\partial t}$$
$$\frac{\partial w}{\partial\sigma} - f_w(\tau, \sigma) = \pm\frac{\partial w}{\partial t}$$
$$\frac{\partial\tau}{\partial\sigma} - f_\tau(\tau, w, \sigma) = \pm\frac{\partial\tau}{\partial t}$$

For now, we don't know yet, which signs to use on the artificially introduced time derivatives.

[H6.9b] Poiseuille Flow Through a Pipe II

- We discretize the PDEs using the method of lines with 50 intervals. We shall use second-order accurate approximations for the three spatial derivatives.
- We apply the correct boundary conditions and suitable initial conditions.
- We shall simulate the system using Matlab's built-in stiff ODE solver, ode15s.
- We would like to simulate an analytically stable problem. To this end, we still need to choose the most suitable sign values for the three artificial time derivatives.
- Try out all eight combinations and look at the distributions of the eigenvalues of the (analytical) Jacobian for $t = 0$. You will find that only one of the eight combinations places the eigenvalues in the left-half complex plane.

Plot the eigenvalue distribution for the chosen set of sign values. What do you conclude about the nature of the embedded PDE problem? Is it parabolic or hyperbolic?

[H6.9b] Poiseuille Flow Through a Pipe III

We wish to simulate the resulting problem using the F-stable trapezoidal rule. To this end, we can simply set the maximal order of the stiff ODE solver to two, as BDF2 is the trapezoidal rule.

Simulate across 3.5 *seconds* and plot the three variables, $\hat{v}$, w, and τ as functions of ρ on three subplots of a single graph.

Repeat the simulation, this time simulating across 10 *seconds* and plot the three variables, $\hat{v}$, w, and τ as functions of ρ on three subplots of a single graph.

What do you conclude?

[H6.9c] Poiseuille Flow Through a Pipe

As we ran into numerical difficulties with our previous attempts, we shall now try to fix our problems by applying *upwind discretization* to all three PDEs.

Plot the eigenvalue distribution of the problem using upwind discretization in space.

Repeat the simulation, simulating across 10 *seconds* and plot the three variables, $\hat{v}$, w, and τ as functions of ρ on three subplots of a single graph.

What do you conclude?