Numerical Simulation of Dynamic Systems: Hw8 - Solution

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[H6.3] Wave Equation

The wave equation has been written as:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}$$

Let us rewrite $u(x, t)$ as $\tilde{u}(v, w)$, where:

$$v = x + c \cdot t$$
$$w = x - c \cdot t$$

What happens?

[H6.3] Wave Equation II

$$\begin{aligned} u_t &= \tilde{u}_v \cdot v_t + \tilde{u}_w \cdot w_t \\ &= c \cdot \tilde{u}_v - c \cdot \tilde{u}_w \\ &= c \cdot (\tilde{u}_v - \tilde{u}_w) \end{aligned}$$

$$\begin{aligned} u_{tt} &= c \cdot (\tilde{u}_{vt} - \tilde{u}_{wt}) \\ &= c \cdot (\tilde{u}_{vv} \cdot v_t + \tilde{u}_{vw} \cdot w_t - \tilde{u}_{wv} \cdot v_t - \tilde{u}_{ww} \cdot w_t) \\ &= c \cdot (c \cdot \tilde{u}_{vv} - c \cdot \tilde{u}_{vw} - c \cdot \tilde{u}_{wv} + c \cdot \tilde{u}_{ww}) \\ &= c^2 \cdot (\tilde{u}_{vv} - 2 \cdot \tilde{u}_{vw} + \tilde{u}_{ww}) \end{aligned}$$

$$\begin{aligned} u_x &= \tilde{u}_v \cdot v_x + \tilde{u}_w \cdot w_x \\ &= \tilde{u}_v + \tilde{u}_w \end{aligned}$$

$$\begin{aligned} u_{xx} &= \tilde{u}_{vx} + \tilde{u}_{wx} \\ &= \tilde{u}_{vv} \cdot v_x + \tilde{u}_{vw} \cdot w_x + \tilde{u}_{wv} \cdot v_x + \tilde{u}_{ww} \cdot w_x \\ &= \tilde{u}_{vv} + 2 \cdot \tilde{u}_{vw} + \tilde{u}_{ww} \end{aligned}$$

[H6.3] Wave Equation III

Plugging these expressions back into the original PDE, we obtain:

$$\tilde{u}_{vw} = 0$$

Hence:

$$\tilde{u}(v, w) = f(v) + g(w)$$

where $f(v)$ and $g(w)$ can be arbitrary functions. They are determined by the initial and boundary conditions of the problem to be simulated.

$$\Rightarrow \quad u(x, t) = f(x + c \cdot t) + g(x - c \cdot t)$$

[H6.9a] Poiseuille Flow Through a Pipe

The following equations describe the stationary flow of an incompressible fluid through a pipe:

$$\frac{d\hat{v}}{d\rho} = \frac{-\sqrt{2\Gamma}}{(\tau_M + 1)^2} \cdot \rho \cdot \tau^2$$

$$\frac{d}{d\rho}\left(\frac{\rho}{T} \cdot \frac{d\tau}{d\rho}\right) = \frac{-\Gamma}{(\tau_M + 1)^3} \cdot \rho^3 \cdot \tau^2$$

where:

$$\rho = \frac{r}{R}$$

$$\tau = \frac{T(r)}{T_W}$$

are two normalized coordinates. r is the distance from the center of the pipe, and R is the radius of the pipe. $T(r)$ is the temperature of the fluid at a distance r from the center, and T_W is the temperature of the pipe wall. T_W is assumed constant. $\hat{v} = k_1 * v$ is the normalized flow velocity, where k_1 is a constant that depends on the viscosity, the thermal conductivity, and the average temperature of the fluid.

[H6.9a] Poiseuille Flow Through a Pipe II

The boundary conditions are:

$$\frac{d\hat{v}}{d\rho}(\rho = 0.0) = 0.0$$
$$\frac{d\tau}{d\rho}(\rho = 0.0) = 0.0$$
$$\hat{v}(\rho = 1.0) = 0.0$$
$$\tau(\rho = 1.0) = 1.0$$

Thus, this is a *boundary value problem*.

The equations contain two yet unknown parameters. Γ is a constant that depends on the fluid. Let us assume that $\Gamma = 10.0$. τ_M is the value of the normalized temperature at the center of the pipe. We shall introduce the momentary value of that temperature into the equation, and adjust that value as the simulation proceeds.

[H6.9a] Poiseuille Flow Through a Pipe III

We wish to simulate this problem across ρ in the range $\rho = [0.0, 1.0]$ with unknown initial conditions $\hat{v}(\rho = 0.0) = \hat{v}_M$ and $\tau(\rho = 0.0) = \tau_M$.

We begin by introducing an additional variable:

$$w = \frac{\rho}{T} \cdot \frac{d\tau}{d\rho}$$

We thus obtain three first-order ODEs in the variables $\hat{v}$, w, and τ.

You can verify easily that all three derivatives are negative everywhere in the pipe except at the center. All three ODEs are analytically unstable when simulating from $\rho = 0$ to $\rho = 1$.

It thus makes sense to substitute:

$$\sigma = 1 - \rho$$

and simulate from $\rho = 1$ to $\rho = 0$, i.e., from $\sigma = 0$ to $\sigma = 1$.

[H6.9a] Poiseuille Flow Through a Pipe IV

We end up with three differential equations of the form:

$$\frac{d\hat{v}}{d\sigma} = f_{\hat{v}}(\tau, \sigma)$$
$$\frac{dw}{d\sigma} = f_w(\tau, \sigma)$$
$$\frac{d\tau}{d\sigma} = f_\tau(\tau, w, \sigma)$$

You may need to do something about the third equation for $\rho = 0$ to avoid a division by zero.

The boundary conditions are:

$$\hat{v}(\sigma = 0) = 0$$
$$\tau(\sigma = 0) = 1$$
$$w(\sigma = 1) = 0$$

[H6.9a] Poiseuille Flow Through a Pipe V

Simulate the system using any guess for the unknown initial value of w using Matlab's built-in non-stiff ODE solver, ode45, and repeat the simulation multiple times using *fixed-point iteration* on this value, until the final value of $w(\sigma = 1) = 0$ is hit.

This method of solving boundary value problems is sometimes referred to as the *artillery method*.

Plot the three variables, $\hat{v}$, w, and τ as functions of ρ on three subplots of a single graph.

[H6.9a] Poiseuille Flow Through a Pipe VI

Introducing the variable, w, we get the following three ODEs:

$$\frac{d\hat{v}}{d\rho} = \frac{-\sqrt{2\Gamma}}{(\tau_M + 1)^2} \cdot \rho \cdot \tau^2$$
$$\frac{dw}{d\rho} = \frac{-\Gamma}{(\tau_M + 1)^3} \cdot \rho^3 \cdot \tau^2$$
$$\frac{d\tau}{d\rho} = \frac{\tau}{\rho} \cdot w$$

- $\tau > 0$ everywhere, since τ is a normalized absolute temperature.
- $\rho > 0$ everywhere except at the center (representing the distance from the center).
- $\frac{d\hat{v}}{d\rho} < 0$ everywhere except at the center, where $\frac{d\hat{v}}{d\rho} = 0$. $\hat{v} > 0$ everywhere except at the perimeter, where $\hat{v} = 0$.
- $\frac{dw}{d\rho} < 0$ everywhere except at the center, where $\frac{dw}{d\rho} = 0$. $w < 0$ everywhere except at the center, where $w = 0$.
- $\frac{d\tau}{d\rho} < 0$ everywhere except at the center. $\tau > 1$ everywhere except at the perimeter, where $\tau = 1$.

[H6.9a] Poiseuille Flow Through a Pipe VII

Substituting $\sigma = 1 - \rho$:

$$\frac{d\hat{v}}{d\sigma} = \frac{\sqrt{2\Gamma}}{(\tau_M + 1)^2} \cdot (1 - \sigma) \cdot \tau^2$$
$$\frac{dw}{d\sigma} = \frac{\Gamma}{(\tau_M + 1)^3} \cdot (1 - \sigma)^3 \cdot \tau^2$$
$$\frac{d\tau}{d\sigma} = \frac{-\tau}{1 - \sigma} \cdot w$$

For $\rho = 0$, we need to set the derivative of the third equation explicitly to zero to avoid a division by zero.

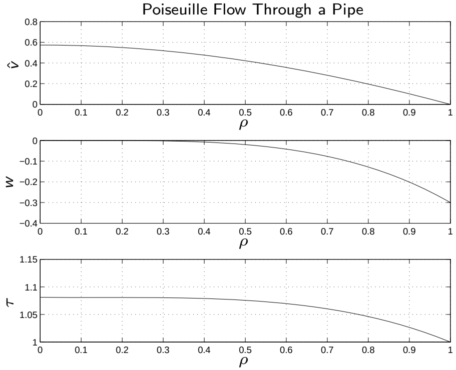
$$\frac{d\hat{v}}{d\sigma} = \frac{\sqrt{2\Gamma}}{(\tau_M + 1)^2} \cdot (1 - \sigma) \cdot \tau^2$$
$$\frac{dw}{d\sigma} = \frac{\Gamma}{(\tau_M + 1)^3} \cdot (1 - \sigma)^3 \cdot \tau^2$$
$$\frac{d\tau}{d\sigma} = \begin{cases} \frac{-\tau}{1-\sigma} \cdot w \;\; ; \;\; \sigma < 1 \\ 0 \;\; ; \;\; \sigma = 1 \end{cases}$$

Initial conditions:

$$\hat{v}(\sigma = 0) = 0$$
$$w(\sigma = 0) = ?$$
$$\tau(\sigma = 0) = 1$$

[H6.9a] Poiseuille Flow Through a Pipe VIII

I chose initial values of $w_0 = -1$ and $\tau_M = 2$ and iterated until $|w_f| < 10^{-4}$ and $|\Delta\tau_M| < 10^{-4}$, where $\Delta\tau_M$ denotes changes in τ_M in subsequent iterations. The program took about 20 iterations to converge.



[H6.9b] Poiseuille Flow Through a Pipe

We now wish to solve this same problem in a different way using *invariant embedding*.

We rewrite the three ODEs in DAE form:

$$\frac{d\hat{v}}{d\sigma} - f_{\hat{v}}(\tau,\sigma) = 0$$
$$\frac{dw}{d\sigma} - f_w(\tau,\sigma) = 0$$
$$\frac{d\tau}{d\sigma} - f_\tau(\tau,w,\sigma) = 0$$

and embed the problem in a set of PDEs:

$$\frac{\partial\hat{v}}{\partial\sigma} - f_{\hat{v}}(\tau,\sigma) = \pm\frac{\partial\hat{v}}{\partial t}$$
$$\frac{\partial w}{\partial\sigma} - f_w(\tau,\sigma) = \pm\frac{\partial w}{\partial t}$$
$$\frac{\partial\tau}{\partial\sigma} - f_\tau(\tau,w,\sigma) = \pm\frac{\partial\tau}{\partial t}$$

For now, we don't know yet, which signs to use on the artificially introduced time derivatives.

[H6.9b] Poiseuille Flow Through a Pipe II

- We discretize the PDEs using the method of lines with 50 intervals. We shall use second-order accurate approximations for the three spatial derivatives.
- We apply the correct boundary conditions and suitable initial conditions.
- We shall simulate the system using Matlab's built-in stiff ODE solver, ode15s.
- We would like to simulate an analytically stable problem. To this end, we still need to choose the most suitable sign values for the three artificial time derivatives.
- Try out all eight combinations and look at the distributions of the eigenvalues of the (analytical) Jacobian for $t = 0$. You will find that only one of the eight combinations places the eigenvalues in the left-half complex plane.

Plot the eigenvalue distribution for the chosen set of sign values. What do you conclude about the nature of the embedded PDE problem? Is it parabolic or hyperbolic?

[H6.9b] Poiseuille Flow Through a Pipe III

We wish to simulate the resulting problem using the F-stable trapezoidal rule. To this end, we can simply set the maximal order of the stiff ODE solver to two, as BDF2 is the trapezoidal rule.

Simulate across 3.5 *seconds* and plot the three variables, $\hat{v}$, w, and τ as functions of ρ on three subplots of a single graph.

Repeat the simulation, this time simulating across 10 *seconds* and plot the three variables, $\hat{v}$, w, and τ as functions of ρ on three subplots of a single graph.

What do you conclude?

[H6.9b] Poiseuille Flow Through a Pipe IV

After embedding, the PDEs present themselves in the form:

$$\frac{\partial \hat{v}}{\partial \sigma} - \frac{\sqrt{2\Gamma}}{(\tau_M + 1)^2} \cdot (1 - \sigma) \cdot \tau^2 = \pm \frac{\partial \hat{v}}{\partial t}$$

$$\frac{\partial w}{\partial \sigma} - \frac{\Gamma}{(\tau_M + 1)^3} \cdot (1 - \sigma)^3 \cdot \tau^2 = \pm \frac{\partial w}{\partial t}$$

$$\frac{\partial \tau}{\partial \sigma} - \frac{-\tau}{1 - \sigma} \cdot w = \pm \frac{\partial \tau}{\partial t}$$

[H6.9b] Poiseuille Flow Through a Pipe V

We discretize using second-order accurate approximations for the spatial derivatives:

$$\frac{d\hat{v}_i}{dt} = s_v \cdot \left(\frac{\hat{v}_{i+1} - \hat{v}_{i-1}}{2\delta\sigma} - \frac{\sqrt{2\Gamma}}{(\tau_M + 1)^2} \cdot (1 - \sigma_i) \cdot \tau_i^2 \right)$$

$$\frac{dw_i}{dt} = s_w \cdot \left(\frac{w_{i+1} - w_{i-1}}{2\delta\sigma} - \frac{\Gamma}{(\tau_M + 1)^3} \cdot (1 - \sigma_i)^3 \cdot \tau_i^2 \right)$$

$$\frac{d\tau_i}{dt} = s_\tau \cdot \left(\frac{\tau_{i+1} - \tau_{i-1}}{2\delta\sigma} - \frac{-\tau_i}{1 - \sigma_i} \cdot w_i \right)$$

except for the boundary values, where we use biased formulae. s_v, s_w, and s_τ denote the still unknown signs of the three equations.

[H6.9b] Poiseuille Flow Through a Pipe VI

The boundary equations are:

$$
\begin{aligned}
\hat{v}_1 &= 0 \\
\frac{dw_1}{dt} &= s_w \cdot \left(\frac{-w_3 + 4w_2 - 3w_1}{2\delta\sigma} - \frac{\Gamma}{(\tau_M + 1)^3} \right) \\
\tau_1 &= 1 \\
\frac{d\hat{v}_{51}}{dt} &= s_v \cdot \left(\frac{3\hat{v}_{51} - 4\hat{v}_{50} + \hat{v}_{49}}{2\delta\sigma} \right) \\
w_{51} &= 0 \\
\frac{d\tau_{51}}{dt} &= s_\tau \cdot \left(\frac{3\tau_{51} - 4\tau_{50} + \tau_{49}}{2\delta\sigma} \right)
\end{aligned}
$$

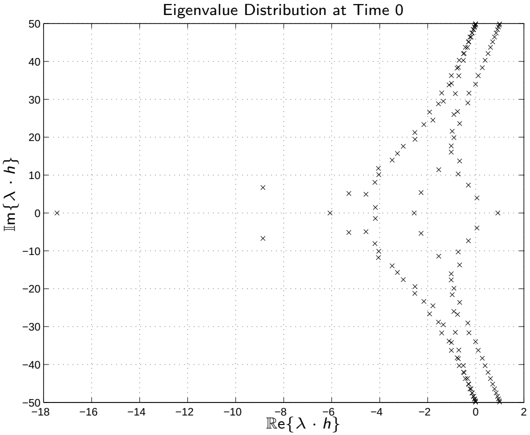
As initial conditions, I chose:

$$
\begin{aligned}
\hat{v}(\sigma, t = 0) &= \sigma \\
w(\sigma, t = 0) &= \sigma - 1 \\
\tau(\sigma, t = 0) &= \sigma + 1
\end{aligned}
$$

Since the equations are band-structured, it was easy to compute the Jacobian analytically.

[H6.9b] Poiseuille Flow Through a Pipe VII

Only one combination of signs gave decent eigenvalues:



[H6.9b] Poiseuille Flow Through a Pipe VIII

The resulting PDEs are:

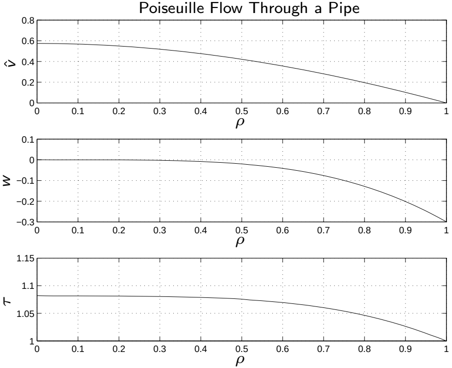
$$\frac{\partial \hat{v}}{\partial t} = -\frac{\partial \hat{v}}{\partial \sigma} + \frac{\sqrt{2\Gamma}}{(\tau_M + 1)^2} \cdot (1 - \sigma) \cdot \tau^2$$
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial \sigma} - \frac{\Gamma}{(\tau_M + 1)^3} \cdot (1 - \sigma)^3 \cdot \tau^2$$
$$\frac{\partial \tau}{\partial t} = -\frac{\partial \tau}{\partial \sigma} + \frac{-\tau}{1 - \sigma} \cdot w$$

which have a good physical interpretation. They simply represent the model of a non-stationary Poiseuille flow through a pipe.

From the eigenvalue distribution, we can see that the PDEs are hyperbolic and not parabolic.

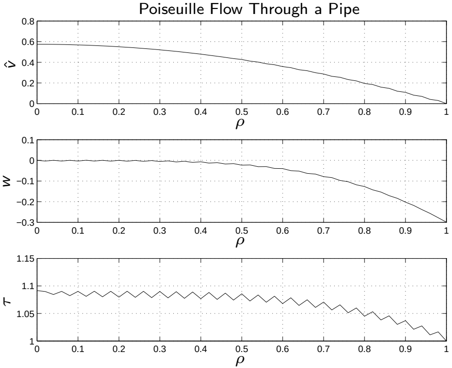
[H6.9b] Poiseuille Flow Through a Pipe IX

We are now ready to simulate:



[H6.9b] Poiseuille Flow Through a Pipe X

We repeat the simulation, this time simulating until $t = 10$ *seconds*.



- **The simulation is mildly unstable. We are starting to accumulate garbage.**

[H6.9c] Poiseuille Flow Through a Pipe

As we ran into numerical difficulties with our previous attempts, we shall now try to fix our problems by applying *upwind discretization* to all three PDEs.

Plot the eigenvalue distribution of the problem using upwind discretization in space.

Repeat the simulation, simulating across 10 *seconds* and plot the three variables, $\hat{v}$, w, and τ as functions of ρ on three subplots of a single graph.

What do you conclude?

[H6.9c] Poiseuille Flow Through a Pipe II

We apply upwind discretization:

$$\frac{d\hat{v}_i}{dt} = -\left(\frac{3\hat{v}_i - 4\hat{v}_{i-1} + \hat{v}_{i-2}}{2\delta\sigma} - \frac{\sqrt{2\Gamma}}{(\tau_M + 1)^2} \cdot (1 - \sigma_i) \cdot \tau_i^2\right)$$

$$\frac{dw_i}{dt} = +\left(\frac{-w_{i+2} + 4w_{i+1} - 3w_i}{2\delta\sigma} - \frac{\Gamma}{(\tau_M + 1)^3} \cdot (1 - \sigma_i)^3 \cdot \tau_i^2\right)$$

$$\frac{d\tau_i}{dt} = -\left(\frac{3\tau_i - 4\tau_{i-1} + \tau_{i-2}}{2\delta\sigma} - \frac{-\tau_i}{1 - \sigma_i} \cdot w_i\right)$$

[H6.9c] Poiseuille Flow Through a Pipe III

The boundary equations are:

$$
\begin{aligned}
\hat{v}_1 &= 0 \\
\frac{dw_1}{dt} &= \left(\frac{-w_3 + 4w_2 - 3w_1}{2\delta\sigma} - \frac{\Gamma}{(\tau_M + 1)^3} \right) \\
\tau_1 &= 1 \\
\frac{d\hat{v}_2}{dt} &= -\left(\frac{\hat{v}_3 - \hat{v}_1}{2\delta\sigma} - \frac{\sqrt{2\Gamma}}{(\tau_M + 1)^2} \cdot (1 - \sigma_2) \cdot \tau_2^2 \right) \\
\frac{dw_2}{dt} &= \left(\frac{-w_4 + 4w_3 - 3w_2}{2\delta\sigma} - \frac{\Gamma}{(\tau_M + 1)^3} \cdot (1 - \sigma_2)^3 \cdot \tau_2^2 \right) \\
\frac{d\tau_2}{dt} &= -\left(\frac{\tau_3 - \tau_1}{2\delta\sigma} - \frac{-\tau_2}{1 - \sigma_2} \cdot w_2 \right)
\end{aligned}
$$

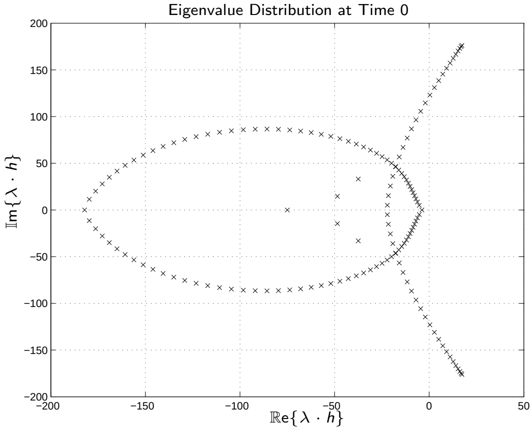
[H6.9c] Poiseuille Flow Through a Pipe IV

and:

$$\begin{aligned}
\frac{d\hat{v}_{50}}{dt} &= -\left(\frac{3\hat{v}_{50} - 4\hat{v}_{49} + \hat{v}_{48}}{2\delta\sigma} - \frac{\sqrt{2\Gamma}}{(\tau_M + 1)^2} \cdot (1 - \sigma_{50}) \cdot \tau_{50}^2\right) \\
\frac{dw_{50}}{dt} &= +\left(\frac{w_{51} - w_{49}}{2\delta\sigma} - \frac{\Gamma}{(\tau_M + 1)^3} \cdot (1 - \sigma_{50})^3 \cdot \tau_{50}^2\right) \\
\frac{d\tau_{50}}{dt} &= -\left(\frac{3\tau_{50} - 4\tau_{49} + \tau_{48}}{2\delta\sigma} - \frac{-\tau_{50}}{1 - \sigma_{50}} \cdot w_{50}\right) \\
\frac{d\hat{v}_{51}}{dt} &= -\left(\frac{3\hat{v}_{51} - 4\hat{v}_{50} + \hat{v}_{49}}{2\delta\sigma}\right) \\
w_{51} &= 0 \\
\frac{d\tau_{51}}{dt} &= -\left(\frac{3\tau_{51} - 4\tau_{50} + \tau_{49}}{2\delta\sigma}\right)
\end{aligned}$$

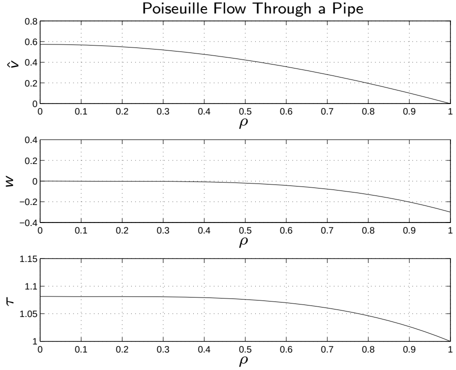
[H6.9c] Poiseuille Flow Through a Pipe V

Let us look at the eigenvalue distribution:



[H6.9c] Poiseuille Flow Through a Pipe VI

We simulate across $t = 10$ *seconds*:



- **The ripple is gone. Upwind discretization did the trick beautifully.**