Numerical Simulation of Dynamic Systems XIV

Prof. Dr. François E. Cellier
Department of Computer Science
ETH Zurich

April 16, 2013

Introduction

Until now, we always assumed that the model to be simulated is available in explicit form, either given as an *explicit set of ODEs* of the type:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

or as an *explicit set of second-derivative equations*:

$$\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{u}, t)$$

or possibly as a *set of time-dependent PDEs*, e.g.:

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f}(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x}, \frac{\partial^2 \mathbf{u}}{\partial x^2}, t)$$

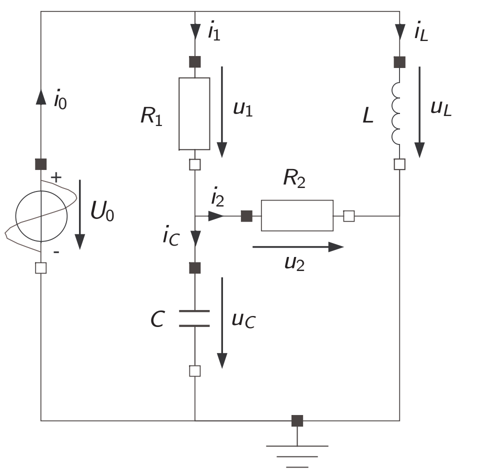
Yet, this is not how we usually obtain models of physical systems initially. Most models present themselves as an *implicitly formulated collection of differential and algebraic equations*, i.e., a set of *differential algebraic equations (DAEs)*.

We then need to either convert these implicitly formulated sets of DAEs to equivalent explicitly formulated sets of ODEs, or alternatively, we need to come up with numerical DAE solvers that are able to simulate DAEs directly.

In the next few presentations, we shall be looking at both of these alternatives.

Introduction II

Let us start with the example of a simple electrical RLC circuit:



$$
\begin{aligned}
1:&\quad u_0 = f(t)\\
2:&\quad u_1 = R_1 \cdot i_1\\
3:&\quad u_2 = R_2 \cdot i_2\\
4:&\quad u_L = L \cdot \frac{di_L}{dt}\\
5:&\quad i_C = C \cdot \frac{du_C}{dt}\\
6:&\quad u_0 = u_1 + u_C\\
7:&\quad u_L = u_1 + u_2\\
8:&\quad u_C = u_2\\
9:&\quad i_0 = i_1 + i_L\\
10:&\quad i_1 = i_2 + i_C
\end{aligned}
$$

⇒ **We got 10 implicitly formulated DAEs in 10 unknowns.**

Introduction III

We wish to convert the set of implicit DAEs to an explicit state-space form.

- We define the outputs of the integrators, u_C and i_L, as our state variables. These can thus be considered known variables, for which no equations need to be found. In contrast, the inputs of the integrators, du_C/dt and di_L/dt, are unknowns, for which equations must be found. These are the state equations of the state-space description.

- The structure of these equations can be captured in the so-called *structure incidence matrix*:

$\mathbf{S} =$

	u_0	i_0	u_1	i_1	u_2	i_2	u_L	$\frac{di_L}{dt}$	$\frac{du_C}{dt}$	i_C
1:	1	0	0	0	0	0	0	0	0	0
2:	0	0	1	1	0	0	0	0	0	0
3:	0	0	0	0	1	1	0	0	0	0
4:	0	0	0	0	0	0	1	1	0	0
5:	0	0	0	0	0	0	0	0	1	1
6:	1	0	1	0	0	0	0	0	0	0
7:	0	0	1	0	1	0	1	0	0	0
8:	0	0	0	0	1	0	0	0	0	0
9:	0	1	0	1	0	0	0	0	0	0
10:	0	0	0	1	0	1	0	0	0	1

Introduction IV

Initially, all of these equations are *acausal*, meaning that the equal sign has to be interpreted in the sense of an equality, rather than in the sense of an assignment. For example, the above set of equations contains two equations that list u_0 to the left of the equal sign. Evidently, only one of those can be used to solve for u_0.

Two simple rules can be formulated that help us decide, which variables to solve for from which of the equations:

1. If an equation contains only a single unknown, i.e., one variable for which no solving equation has been found yet, we need to use that equation to solve for this variable. For example, Eq.(1) contains only one unknown, u_0, hence that equation must be used to solve for u_0, and consequently, Eq.(1) has now become a *causal equation*, and u_0 can henceforth be considered a known variable in all remaining equations.
2. If an unknown only appears in a single equation, that equation must be used to solve for it. For example, i_0 only appears in Eq.(9). Hence we must use Eq.(9) to solve for i_0.

Introduction V

- The two rules can be easily visualized in the structure incidence matrix.
- If a row contains a single element with a value of 1, that equation needs to be solved for the corresponding variable, and both the row and the column can be eliminated from the structure incidence matrix.
- If a column contains a single element with a value of 1, that variable must be solved for using the corresponding equation, and both the column and the row can be eliminated from the structure incidence matrix.
- The algorithm proceeds iteratively, until no more rows and columns can be eliminated from the structure incidence matrix.

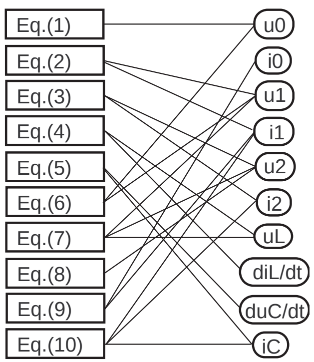
While this algorithm could in theory be used, another *graph-theoretical algorithm* is more common. This algorithm shall be introduced next.

Causalization of Equations

We describe the topology of the DAE set by means of a so-called *structure digraph*.

The structure digraph lists the equations on the left side and the unknowns on the right side. A connection between an equation and a variable indicates that the variable appears in the equation.

$$
\begin{aligned}
u_0 &= f(t) \\
u_1 &= R_1 \cdot i_1 \\
u_2 &= R_2 \cdot i_2 \\
u_L &= L \cdot di_L/dt \\
i_C &= C \cdot du_C/dt \\
u_0 &= u_1 + u_C \\
u_L &= u_1 + u_2 \\
u_C &= u_2 \\
i_0 &= i_1 + i_L \\
i_1 &= i_2 + i_C
\end{aligned}
$$



Causalization of Equations II

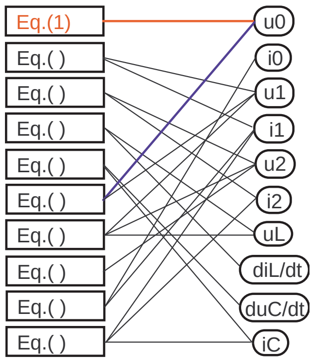
Let us reinterpret our two rules in terms of this new representation:

1. For all acausal equations, if an equation has only one black line attached to it, color that line red, follow it to the variable it points at, and color all other connections ending in that variable in blue. Renumber the equation using the lowest free number starting from 1.
2. For all unknown variables, if a variable has only one black line attached to it, color that line red, follow it back to the equation it points at, and color all other connections emanating from that equation in blue. Renumber the equation using the highest free number starting from n, where n is the number of equations.

The two rules are applied iteratively, until there are no longer either equations or variables with a single black line attached to them.

Causalization of Equations III

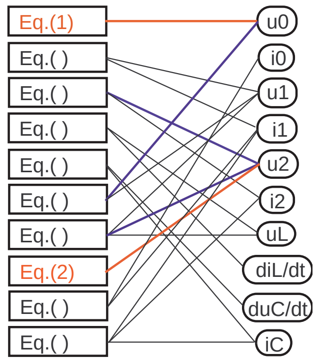
$u_0 = f(t)$
$u_1 = R_1 \cdot i_1$
$u_2 = R_2 \cdot i_2$
$u_L = L \cdot di_L/dt$
$i_C = C \cdot du_C/dt$
$u_0 = u_1 + u_C$
$u_L = u_1 + u_2$
$u_C = u_2$
$i_0 = i_1 + i_L$
$i_1 = i_2 + i_C$



red variable: variable to be solved for from that equation
blue variable: variable already known, not to be solved for from that equation
black variable: causality of variable still unknown

Causalization of Equations IV

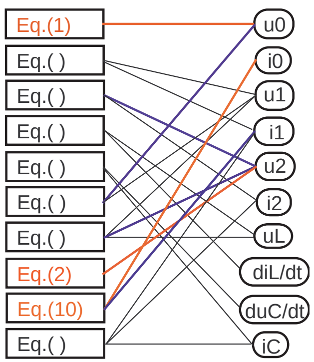
$u_0 = f(t)$
$u_1 = R_1 \cdot i_1$
$u_2 = R_2 \cdot i_2$
$u_L = L \cdot di_L/dt$
$i_C = C \cdot du_C/dt$
$u_0 = u_1 + u_C$
$u_L = u_1 + u_2$
$u_C = u_2$
$i_0 = i_1 + i_L$
$i_1 = i_2 + i_C$



red variable: variable to be solved for from that equation
blue variable: variable already known, not to be solved for from that equation
black variable: causality of variable still unknown

Causalization of Equations V

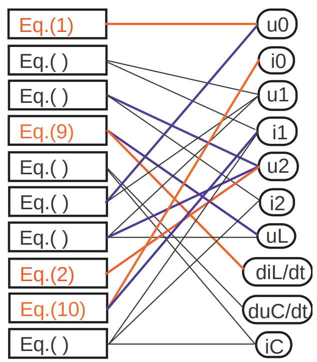
$u_0 = f(t)$
$u_1 = R_1 \cdot i_1$
$u_2 = R_2 \cdot i_2$
$u_L = L \cdot di_L/dt$
$i_C = C \cdot du_C/dt$
$u_0 = u_1 + u_C$
$u_L = u_1 + u_2$
$u_C = u_2$
$i_0 = i_1 + i_L$
$i_1 = i_2 + i_C$



red variable: variable to be solved for from that equation
blue variable: variable already known, not to be solved for from that equation
black variable: causality of variable still unknown

Causalization of Equations VI

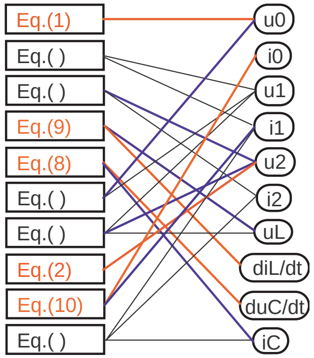
$u_0 = f(t)$
$u_1 = R_1 \cdot i_1$
$u_2 = R_2 \cdot i_2$
$u_L = L \cdot di_L/dt$
$i_C = C \cdot du_C/dt$
$u_0 = u_1 + u_C$
$u_L = u_1 + u_2$
$u_C = u_2$
$i_0 = i_1 + i_L$
$i_1 = i_2 + i_C$



red variable: variable to be solved for from that equation
blue variable: variable already known, not to be solved for from that equation
black variable: causality of variable still unknown

Causalization of Equations VII

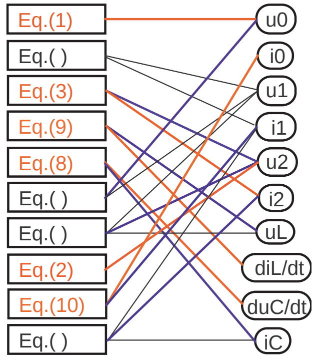
$$u_0 = f(t)$$
$$u_1 = R_1 \cdot i_1$$
$$u_2 = R_2 \cdot i_2$$
$$u_L = L \cdot di_L/dt$$
$$i_C = C \cdot du_C/dt$$
$$u_0 = u_1 + u_C$$
$$u_L = u_1 + u_2$$
$$u_C = u_2$$
$$i_0 = i_1 + i_L$$
$$i_1 = i_2 + i_C$$



red variable: variable to be solved for from that equation
blue variable: variable already known, not to be solved for from that equation
black variable: causality of variable still unknown

Causalization of Equations VIII

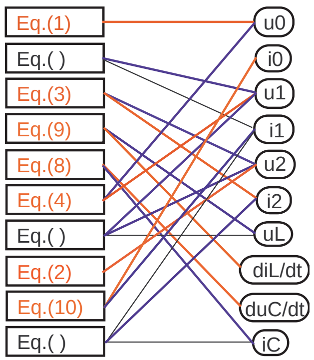
$$u_0 = f(t)$$
$$u_1 = R_1 \cdot i_1$$
$$u_2 = R_2 \cdot i_2$$
$$u_L = L \cdot di_L/dt$$
$$i_C = C \cdot du_C/dt$$
$$u_0 = u_1 + u_C$$
$$u_L = u_1 + u_2$$
$$u_C = u_2$$
$$i_0 = i_1 + i_L$$
$$i_1 = i_2 + i_C$$



red variable: variable to be solved for from that equation
blue variable: variable already known, not to be solved for from that equation
black variable: causality of variable still unknown

Causalization of Equations IX

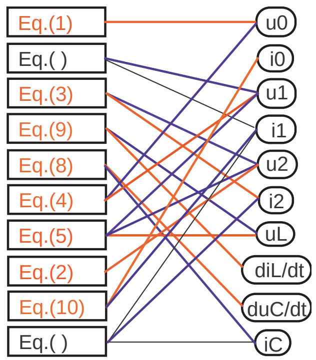
$$u_0 = f(t)$$
$$u_1 = R_1 \cdot i_1$$
$$u_2 = R_2 \cdot i_2$$
$$u_L = L \cdot di_L/dt$$
$$i_C = C \cdot du_C/dt$$
$$u_0 = u_1 + u_C$$
$$u_L = u_1 + u_2$$
$$u_C = u_2$$
$$i_0 = i_1 + i_L$$
$$i_1 = i_2 + i_C$$



red variable: variable to be solved for from that equation
blue variable: variable already known, not to be solved for from that equation
black variable: causality of variable still unknown

Causalization of Equations X

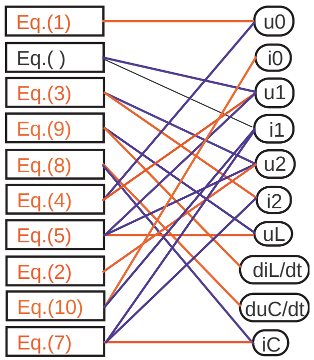
$$u_0 = f(t)$$
$$u_1 = R_1 \cdot i_1$$
$$u_2 = R_2 \cdot i_2$$
$$u_L = L \cdot di_L/dt$$
$$i_C = C \cdot du_C/dt$$
$$u_0 = u_1 + u_C$$
$$u_L = u_1 + u_2$$
$$u_C = u_2$$
$$i_0 = i_1 + i_L$$
$$i_1 = i_2 + i_C$$



red variable: variable to be solved for from that equation
blue variable: variable already known, not to be solved for from that equation
black variable: causality of variable still unknown

Causalization of Equations XI

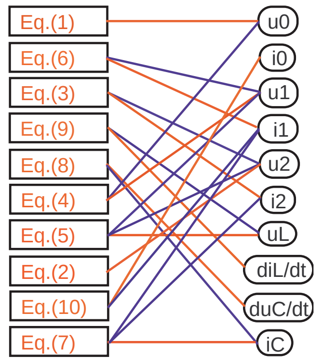
$$u_0 = f(t)$$
$$u_1 = R_1 \cdot i_1$$
$$u_2 = R_2 \cdot i_2$$
$$u_L = L \cdot di_L/dt$$
$$i_C = C \cdot du_C/dt$$
$$u_0 = u_1 + u_C$$
$$u_L = u_1 + u_2$$
$$u_C = u_2$$
$$i_0 = i_1 + i_L$$
$$i_1 = i_2 + i_C$$



red variable: variable to be solved for from that equation
blue variable: variable already known, not to be solved for from that equation
black variable: causality of variable still unknown

Causalization of Equations XII

$$u_0 = f(t)$$
$$u_1 = R_1 \cdot i_1$$
$$u_2 = R_2 \cdot i_2$$
$$u_L = L \cdot di_L/dt$$
$$i_C = C \cdot du_C/dt$$
$$u_0 = u_1 + u_C$$
$$u_L = u_1 + u_2$$
$$u_C = u_2$$
$$i_0 = i_1 + i_L$$
$$i_1 = i_2 + i_C$$



red variable: variable to be solved for from that equation
blue variable: variable already known, not to be solved for from that equation
black variable: causality of variable still unknown

Causalization of Equations XIII

We now *sort the equations vertically* in accordance with their new numbering scheme:

1:	u_0	=	$f(t)$		1:	u_0	=	$f(t)$
6:	u_1	=	$R_1 \cdot i_1$		2:	u_C	=	u_2
3:	u_2	=	$R_2 \cdot i_2$		3:	u_2	=	$R_2 \cdot i_2$
9:	u_L	=	$L \cdot di_L/dt$		4:	u_0	=	$u_1 + u_C$
8:	i_C	=	$C \cdot du_C/dt$	$\Rightarrow$	5:	u_L	=	$u_1 + u_2$
4:	u_0	=	$u_1 + u_C$		6:	u_1	=	$R_1 \cdot i_1$
5:	u_L	=	$u_1 + u_2$		7:	i_1	=	$i_2 + i_C$
2:	u_C	=	u_2		8:	i_C	=	$C \cdot du_C/dt$
10:	i_0	=	$i_1 + i_L$		9:	u_L	=	$L \cdot di_L/dt$
7:	i_1	=	$i_2 + i_C$		10:	i_0	=	$i_1 + i_L$

Causalization of Equations XIV

We then *sort the equations horizontally* in accordance with their assigned causality:

1:	u_0	=	$f(t)$		1:	u_0	=	$f(t)$
2:	u_C	=	u_2		2:	u_2	=	u_C
3:	u_2	=	$R_2 \cdot i_2$		3:	i_2	=	u_2/R_2
4:	u_0	=	$u_1 + u_C$		4:	u_1	=	$u_0 - u_C$
5:	u_L	=	$u_1 + u_2$	$\Rightarrow$	5:	u_L	=	$u_1 + u_2$
6:	u_1	=	$R_1 \cdot i_1$		6:	i_1	=	u_1/R_1
7:	i_1	=	$i_2 + i_C$		7:	i_C	=	$i_1 - i_2$
8:	i_C	=	$C \cdot du_C/dt$		8:	du_C/dt	=	i_C/C
9:	u_L	=	$L \cdot di_L/dt$		9:	di_L/dt	=	u_L/L
10:	i_0	=	$i_1 + i_L$		10:	i_0	=	$i_1 + i_L$

The sorted equations can be coded in **Matlab** directly.

Causalization of Equations XV

If we wish to obtain a state-space form directly, we eliminate the algebraic variables by substitution:

$$
\begin{aligned}
\frac{du_C}{dt} &= \frac{1}{C} \cdot i_C \\
&= \frac{1}{C} \cdot (i_1 - i_2) \\
&= \frac{1}{C \cdot R_1} \cdot u_1 - \frac{1}{C \cdot R_2} \cdot u_2 \\
&= \frac{1}{C \cdot R_1} \cdot (u_0 - u_C) - \frac{1}{C \cdot R_2} \cdot u_C \\
&= -\left(\frac{1}{C \cdot R_1} + \frac{1}{C \cdot R_2}\right) \cdot u_C + \frac{1}{C \cdot R_1} \cdot u_0 \\
\frac{di_L}{dt} &= \frac{1}{L} \cdot u_L \\
&= \frac{1}{L} \cdot (u_1 + u_2) \\
&= \frac{1}{L} \cdot (u_0 - u_C + u_C) \\
&= \frac{1}{L} \cdot u_0
\end{aligned}
$$

Causalization of Equations XVI

The structure incidence matrix of the sorted equations is in *lower triangular form*:

$$
\mathbf{S} = \begin{array}{r}
 \\ 1: \\ 2: \\ 3: \\ 4: \\ 5: \\ 6: \\ 7: \\ 8: \\ 9: \\ 10:
\end{array}
\begin{array}{cccccccccc}
u_0 & u_2 & i_2 & u_1 & u_L & i_1 & i_C & \frac{du_C}{dt} & \frac{di_L}{dt} & i_0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}
$$

Conclusions

- In this presentation, we showed that models extracted from an object-oriented description of physical systems usually present themselves in the form of implicitly formulated sets of differential and algebraic equations.
- We presented two techniques for capturing the topology of a set of DAEs: the structure incidence matrix and the structure digraph.
- We then introduced a first algorithm for converting an implicit DAE system to an equivalent explicit ODE system using the structure digraph.

References

1. Cellier, F.E., and H. Elmqvist (1993), "Automated Formula Manipulation Supports Object-Oriented Continuous-System Modeling," *IEEE Control Systems*, **13**(2), pp.28-38.
2. Zimmer, Dirk (2010), *Equation-based Modeling of Variable-structure Systems*, Ph.D. Dissertation, Dept. of Computer Science, ETH Zurich, Switzerland.