Vertex Generation

Hunt for a Hard Case

Related Problems

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# Generating Vertices of Polyhedra is Hard

#### ... and other related problems ...

#### Endre Boros

Joint research with K. Borys, K. Elbassioni, V. Gurvich, and L. Khachiyan<sup>1</sup>

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## Outline

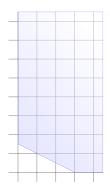
1 Polyhedra and Vertices • What is a polyhedron? • What is a vertex? • What is vertex generation? • When is generation hard? • Hypergraph dualization • A polyhedral application • Matching polytopes • Yet another reformulation • The hunt resumed ... • The hunt is over • Simplices and Bodies

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### Intersection of half-spaces



$x_1$	$+2x_{2}$	$\geq$	3

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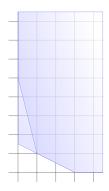
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#### Intersection of half-spaces



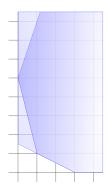
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### Intersection of half-spaces



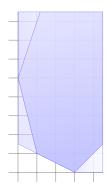
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#### Intersection of half-spaces



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### Intersection of half-spaces



$$P = \left\{ (x_1, x_2) \in \mathbb{R}^2 \middle| \begin{array}{ccc} x_1 & +2x_2 & \ge & 3\\ 4x_1 & +x_2 & \ge & 5\\ 3x_1 & -x_2 & \ge & -5\\ -x_1 & +x_2 & \ge & -3 \end{array} \right\}$$

 $\begin{array}{c} Polyhedra\\ \circ\circ\bullet\circ\circ\end{array}$ 

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### Those prickly corners



$$P = \left\{ x \in \mathbb{R}^d \mid Ax \ge b \right\}$$

 $v \in P$  is a vertex if there are no  $u, w \in P$  such that

$$v=\frac{1}{2}u+\frac{1}{2}w$$

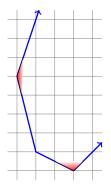
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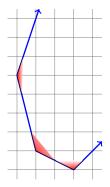
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## Polynomially decidable questions



Given a polyhedron  $P = \{x \in \mathbb{R}^d \mid Ax \ge b\}$ , let V(P) denote its set of vertices.

• Is  $P \neq \emptyset$ ?

Is  $V(P) \neq \emptyset$ ?

• Is  $\operatorname{conv}(V(P)) = P?$ 

All these, and many other related questions, can be decided efficiently by solving linear programming problems.

(Khachiyan, 1979)

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## Outline

Polyhedra and Vertices
What is a polyhedron?
What is a vertex?

#### 2 Vertex Generation

#### • What is vertex generation?

- When is generation hard?
- Hypergraph dualization
- A polyhedral application
- **3** Hunt for a Hard Case
  - Matching polytopes
  - Yet another reformulation
  - The hunt resumed ...
  - The hunt is over

#### Related Problems

• Simplices and Bodies

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## Vertex generation

First formulation (Mr. Folklore, Age of Pisces):

Given 
$$P = \{x \in \mathbb{R}^d \mid Ax \ge b\}$$
 generate  $V(P)$ .

- Output maybe exponentially larger than input!
- Long history ... (Motzkin, Raiffa, Thompson and Thrall, 1953) (Charnes and Cooper, 1953) (Balinski, 1961)
- Well solved for many special cases
  - Simple polytopes
  - In fixed dimension
  - Network polytopes
  - Zero-one polytopes

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Second formulation (Lovász, 1992):

Given  $P = \left\{ x \in \mathbb{R}^d | Ax \ge b \right\}$  and  $\mathcal{A} \subseteq \mathbb{R}^d$  decide if  $\mathcal{A} = V(P)$ .

- V(P) can be generated by repeatedly solving the above decision problem.
- $\operatorname{conv}(\mathcal{A}) \subseteq P$  is easy to check
- $P \subseteq \operatorname{conv}(A)$  is co-NP-complete (Freund and Orlin 1985)
- Yet, if  $\mathcal{A} \subseteq V(P)$ , then  $P \subseteq \operatorname{conv}(\mathcal{A})$  was open .....

Given  $P = \{x \in \mathbb{R}^d | Ax \ge b\}$  and  $A \subseteq V(P)$ , deciding whether  $V(P) \subseteq A$  or not is co-NP-complete.

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### Equivalent vertex definitions

### Assume $A \in \mathbb{R}^{m \times d}$ , $b \in \mathbb{R}^m$ , $I \subseteq [m] = \{1, ..., m\}$ , and let

- $A_I$  be the submatrix of S formed by the rows  $i \in I$ ;
- $b_I$  be the subvector of b formed by the components  $i \in I$ ;

• 
$$\overline{I} = \{1, ..., m\} \setminus I;$$

• 
$$P_I = \{x \in \mathbb{R}^d \mid A_I x = b_I, \ A_{\bar{I}} x \ge b_{\bar{I}} \}.$$

#### Claim

For  $P = \{x \in \mathbb{R}^d \mid Ax \ge b\}$  such that  $V(P) \ne \emptyset$ , there is a one-to-one correspondence between vertices of P and the maximal tight feasible subsets of the inequalities

 $MaxTF(P) = \{ max'l I \subseteq [m] \mid P_I \neq \emptyset \}.$ 

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### Monotone properties

#### Assume $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$ .

Let us define a property  $\Pi \subseteq 2^{\{1,\dots,m\}}$  such that  $I \in \Pi$  iff

$$P_I = \left\{ x \in \mathbb{R}^d \mid A_I x = b_I, \ A_{\bar{I}} x \ge b_{\bar{I}} \right\} \neq \emptyset.$$

• Then,  $\Pi$  is a monotone property:

 $I \subseteq I' \in \mathbf{\Pi}$  implies  $I \in \mathbf{\Pi}$ .

• Generating V(P) is equivalent with generating

 $Max(\mathbf{\Pi}) = MaxTF(P) = \{ \max' l \text{ subsets } I \in \mathbf{\Pi} \}.$ 

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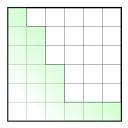
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### Monotone generation



Consider a monotone property  $\Pi$  in a lattice (e.g.,  $\{0,1\}^m$ ) •  $Max(\Pi) = \{ \text{ max'l elements } v \in \Pi \}.$ •  $Min(\overline{\Pi}) = \{ \text{ min'l elements } v \notin \Pi \}.$ 

#### Given a monotone system 11, generate

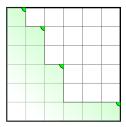
- $\sim Max(2)$  (or Min(2) or both).
- $\circ$  (Typically size(12) << (Max(12)).

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#### Monotone generation



Consider a monotone property II in a lattice (e.g., {0,1}<sup>m</sup>)
Max(II) = { max'l elements v ∈ II}.
Min(II) = { min'l elements v ∉ II}.

- Typically size( $\square$ )  $\ll |Max(\square)|$ .
- Blow to measure efficiency of

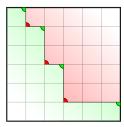
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Given a monotone system II, generate

- $Max(\Pi)$  (or  $Min(\overline{\Pi})$  or both).
- Typically  $size(\Pi) \ll |Max(\Pi)|$ .
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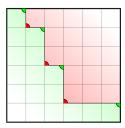
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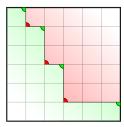
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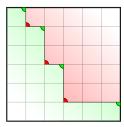
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- How to measure efficiency of generation?

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# Outline

- What is a polyhedron? • What is a vertex? 2 Vertex Generation • What is vertex generation? • When is generation hard? • Hypergraph dualization • A polyhedral application • Matching polytopes • Yet another reformulation • The hunt resumed ... • The hunt is over
  - Simplices and Bodies

Vertex Generation

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Related Problems 000

# Complexity of generation

#### Sequential generation

• Given a monotone system  $\Pi$  of input size  $|\Pi| = N$ , an algorithm  $\mathfrak{A}$  generates one-by-one the elements

 $Max(\mathbf{\Pi}) = \{v_1, v_2, ..., v_M\},\$ 

outputting  $v_k$  at time  $t_k$   $(t_1 \le t_2 \le \cdots \le t_M)$ .

• Algorithm  $\mathfrak{A}$  is said to work

in total polynomial time, if  $t_M \leq poly(N)$ 

 $t_k \leq poly(N,k)$  for all  $k \leq M$ 

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 $t_{k+1} - t_k \le poly(N)$  for all k < M

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# Complexity of generation

#### $\mathrm{NEXT}(\Pi,\mathcal{M})$

Given a monotone system  $\Pi$  and  $\mathcal{M} \subseteq Max(\Pi)$ , decide if  $\mathcal{M} = Max(\Pi)$ , or find  $v \in Max(\Pi) \setminus \mathcal{M}$  if not.

#### Theorem (Ms. Folklore, 19??)

 $Max(\Pi)$  can be generated in incremental polynomial time (total polynomial time) if and only if problem  $NEXT(\Pi, \mathcal{M})$  can be solved in polynomial time for all  $\mathcal{M} \subseteq Max(\Pi)$ .

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### Prime example for monotone generation

#### Hypergraph transversals

Let |U| = m and  $\mathcal{H} \subseteq 2^U$  be a hypergraph. Associate to it a property  $\Pi = \Pi_{\mathcal{H}} \subseteq 2^U$  by

 $S \in \Pi \iff \begin{cases} \nexists H \in \mathcal{H} : H \subseteq S \\ S \text{ is independent} \end{cases} \Leftrightarrow (U \setminus S) \cap H \neq \emptyset \ \forall H \in \mathcal{H} \\ (U \setminus S) \text{ is a transversal} \end{cases}$ 

- $\mathcal{H}^* = Max(\Pi_{\mathcal{H}})$  is the family of maximal independent sets of  $\mathcal{H}$ .
- $\mathcal{H}^d = \{U \setminus S \mid S \in Max(\Pi_{\mathcal{H}})\}$  is the family of minimal transversals of  $\mathcal{H}$ .
- $\mathcal{H} \to \mathcal{H}^d$  (or  $\mathcal{H} \to \mathcal{H}^*$ ) are known as the hypergraph transversal or monotone dualization problems.

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Vertex Generation

Hunt for a Hard Case

Related Problems

## Generating hypergraph transversals

Theorem (Fredmand and Khachiyan, 1996)

For any hypergraph  $\mathcal{H}$  and an arbitrary family  $\mathcal{M} \subseteq \mathcal{H}^d$  of its minimal transversals, problem  $NEXT(\mathcal{H}, \mathcal{M})$  can be solved in  $O\left((|\mathcal{H}| + |\mathcal{H}^d|)^{o(\log |\mathcal{H}| + |\mathcal{H}^d|)}\right)$  time.

... many-many special cases ...

#### Claim (Eiter and Gottlob, 1995)

If for all hyperedges  $H \in \mathcal{H}$  we have  $|H| \leq k$ , where k is fixed, then  $\mathcal{H}^d$  can be generated with polynomial delay.

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# Outline

- Polyhedra and VerticesWhat is a polyhedron?
  - What is a vertex?

## 2 Vertex Generation

- What is vertex generation?
- When is generation hard?
- Hypergraph dualization

## • A polyhedral application

- **3** Hunt for a Hard Case
  - Matching polytopes
  - Yet another reformulation
  - The hunt resumed ...
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## Related Problems

• Simplices and Bodies

Hunt for a Hard Case

Related Problems 000

## Vertex generation in fixed dimension

#### Assume $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$ , and recall:

Generating the vertices of  $P = \{x \in \mathbb{R}^d \mid Ax \ge b\}$  is equivalent with generating MaxTF (maximal subsets  $I \subseteq [m] = \{1, ..., m\}$ for which  $P_I = \{x \in \mathbb{R}^d \mid A_I x = b_I, A_{\bar{I}} x \ge b_{\bar{I}}\} \neq \emptyset$ ).

• Think of MaxTF as the family  $\mathcal{H}^*$  of maximal independent sets of a hypergraph  $\mathcal{H}$ .

- $\mathcal{H} = MinTI = \{ \min' \mid J \subseteq [m] \mid P_J = \emptyset \}.$
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- A polynomial delay vertex generation in fixed dimension
  - Generate  $\mathcal{H} = MinTI$  in  $O(m^d)$  time.
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Vertex Generation

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Vertex Generation

Related Problems 000

# Bipartite matching polytope



Let 
$$G = (V, E)$$
 be a bipartite graph, and consider  

$$P = \left\{ x \in \mathbb{R}^E \mid \sum_{e \ni v} x_e \leq 1 \quad \forall v \in V \\ x_e \geq 0 \quad \forall e \in E \end{array} \right\}$$

$$MaxTF = \left\{ \max' \mathbf{I} \mathbf{I} \subseteq V \cup E \mid \begin{bmatrix} E \setminus \mathbf{I} \text{ is a matching} \\ \text{covering } V \cap \mathbf{I} \end{bmatrix}$$

$$V(P) \longleftrightarrow MaxTF \longleftrightarrow \mathcal{M} = \left\{ \max' \mathbf{I} \text{ matchings of } G \right\}$$

Polynomial delay generation (Fukuda and Matsui, 1992) (Uno, 1997)

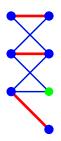
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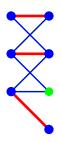
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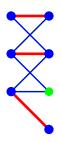
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Vertex Generation

Related Problems

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Polynomial delay generation (Fukuda and Matsui, 1992) (Uno, 1997)

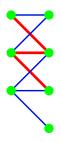
 $\mathcal{M}^{a}$  can also be generated with polynomial delay (Boros, Elbassioni, and Gurvich, 200

Vertex Generation

Related Problems

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## Bipartite matching polytope



Let 
$$G = (V, E)$$
 be a bipartite graph, and consider  

$$P = \left\{ x \in \mathbb{R}^E \mid \begin{array}{c} \sum_{e \ni v} x_e & \leq 1 \quad \forall v \in V \\ \\ x_e & \geq 0 \quad \forall e \in E \end{array} \right\}$$

$$MaxTF = \left\{ \text{max'l } \mathbf{I} \subseteq V \cup E \mid \begin{array}{c} E \setminus \mathbf{I} \text{ is a matching} \\ \\ \text{covering } V \cap \mathbf{I} \end{array} \right\}$$

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Polynomial delay generation (Fukuda and Matsui, 1992) (Uno, 1997)

 $\mathcal{M}^d$  can also be generated with polynomial delay (Boros, Elbassioni, and Gurvich, 2004)

Vertex Generation

Related Problems 000

### Non-bipartite matching polytope



Let 
$$G = (V, E)$$
 be a connected graph, and consider  

$$P = \left\{ x \in \mathbb{R}^E \mid \begin{array}{c} \sum_{e \ni v} x_e & \leq 1 \quad \forall v \in V \\ \\ x_e & \geq 0 \quad \forall e \in E \end{array} \right\}$$

$$MaxTF = \left\{ \max \exists \mathbf{I} \subseteq V \cup E \mid \begin{array}{c} E \setminus \mathbf{I} \text{ is a 2-matching} \\ \\ \text{covering } V \cap \mathbf{I} \end{array} \right\}$$

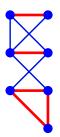
$$V(B) \leftarrow MaxTE \leftarrow MaxTE \in C \text{ and the proof } C$$

With polynomial delay (Boros, Elbassioni, and Gurvich, 2004)

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Vertex Generation 00000000000 Related Problems 000

## Non-bipartite matching polytope



Let 
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$$P = \left\{ x \in \mathbb{R}^E \mid \sum_{e \ni v} x_e \leq 1 \quad \forall v \in V \\ x_e \geq 0 \quad \forall e \in E \right\}$$

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 $V(P) \longleftrightarrow MaxTF \longleftrightarrow \mathcal{M} = \{ 2\text{-matchings of } G \}$ 

With polynomial delay

(Boros, Elbassioni, and Gurvich, 2004)

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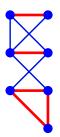
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### Non-bipartite matching polytope



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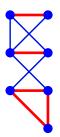
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# Outline

- What is a polyhedron? • What is a vertex? • What is vertex generation? • When is generation hard? • Hypergraph dualization • A polyhedral application 3 Hunt for a Hard Case • Matching polytopes • Yet another reformulation • The hunt resumed ... • The hunt is over
  - Simplices and Bodies

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### Irreducible Inconsistent Subsystems (IIS)

Consider  $A \in \mathbb{R}^{m \times d}$  and  $b \in \mathbb{R}^d$  such that  $Ax \ge b$  is inconsistent.

•  $MinIS(A, b) = \{\min' \mid I \subseteq [m] \mid A_I x \ge b_I \text{ is inconsistent} \}$ •  $MaxFS(A, b) = \{\max' \mid I \subseteq [m] \mid A_I x \ge b_I \text{ is feasible} \}$ 

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#### Facts and History

•  $MinIS(A, b)^* = MaxFS(A, b)$ 

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#### Facts and History

- $MinIS(A, b)^* = MaxFS(A, b)$
- Lots of attention ... machine learning applications

(Gleason and Ryan, 1990)

(Ryan, 1996)

(Pfetsch, 2002)

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(Amaldi, Pfetsch and Trotter, 2003)

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## Irreducible Inconsistent Subsystems (IIS)

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#### Facts and History

- $MinIS(A, b)^* = MaxFS(A, b)$
- Problems  $\min\{|I| \mid I \in MinIS(A, b)\}$  and  $\max\{|I| \mid I \in MaxFS(A, b)\}$  are both NP-hard

(Johnson and Preparata, 1978)

(Chakravarty, 1994)

(Pfetsch, 2002)

(Amaldi, Pfetsch and Trotter, 2003)

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#### Alternative Polyhedron

(Gleason and Ryan, 1990)

$$Q_{A,b} = \{ y \in \mathbb{R}^m \mid y^T A = 0, \ y^T b = 1, \ y \ge 0 \}$$

Claim

= Farkas' lemma, 1901

$$MinIS(A,b) \iff V(Q_{A,b})$$

Another monotone formulation of vertex generation!

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## Irreducible Inconsistent Subsystems (IIS)

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Vertex Generation

Related Problems 000

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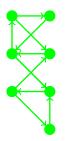
# Outline

• What is a polyhedron? • What is a vertex? • What is vertex generation? • When is generation hard? • Hypergraph dualization • A polyhedral application 3 Hunt for a Hard Case • Matching polytopes • Yet another reformulation • The hunt resumed ... • The hunt is over • Simplices and Bodies

Vertex Generation

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## Acyclic subgraph polyhedron



Let G = (V, E) be a directed graph,  $x \in \mathbb{R}^V$ , and consider the linear system  $\{x_i - x_j \ge 1 \ \forall \ (i, j) \in E\}$ 

 $MinIS \leftrightarrow \{ \text{ simple cycles of } G \}$ 

With polynomial delay

(Read and Tarjan 1975)

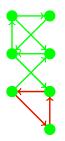
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With polynomial delay (Schwikowski and Speckenmeyer, 2002)

Vertex Generation

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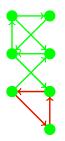
 $MaxFS \leftrightarrow \left\{ \begin{array}{c} \max'' \\ \operatorname{acyclic subgraphs} \\ \operatorname{of} G \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \min'' \\ \operatorname{feedback \ are \ sets} \\ \operatorname{of} G \end{array} \right\}$ 

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Related Problems 000

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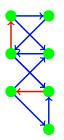
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With polynomial delay (Schwikowski and Speckenmeyer, 2002)

Vertex Generation 00000000000 Related Problems 000

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# Acyclic subgraph polyhedron



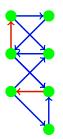
Let $G = (V, E)$ by the linear system					nsider	
Min	$nIS \leftrightarrow \{ sim \}$	ple cy	v <mark>cles</mark> of	G		
With polynomial delay			(Read and Tarjan 1975)			

$$MaxFS \leftrightarrow \left\{ \begin{array}{c} max'l \\ acyclic \ subgraphs \\ of \ G \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} min'l \\ feedback \ arc \ sets \\ of \ G \end{array} \right\}$$
  
With polynomial delay (Schwikowski and Speckenmeyer, 2002)

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# Acyclic subgraph polyhedron



Let $G = (V, E)$ be a directed graph the linear system $\{x_i - x_j \ge 1\}$	- / /
$MinIS \leftrightarrow \{ \text{ simple} \}$	cycles of $G$ }
With polynomial delay	(Read and Tarjan 1975)
$MarES \leftrightarrow \begin{cases} max'l \\ acyclic subgraphs \end{cases}$	$\int \min' l$

$$MaxFS \leftrightarrow \left\{ \begin{array}{c} max'l \\ acyclic \ subgraphs \\ of \ G \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} min'l \\ feedback \ arc \ sets \\ of \ G \end{array} \right\}$$
  
With polynomial delay (Schwikowski and Speckenmeyer, 2002)

Vertex Generation

Hunt for a Hard Case

Related Problems 000

### Strongly connected subgraphs' polyhedron

Let G = (V, E) be a strongly connected directed graph,  $x \in \mathbb{R}^V$ , and consider the system of linear inequalities  $x_j - x_i \ge 0 \quad \forall \ (i, j) \in E$  $\sum_{(i,j) \in E} (x_j - x_i) \ge 1$  $(i,j) \in E$ 

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 $MinIS \iff \{ \min' \mathbf{I} \subseteq E \mid (V, \mathbf{I}) \text{ is strongly connected} \}$ 

Incrementally polynomial (Boros, Elbassioni, Gurvich and Kh

 $Max FS \longrightarrow \{\max' | 1 \subseteq E | (V, 1) \text{ is not strongly connected} \}$ NP-hard (Boros, Elbassioni, Gurvich and Khachiyan, 2004)

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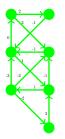
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NP-hard (Boros, Elbassioni, Gurvich and Khachiyan, 2004)

Vertex Generation

Related Problems

## Negative cycle free subgraphs' polyhedron



Let G = (V, E) be a directed graph,  $w : E \to \mathbb{R}, x \in \mathbb{R}^V$ , and consider the system of linear inequalities  $\{x_i - x_j \le w_{ij} \ \forall \ (i, j) \in E\}$ MinIS  $\rightsquigarrow \{\mathbf{C} \subseteq E \mid \mathbf{C} \text{ is a negative cycle }\}$ 

Theorem (Boros, Borys, Elbassioni, Gurvich and Khachiyan, 2005)

Given a directed graph G with real weights on its arcs, generating all negative cycles of G is **NP-hard**.

 $MaxFS \iff \{\max' | \mathbf{I} \subseteq E \mid (V, \mathbf{I}) \text{ is negative cycle free } \}$ 

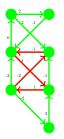
**Open** *~~ ???* **blocking short paths** ???

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Vertex Generation

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Given a directed graph G with real weights on its arcs, generating all negative cycles of G is **NP-hard**. Even if  $w_{ij} \in \{\pm 1\}$  for all arcs  $(i, j) \in B$ .

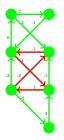
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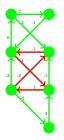
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Vertex Generation

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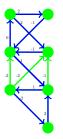


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## Negative cycle free subgraphs' polyhedron



Let G = (V, E) be a directed graph,  $w : E \to \mathbb{R}, x \in \mathbb{R}^V$ , and consider the system of linear inequalities  $\{x_i - x_j \le w_{ij} \ \forall \ (i, j) \in E\}$ MinIS  $\iff \{\mathbf{C} \subseteq E \mid \mathbf{C} \text{ is a negative cycle }\}$ 

Theorem (Boros, Borys, Elbassioni, Gurvich and Khachiyan, 2005)

Given a directed graph G with real weights on its arcs, generating all negative cycles of G is **NP-hard**. Even if  $w_{ij} \in \{\pm 1\}$  for all arcs  $(i, j) \in E$ .

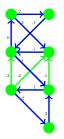
 $MaxFS \iff \{\max' | \mathbf{I} \subseteq E \mid (V, \mathbf{I}) \text{ is negative cycle free } \}$ 

Vertex Generation

Related Problems

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**Open** *« ???* blocking short paths ???

Vertex Generation

Related Problems 000

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# Outline

• What is a polyhedron? • What is a vertex? • What is vertex generation? • When is generation hard? • Hypergraph dualization • A polyhedral application 3 Hunt for a Hard Case • Matching polytopes • Yet another reformulation • The hunt resumed ... • The hunt is over • Simplices and Bodies

Vertex Generation

Related Problems 000

### The hunt is over ...

For a directed graph G = (V, E) and edge weights  $w_{ij} \in \{-1, +1\}$  for all arcs  $(i, j) \in E$ , define

$$\mathcal{S}_{G,w} = \{ x_i - x_j \le w_{ij} \ \forall \ (i,j) \in E \}$$

$$P_{G,w} = \left\{ y \in \mathbb{R}^E \; \middle| \; \begin{array}{l} \sum_{i:(i,j)\in E} y_{ij} - \sum_{k:(j,k)\in E} y_{jk} = 0 \; \forall \; j \in V \\ \sum_{i:(i,j)\in E} w_{ij}y_{ij} = -1 \end{array} \right\}$$

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Vertex Generation

Related Problems 000

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## The hunt is over ...

- (i) The problem of generating all minimal inconsistent subsystems of linear inequalities is NP-hard already for the family {S<sub>G,w</sub>}.
- (ii) The problem of generating vertices of polyhedra is NP-hard

Vertex Generation

Related Problems 000

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Vertex Generation

Related Problems 000

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Vertex Generation

Related Problems 000

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Vertex Generation

Related Problems 000

### Further consequences

Theorem (Fukuda, Liebling and Margot, 1997)

Given a polyhedron P and an open half-space H, deciding if  $V(P) \cap H \neq \emptyset$  is **NP-hard**.

The same problem for a polytope P is **polynomial** (LP)

#### Corollary (BBEGK, 2005)

(iii) Given a polytope P and an open half-space H, generating V(P) ∩ H is NP-hard.

Generating extremal rays of polyhedra is NP-hard.

Generating both vertices and extreme rays of polyhedra is equivalent with generating vertices of polytopes, and the complexity of this is still **open**.

Related Problems 000

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Related Problems 000

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Related Problems 000

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Related Problems 000

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Vertex Generation

Hunt for a Hard Case

Related Problems ••••

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## Outline

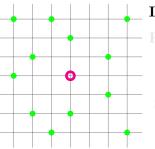
- What is a polyhedron? • What is a vertex? • What is vertex generation? • When is generation hard? • Hypergraph dualization • A polyhedral application • Matching polytopes • Yet another reformulation • The hunt resumed ... • The hunt is over 4 Related Problems
  - Simplices and Bodies

Vertex Generation

Hunt for a Hard Case

Related Problems 000

# Simplices



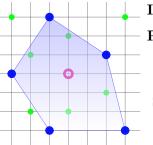
 $\mathcal{A}\subseteq\mathbb{R}^{d}$ and  $\mathbf{o} \in \mathbb{R}^d$ Input:

Vertex Generation

Hunt for a Hard Case

Related Problems  $0 \bullet 0$ 

## Simplices



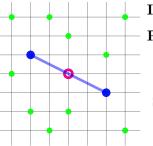
 $\mathcal{A} \subseteq \mathbb{R}^d$  and  $\mathbf{o} \in \mathbb{R}^d$ Input: **Property:**  $\Pi \subseteq 2^{\mathcal{A}}$  $\mathcal{X} \in \Pi$  iff  $\mathbf{o} \in \operatorname{conv}(\mathcal{X})$ 

Vertex Generation

Hunt for a Hard Case

Related Problems  $0 \bullet 0$ 

# Simplices



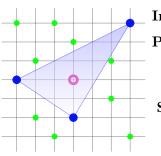
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Vertex Generation

Hunt for a Hard Case

Related Problems  $0 \bullet 0$ 

## Simplices



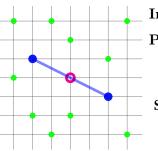
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Vertex Generation

Hunt for a Hard Case

Related Problems  $0 \bullet 0$ 

# Simplices



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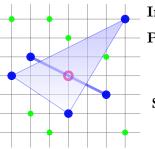
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Vertex Generation

Hunt for a Hard Case

Related Problems  $0 \bullet 0$ 

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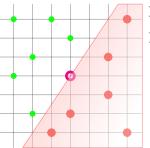
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Vertex Generation

Hunt for a Hard Case

Related Problems  $0 \bullet 0$ 

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 $\mathcal{S}(\mathcal{A})^* = \{ \max' l \ \mathcal{X} \subseteq \mathcal{A} \mid o \notin \operatorname{conv}(\mathcal{X}) \}$ 

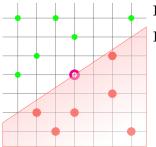
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Vertex Generation

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Related Problems  $0 \bullet 0$ 

# Simplices



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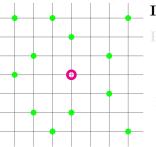
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Vertex Generation

Hunt for a Hard Case

Related Problems 000

## Bodies



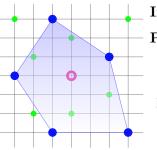
 $\mathcal{A} \subseteq \mathbb{R}^d$  and  $\mathbf{o} \in \mathbb{R}^d$ Input:

Vertex Generation

Hunt for a Hard Case

Related Problems

## Bodies



**Input**:  $\mathcal{A} \subseteq \mathbb{R}^d$  and  $\mathbf{o} \in \mathbb{R}^d$ **Property:**  $\Pi \subseteq 2^{\mathcal{A}}$  $\mathcal{Y} \in \Pi$  iff  $\mathbf{o} \in \operatorname{intconv}(\mathcal{Y})$ 

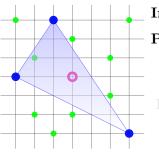
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Vertex Generation

Hunt for a Hard Case

Related Problems

## Bodies



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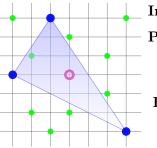
Vertex Generation

Hunt for a Hard Case

Related Problems

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## Bodies



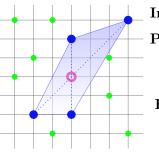
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Vertex Generation

Hunt for a Hard Case

Related Problems

## Bodies



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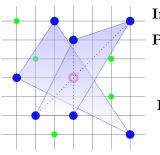
Vertex Generation

Hunt for a Hard Case

Related Problems

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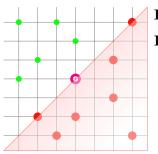
Vertex Generation

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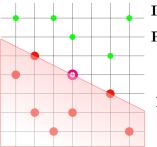
Vertex Generation

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Related Problems

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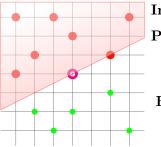
Vertex Generation

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