Relationship between Laplace and Fourier Transforms

We have seen that, in a narrow sense, the Fourier transform only exists for functions that decay to zero for sufficiently large positive and negative values of time.

We allowed a generalization that would define generalized Fourier transforms for at least a few functions that do not satisfy the above constraints.

Fourier transform can also be used to describe systems (by means of transfer functions), but only if their impulse response decays, i.e., if the system is stable. This makes the technique useless for control engineers whose primary job is to stabilize unstable systems.

Idea: Transform the function in the time domain before applying the Fourier transform:

Given $x(t)$ that doesn't grow faster than $e^{\alpha t}$ for large values of time.

$$\Rightarrow \boxed{y(t) = x(t) \cdot e^{-\sigma t} \cdot \varepsilon(t)}$$

is a function that decays for $t<0$ (because of $\varepsilon(t)$), and also for large positive values of t, if $\sigma>\alpha$.

$\Rightarrow$ $y(t)$ has a Fourier transform.

$$Y(m) = \int_{-\infty}^{+\infty} y(t) \cdot e^{-mt} \cdot dt$$

$$\Rightarrow Y(j\omega) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-\sigma t} \cdot \varepsilon(t) \cdot e^{-j\omega t} \cdot dt$$

$$\Rightarrow Y(j\omega) = \int_{-\infty}^{+\infty} x(t) \cdot \varepsilon(t) \cdot e^{-(\sigma+j\omega)t} \cdot dt$$

$$= \int_{0-}^{\infty} x(t) \cdot e^{-(\sigma+j\omega)t} dt$$

We introduce a new variable:

$$s = \sigma + j\omega$$

$$\Rightarrow Y(j\omega) = \int_{0-}^{\infty} x(t) \cdot e^{-st} dt$$

By treating σ as a variable rather than as a constant, we can imbed this into:

$$\boxed{X(s) = \int_{0-}^{\infty} x(t) \cdot e^{-st} dt}$$

The embedded Fourier transform of $y(t)$ is defined as the Laplace transform of $x(t)$.

$$y(t) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} Y(m) \cdot e^{+mt} \cdot dm$$

$$\Rightarrow x(t) \cdot e^{-\sigma t} \cdot \varepsilon(t) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} Y(j\omega) \cdot e^{+j\omega t} \cdot d(j\omega)$$

If we assume that

$$x(t) = \emptyset \; ; \; \forall t < \emptyset$$

we can ignore the $\varepsilon(t)$ term:

$$x(t) \cdot e^{-\sigma t} = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} Y(j\omega) \cdot e^{+j\omega t} \cdot d(j\omega)$$

$$\Rightarrow x(t) = \frac{1}{2\pi j} \cdot e^{+\sigma t} \cdot \int_{-j\infty}^{+j\infty} Y(j\omega) \cdot e^{+j\omega t} \cdot d(j\omega)$$

$$= \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} Y(j\omega) \cdot e^{\sigma t} \cdot e^{j\omega t} \cdot d(j\omega)$$

$$= \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} Y(j\omega) \cdot e^{+(\sigma + j\omega) t} \cdot d(j\omega)$$

$$\Rightarrow \quad x(t) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} Y(j\omega) \cdot e^{+st} \cdot d(j\omega)$$

Embedding:

$$s = \sigma + j\omega$$

$$ds = d(j\omega)$$

because σ is treated as a constant

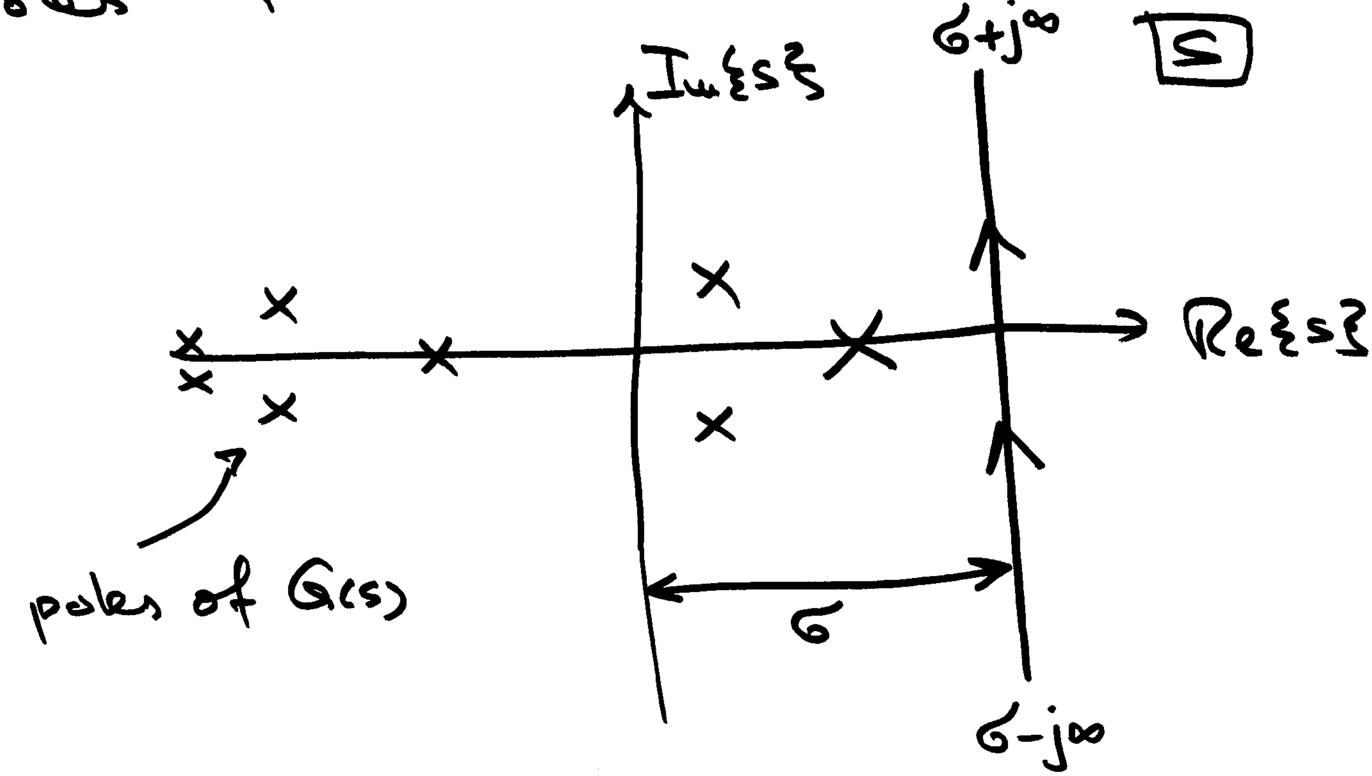
$j\omega$	s
$-j\infty$	$\sigma - j\infty$
$+j\infty$	$\sigma + j\infty$

$$\Rightarrow \quad \boxed{x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) \cdot e^{+st} \cdot ds}$$

is the inverse Laplace transform.

Let $G(s)$ be a transfer function of a linear system, i.e., a rational function of s.

To compute $g(t) = \mathcal{L}^{-1}\{G(s)\}$, we need to choose σ such that it is larger than any poles of $G(s)$:



The path must be to the right of all the poles of $G(s)$.

The Laplace transform exists for all functions that decay for sufficiently large negative values of time.

Let $x(t) = \emptyset \; ; \; \forall t < t_0 < \emptyset$

$$\Rightarrow \tau = t + t_0$$

$$\Rightarrow x(\tau) = \emptyset \; ; \; \forall \tau < \emptyset$$

$x(\tau)$ has a Laplace transform.

It is <u>more powerful</u> than the Fourier transform, because the restriction on positive values of time (stability) no longer applies.

It is also <u>less powerful</u> than the Fourier transform, because there is no Laplace transform of periodic signals.

Let $x(t)$ be a function:

$$x(t) = 0 \; ; \; \forall t < 0$$

$$\lim_{t \to \infty} x(t) = 0$$

In this case, $x(t)$ has a Fourier transform. It also has a Laplace transform, and we can choose $\sigma = 0$.

$$y(t) = x(t) \cdot e^{-\sigma t} \cdot \varepsilon(t) \equiv x(t)$$

$$X(s) = \int_{0-}^{\infty} x(t) \cdot e^{-st} \, dt$$

$$\equiv \int_{-\infty}^{+\infty} x(t) \, e^{-(\sigma + j\omega) t} \, dt$$

$$\equiv \int_{-\infty}^{+\infty} x(t) \, e^{-j\omega t} \, dt \equiv X(j\omega)$$

If both transforms exist, the Fourier transform <u>is</u> the Laplace transform for $\sigma = 0$.

Some old books define the Laplace transform differently:

$$X(p) = \int_{0+}^{\infty} x(t) \cdot e^{-pt} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(p) \cdot e^{+pt} \cdot dp$$

Notice that:

$$\mathcal{L}_s \{ \delta(t) \} \equiv 1$$

$$\mathcal{L}_p \{ \delta(t) \} \equiv \phi$$

The so-called "ϕ+ Laplace transform" cannot handle δ-functions. It therefore has difficulties with the correct handling of initial conditions. It is therefore not recommended.