Let us now look at two linear systems sharing the same A matrix:

$$A \cdot \underline{x}_1 = \underline{b}_1$$
$$A \cdot \underline{x}_2 = \underline{b}_2$$

Obviously:

$$\underline{x}_1 = A^{-1} \cdot \underline{b}_1$$
$$\underline{x}_2 = A^{-1} \cdot \underline{b}_2$$

We can naturally extend our shorthand notation in the following way:

$$A \cdot \left[\underline{x}_1 \,\vdots\, \underline{x}_2 \right] = \left[\underline{b}_1 \,\vdots\, \underline{b}_2 \right]$$

concatenation

or: $$A \cdot X = B$$

where: $X = \left[\underline{x}_1 \,\vdots\, \underline{x}_2 \right]$; $B = \left[\underline{b}_1 \,\vdots\, \underline{b}_2 \right]$

and: $\quad X, B \in \mathbb{R}^{n \times 2}$

$$\Rightarrow X = A^{-1} \cdot B$$

In general:

$$\begin{array}{cc} & n \\ p & \begin{bmatrix} \quad A \quad \end{bmatrix} \end{array} \cdot \begin{array}{cc} & m \\ n & \begin{bmatrix} \\ \quad X \quad \\ \\ \end{bmatrix} \end{array} = \begin{array}{cc} & m \\ p & \begin{bmatrix} \quad B \quad \end{bmatrix} \end{array}$$

Multiplication of A with X is possible, iff # colums of A = # rows of X. The resulting matrix B has the same number of rows as A, and the same number of columns as X.

$$b_{ij} = \sum_{k=1}^{n} a_{ik} \cdot x_{kj}$$

In physics, ~~one~~ can also find the notation:

$$b_{ij} = a_{ik} \cdot x_{kj}$$

with a repeated index (k) being implied to mean summation. This is referred to as the "Einstein" notation, because Einstein used it frequently in his notes. In engineering books, this notation has not been used much.

Let $\underline{a}_i'$ be a row vector consisting of the elements of the i^{th} row of A, and

$\underline{a}_j$ be a column vector consisting of the elements of the j^{th} column of A.

We use the apostrophe or sometimes a superindex ^{T}:

$$\underline{a}_i' = \underline{a}_i^T$$

to denote the fact that we took a row rather than a column. Then:

$$b_{ij} = \underline{a}_i' \cdot \underline{x}_j$$

would be the preferred engineering notation avoiding the explicit summation symbol.

However, we have to be careful with this notation, because the apostrophe (or superindex ^{T}) have been introduced also to denote the transpose of A matrix:

$$A' = A^T = \{ a_{ji} \}$$

Let
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\Rightarrow \quad A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Thus, $\underline{a}_3'$ can then either mean the transpose of $\underline{a}_3$, i.e.

$$\underline{a}_3' = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}' = \begin{bmatrix} 3 & 6 & 9 \end{bmatrix}$$

or it can mean the 3rd row vector:

$$\underline{a}_3' = \begin{bmatrix} 7 & 8 & 9 \end{bmatrix}$$

which are not the same.

<u>Matlab</u>: knows addition, subtraction and multiplication of matrices

A*B + C .

It also knows multiplication of a matrix with a scalar k*A

[This had implicitly been used already:

$$A^{-1} = \frac{A^{+}}{|A|}$$

$$\Rightarrow \underbrace{A^{+}}_{\text{matrix}} = \underbrace{|A|}_{\text{scalar}} \cdot \underbrace{A^{-1}}_{\text{matrix}}$$

]

Matlab knows the meaning of the apostrophe to denote the transpose:

A'*B

and it knows inverses

inv(A)

and determinants

det(A)

but <u>not</u> adjugates.

Let:

$$A \cdot \underline{x} = \underline{b}$$

$$\Rightarrow \underline{x} = A^{-1} \cdot \underline{b}$$

$$= A^{-1} \cdot (A \cdot \underline{x})$$

$$= (A^{-1} \cdot A) \cdot \underline{x}$$

$A^{-1} \cdot A$ is an $n \times n$ matrix. Multiplying $\underline{x}$ from the left with $A^{-1} \cdot A$, results in $\underline{x}$. The only matrix that does this is the <u>identity</u> <u>matrix</u>, $I^{(n)}$.

$$\Rightarrow \boxed{A^{-1} \cdot A = I^{(n)}} \quad ; \quad I^{(3)} = \begin{bmatrix} 1 & \phi & \phi \\ \phi & 1 & \phi \\ \phi & \phi & 1 \end{bmatrix}$$

From the definitions as shorthand, one can easily verify that:

$$\boxed{(A \cdot B)' = B' \cdot A'}$$

$$\Rightarrow \quad (A^{-1} \cdot A)' = (I^{(n)})'$$

$$\Rightarrow \quad A' \cdot (A^{-1})' = I^{(n)}$$

However: $\boxed{(A^{-1})' \equiv (A')^{-1}}$

$$\Rightarrow \quad A' \cdot (A')^{-1} = I^{(n)}$$

Let: $\quad B = A'$

$$\Rightarrow \quad \boxed{B \cdot B^{-1} = I^{(n)}}$$

$\Rightarrow$ Any non-singular matrix, A, can be multiplied either from the left or from the right with its inverse, yielding the identity matrix.

Let:
$$A \cdot \underline{x} = \underline{b}$$

$$\Rightarrow \quad A^{-1} \cdot (A \cdot \underline{x}) = A^{-1} \cdot \underline{b}$$

$$\Rightarrow \quad (A^{-1} \cdot A) \cdot \underline{x} = A^{-1} \cdot \underline{b}$$

$$\Rightarrow \quad I^{(n)} \cdot \underline{x} = A^{-1} \cdot \underline{b}$$

$$\Rightarrow \quad \underline{x} = A^{-1} \cdot \underline{b}$$

Notice that:

$$\boxed{(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}}$$