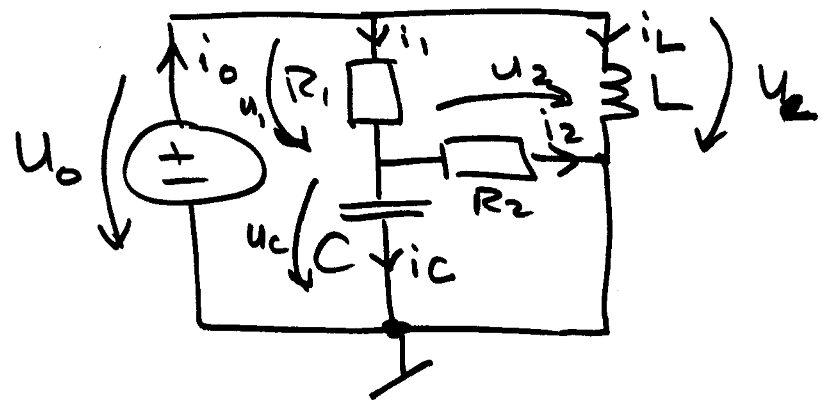
Modeling circuits in the time domain

Given a circuit:



We name all voltages and currents. Since we have n-branches in the circuit, there are now 2n variables.

$\Rightarrow$ We need 2n equations to compute them.

We start with the elemental laws of each branch:

$$\left.\begin{aligned} U_0 &= f(t) \\ u_1 &= R_1 \cdot i_1 \\ U_2 &= R_2 \cdot i_2 \\ i_c &= C \cdot \frac{du_c}{dt} \\ u_L &= L \cdot \frac{di_L}{dt} \end{aligned}\right\} \text{n equations (one per branch)}$$

We then formulate Kirchhoff's Voltage law for each mesh:

$$U_0 = u_1 + u_c$$
$$u_L = u_1 + u_2$$
$$u_c = u_2$$

and Kirchhoff's current law for each node <u>except</u> the ground node:

$$i_0 = i_1 + i_L$$
$$i_1 = i_2 + i_c$$

Since in every circuit:

$$\#_{meshes} + \#_{nodes} \equiv \#_{branches} + 1$$

we now got $2n$ equations.

We assume the state variables (outputs of integrators) to be known. We underline all known variables.

We now visit each equation. If an equation only contains one unknown, we solve for that unknown. If an unknown shows up in only one unused equation, we need to solve for it from that equation. We place the variables we solve for in square brackets.

$$[U_0] = f(\underline{t})$$
$$\underline{u_1} = R_1 \cdot [i_1]$$
$$\underline{u_2} = R_2 \cdot [i_2]$$
$$i_c = C \cdot \left[\frac{du_c}{dt}\right]$$
$$u_L = L \cdot \left[\frac{di_L}{dt}\right]$$

$$\underline{U_0} = [u_1] + \underline{u_c}$$
$$[u_L] = \underline{u_1} + \underline{u_2}$$
$$\underline{u_c} = [u_2]$$
$$[i_0] = i_1 + \underline{i_L}$$
$$\underline{i_1} = \underline{i_2} + [i_c]$$

$\Rightarrow$

$$U_0 = f(t)$$
$$i_1 = u_1 / R_1$$
$$i_2 = u_2 / R_2$$
$$\frac{du_c}{dt} = i_c / C$$
$$\frac{di_L}{dt} = u_L / L$$

$$u_1 = U_0 - u_c$$
$$u_L = u_1 + u_2$$
$$u_2 = u_c$$
$$i_0 = i_1 + i_L$$
$$i_c = i_1 - i_2$$

are the solved equations.

We can now construct a state-space model:

$$u = U_0$$
$$x_1 = u_C$$
$$x_2 = i_L$$
$$y = u_2 \quad \text{(why not!)}$$

$$\dot{x}_1 = \frac{dx_1}{dt} = \frac{du_C}{dt} = \frac{1}{C} \cdot i_C = \frac{1}{C} \cdot i_1 - \frac{1}{C} \cdot i_2$$

$$= \frac{1}{R_1 C} \cdot u_1 - \frac{1}{R_2 C} \cdot u_2$$

$$= \frac{1}{R_1 C} \cdot U_0 - \frac{1}{R_1 C} \cdot u_C - \frac{1}{R_2 C} \cdot u_C$$

$$= \frac{1}{R_1 C} \cdot u - \frac{1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) x_1$$

$$\dot{x}_2 = \frac{dx_2}{dt} = \frac{di_L}{dt} = \frac{1}{L} \cdot u_L = \frac{1}{L} u_1 + \frac{1}{L} \cdot u_2$$

$$= \frac{1}{L} \cdot U_0 \cancel{- \frac{1}{L} \cdot u_C} \cancel{+ \frac{1}{L} \cdot u_C} = \frac{1}{L} \cdot u$$

$$y = u_2 = u_C = x_1$$

$$
\left|
\begin{aligned}
\dot{x}_1 &= -\frac{R_1+R_2}{C(R_1R_2)}\,x_1 + \frac{1}{R_1C}\,u \\
\dot{x}_2 &= \frac{1}{L}\cdot u \\
y &= x_1
\end{aligned}
\right|
$$

In matrix form:

$$
\underline{\dot{x}} = \begin{bmatrix} -\frac{R_1+R_2}{R_1R_2C} & 0 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} \frac{1}{R_1C} \\ \frac{1}{L} \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \end{bmatrix} u
$$

Of course, if we are only interested in u_2, we wouldn't need to compute i_L at all.

$$\left|\begin{aligned} \dot{x}_1 &= -\frac{R_1+R_2}{R_1R_2C}\,x_1 + \frac{1}{R_1C}\,u \\ y &= x_1 \end{aligned}\right|$$

$$\Rightarrow \left|\begin{aligned} \dot{x}_1 &= \left[-\frac{R_1+R_2}{R_1R_2C}\right] x_1 + \left[\frac{1}{R_1C}\right] u \\ y &= \left[\quad 1 \quad\right] x_1 + \left[\,\phi\,\right] u \end{aligned}\right|$$