Generating Vertices of Polyhedra is Hard

... and other related problems ...

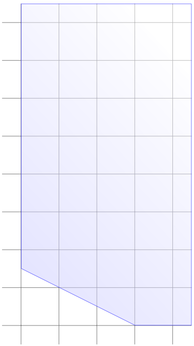
Endre Boros

Joint research with K. Borys, K. Elbassioni, V. Gurvich, and L. Khachiyan[1]

[1] (1952-2005)

Outline

Intersection of half-spaces



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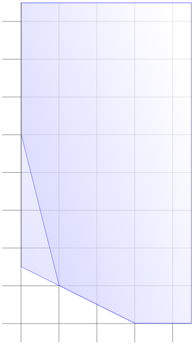

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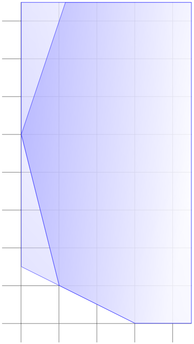

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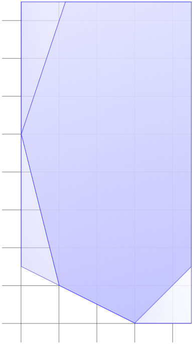

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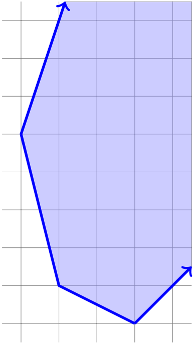
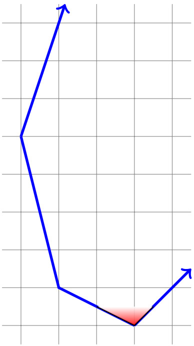


$$P = \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{rrcr} x_1 & +2x_2 & \geq & 3 \\ 4x_1 & +x_2 & \geq & 5 \\ 3x_1 & -x_2 & \geq & -5 \\ -x_1 & +x_2 & \geq & -3 \end{array} \right\}$$

Outline

1. Polyhedra and Vertices
   - What is a polyhedron?
   - What is a vertex?
2. Vertex Generation
   - What is vertex generation?
   - When is generation hard?
   - Hypergraph dualization
   - A polyhedral application
3. Hunt for a Hard Case
   - Matching polytopes
   - Yet another reformulation
   - The hunt resumed ...
   - The hunt is over
4. Related Problems
   - Simplices and Bodies

Those prickly corners



$$P = \left\{ x \in \mathbb{R}^d \;\middle|\; Ax \geq b \right\}$$

$v \in P$ is a vertex if there are no $u, w \in P$ such that

$$v = \frac{1}{2}u + \frac{1}{2}w$$

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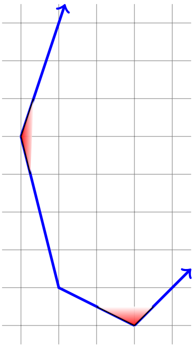

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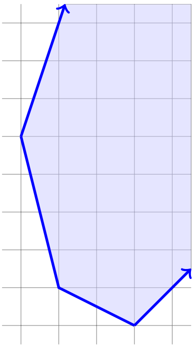
Polynomially decidable questions



Given a polyhedron $P = \{x \in \mathbb{R}^d \mid Ax \geq b\}$, let $V(P)$ denote its set of vertices.

- Is $P \neq \emptyset$?
- Is $V(P) \neq \emptyset$?
- Is $\mathrm{conv}(V(P)) = P$?

...

All these, and many other related questions, can be decided efficiently by solving linear programming problems.

(Khachiyan, 1979)

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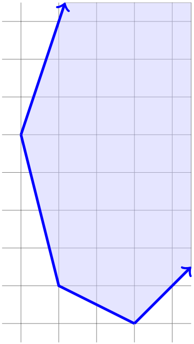

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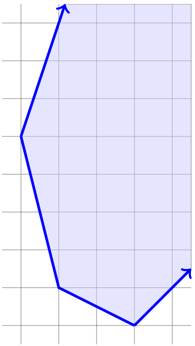

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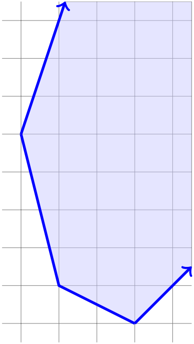

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Vertex generation

First formulation (Mr. Folklore, Age of Pisces):

Given $P = \{x \in \mathbb{R}^d \mid Ax \geq b\}$ generate $V(P)$.

- Output maybe exponentially larger than input!
- Long history … (Motzkin,Raiffa, Thompson and Thrall, 1953)
  (Charnes and Cooper, 1953)
  (Balinski, 1961)
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- Well solved for many special cases
  - Simple polytopes (Avis and Fukuda, 1992)
  - In fixed dimension (Chazelle, 1993)
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- $V(P)$ can be generated by repeatedly solving the above decision problem.
- $\text{conv}(\mathcal{A}) \subseteq P$ is easy to check
- $P \subseteq \text{conv}(\mathcal{A})$ is co-NP-complete (Freund and Orlin 1985)
- Yet, if $\mathcal{A} \subseteq V(P)$, then $P \subseteq \text{conv}(\mathcal{A})$ was open ...

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Equivalent vertex definitions

Assume $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$, $I \subseteq [m] = \{1, ..., m\}$, and let

- A_I be the submatrix of S formed by the rows $i \in I$;
- b_I be the subvector of b formed by the components $i \in I$;
- $\bar{I} = \{1, ..., m\} \setminus I$;
- $P_I \;=\; \{x \in \mathbb{R}^d \mid A_I x = b_I,\; A_{\bar{I}} x \geq b_{\bar{I}}\}$.

Claim

For $P = \{x \in \mathbb{R}^d \mid Ax \geq b\}$ such that $V(P) \neq \emptyset$, there is a one-to-one correspondence between vertices of P and the maximal tight feasible subsets of the inequalities

$$MaxTF(P) \;=\; \{\; max'l\; I \subseteq [m] \mid P_I \neq \emptyset\}.$$

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Monotone properties

Assume $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$.

Let us define a property $\mathbf{\Pi} \subseteq 2^{\{1,\ldots,m\}}$ such that $I \in \mathbf{\Pi}$ iff

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- Then, $\mathbf{\Pi}$ is a monotone property:

$$I \subseteq I' \in \mathbf{\Pi} \quad \text{implies} \quad I \in \mathbf{\Pi}.$$

- Generating $V(P)$ is equivalent with generating

$$Max(\mathbf{\Pi}) \;=\; MaxTF(P) \;=\; \{\text{ max'l subsets } I \in \mathbf{\Pi}\}.$$

- Typically $|\mathbf{\Pi}| = size(A, b) \ll |Max(\mathbf{\Pi})|$.

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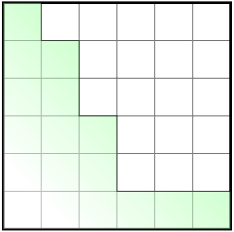

Consider a monotone property Π in a lattice (e.g., $\{0,1\}^m$)

- $Max(\Pi) = \{$ max'l elements $v \in \Pi\}$.
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Given a monotone system Π, generate

- $Max(\Pi)$ (or $Min(\overline{\Pi})$ or both).
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- How to measure efficiency of generation?

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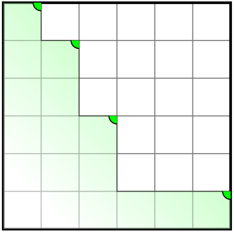

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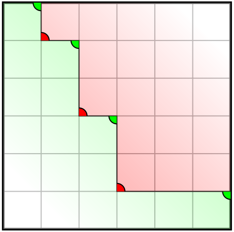

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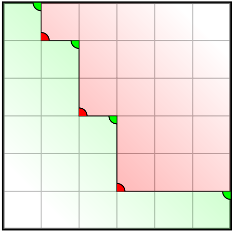

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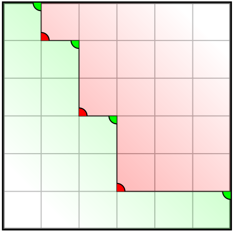

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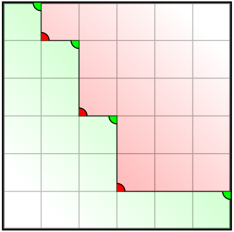

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Complexity of generation

Sequential generation

- Given a monotone system $\mathbf{\Pi}$ of input size $|\mathbf{\Pi}| = N$, an algorithm $\mathfrak{A}$ generates one-by-one the elements

$$Max(\mathbf{\Pi}) = \{v_1, v_2, ..., v_M\},$$

outputting v_k at time t_k $\quad (t_1 \leq t_2 \leq \cdots \leq t_M)$.

- Algorithm $\mathfrak{A}$ is said to work
  - in total polynomial time, if $t_M \leq poly(N, M)$
  - in incremental polynomial time, if

$$t_k \leq poly(N, k) \quad \text{for all} \quad k \leq M$$

  - with polynomial delay, if

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$$t_{k+1} - t_k \leq poly(N) \quad \text{for all} \quad k < M$$

Complexity of generation

Sequential generation

- Given a monotone system $\mathbf{\Pi}$ of input size $|\mathbf{\Pi}| = N$, an algorithm $\mathfrak{A}$ generates one-by-one the elements

$$Max(\mathbf{\Pi}) = \{v_1, v_2, ..., v_M\},$$

outputting v_k at time t_k $\quad (t_1 \leq t_2 \leq \cdots \leq t_M)$.

- Algorithm $\mathfrak{A}$ is said to work
  - in total polynomial time, if $t_M \leq poly(N, M)$
  - in incremental polynomial time, if

$$t_k \leq poly(N, k) \quad \text{for all} \quad k \leq M$$

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NEXT($\Pi, \mathcal{M}$)

Given a monotone system Π and $\mathcal{M} \subseteq Max(\Pi)$, decide if $\mathcal{M} = Max(\Pi)$, or find $v \in Max(\Pi) \setminus \mathcal{M}$ if not.

Theorem (Ms. Folklore,19??)

$Max(\Pi)$ can be generated in incremental polynomial time (total polynomial time) if and only if problem NEXT($\Pi, \mathcal{M}$) can be solved in polynomial time for all $\mathcal{M} \subseteq Max(\Pi)$.

(Lawler, Lenstra, and Rinnooy Kann, 1980)

Generation is hard if NEXT($\Pi, \mathcal{M}$) is NP-hard.

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Outline

Prime example for monotone generation

Hypergraph transversals

Let $|U| = m$ and $\mathcal{H} \subseteq 2^U$ be a hypergraph. Associate to it a property $\mathbf{\Pi} = \mathbf{\Pi}_{\mathcal{H}} \subseteq 2^U$ by

$$S \in \mathbf{\Pi} \quad \Leftrightarrow \quad \begin{array}{c} \nexists H \in \mathcal{H} : H \subseteq S \\ \\ S \textit{ is independent} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} (U \setminus S) \cap H \neq \emptyset \; \forall H \in \mathcal{H} \\ \\ (U \setminus S) \textit{ is a transversal} \end{array}$$

- $\mathcal{H}^* = Max(\mathbf{\Pi}_{\mathcal{H}})$ is the family of maximal independent sets of $\mathcal{H}$.
- $\mathcal{H}^d = \{U \setminus S \mid S \in Max(\mathbf{\Pi}_{\mathcal{H}})\}$ is the family of minimal transversals of $\mathcal{H}$.
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Theorem (Fredmand and Khachiyan, 1996)

For any hypergraph $\mathcal{H}$ and an arbitrary family $\mathcal{M} \subseteq \mathcal{H}^d$ of its minimal transversals, problem NEXT$(\mathcal{H}, \mathcal{M})$ can be solved in $O\left((|\mathcal{H}| + |\mathcal{H}^d|)^{o(\log|\mathcal{H}|+|\mathcal{H}^d|)}\right)$ time.

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Outline

Vertex generation in fixed dimension

Assume $A \in \mathbb{R}^{m\times d}$ and $b \in \mathbb{R}^m$, and recall:

Generating the vertices of $P = \{x \in \mathbb{R}^d \mid Ax \geq b\}$ is equivalent with generating MaxTF (maximal subsets $I \subseteq [m] = \{1, ..., m\}$ for which $P_I = \{x \in \mathbb{R}^d \mid A_I x = b_I,\ A_{\bar{I}} x \geq b_{\bar{I}}\} \neq \emptyset$).

- Think of $MaxTF$ as the family $\mathcal{H}^*$ of maximal independent sets of a hypergraph $\mathcal{H}$.
- $\mathcal{H} = MinTI = \{\text{ min'l } J \subseteq [m] \mid P_J = \emptyset\}$.
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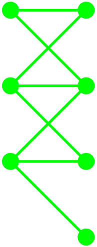
Outline

Bipartite matching polytope



Let $G = (V, E)$ be a bipartite graph, and consider

$$P = \left\{ x \in \mathbb{R}^E \;\middle|\; \begin{array}{lll} \sum_{e \ni v} x_e & \leq 1 & \forall v \in V \\ x_e & \geq 0 & \forall e \in E \end{array} \right\}$$

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Polynomial delay generation (Fukuda and Matsui, 1992)

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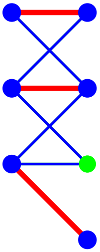

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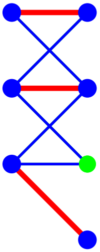

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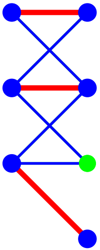

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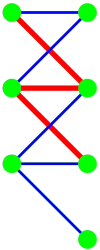

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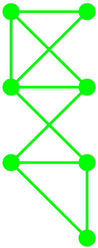

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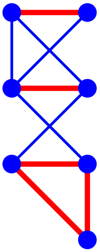

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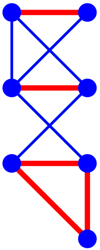

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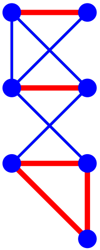

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$$P = \left\{ x \in \mathbb{R}^E \;\middle|\; \begin{array}{lll} \sum_{e \ni v} x_e & \leq 1 & \forall v \in V \\ x_e & \geq 0 & \forall e \in E \end{array} \right\}$$

$$MaxTF = \left\{ \text{max'l } \mathbf{I} \subseteq V \cup E \;\middle|\; \begin{array}{c} E \setminus \mathbf{I} \text{ is a } \textbf{2-matching} \\ \text{covering } V \cap \mathbf{I} \end{array} \right\}$$

$$V(P) \longleftrightarrow MaxTF \longleftrightarrow \mathcal{M} = \{ \text{2-matchings of } G \}$$

With polynomial delay (Boros, Elbassioni, and Gurvich, 2004)

Outline

Irreducible Inconsistent Subsystems (IIS)

Consider $A \in \mathbb{R}^{m\times d}$ and $b \in \mathbb{R}^d$ such that $Ax \geq b$ is inconsistent.

- $MinIS(A,b) = \{\text{min'l } I \subseteq [m] \mid A_I x \geq b_I \text{ is inconsistent}\}$
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- $MinIS(A,b)^* \;=\; MaxFS(A,b)$

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Facts and History

- $MinIS(A, b)^* \;=\; MaxFS(A, b)$
- Lots of attention ... machine learning applications

(Gleason and Ryan, 1990)

(Ryan, 1996)

(Pfetsch, 2002)

(Amaldi, Pfetsch and Trotter, 2003)

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Facts and History

- $MinIS(A,b)^* \; = \; MaxFS(A,b)$
- Problems $\min\{|I| \mid I \in MinIS(A,b)\}$ and $\max\{|I| \mid I \in MaxFS(A,b)\}$ are both NP-hard

(Johnson and Preparata, 1978)

(Chakravarty, 1994)

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Alternative Polyhedron (Gleason and Ryan, 1990)

$$Q_{A,b} \;=\; \{y \in \mathbb{R}^m \mid y^T A = 0,\; y^T b = 1,\; y \geq 0\}$$

Claim $\Longleftarrow$ **Farkas' lemma, 1901**

$$MinIS(A,b) \;\longleftrightarrow\; V(Q_{A,b})$$

Another monotone formulation of vertex generation!

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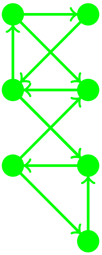
Outline

Acyclic subgraph polyhedron



Let $G = (V, E)$ be a directed graph, $x \in \mathbb{R}^V$, and consider the linear system $\{x_i - x_j \geq 1 \ \ \forall\ (i,j) \in E\}$

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With polynomial delay (Read and Tarjan 1975)

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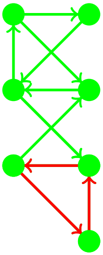

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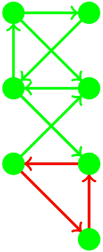

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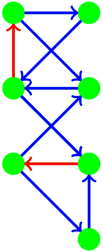

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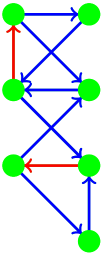

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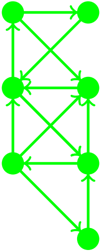
Strongly connected subgraphs' polyhedron



Let $G = (V, E)$ be a strongly connected directed graph, $x \in \mathbb{R}^V$, and consider the system of linear inequalities

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$MinIS \rightsquigarrow \{ \min\text{'l } \mathbf{I} \subseteq E \mid (V, \mathbf{I}) \text{ is strongly connected} \}$

Incrementally polynomial

(Boros, Elbassioni, Gurvich and Khachiyan, 2004)

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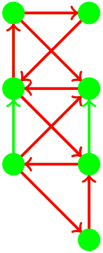

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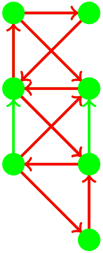

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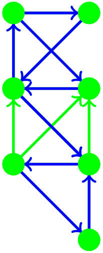

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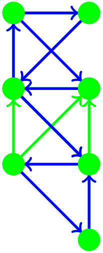

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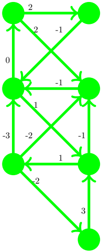
Negative cycle free subgraphs' polyhedron



Let $G = (V, E)$ be a directed graph, $w : E \to \mathbb{R}$, $x \in \mathbb{R}^V$, and consider the system of linear inequalities

$$\{x_i - x_j \leq w_{ij} \quad \forall\ (i,j) \in E\}$$

$MinIS \quad \leftrightsquigarrow \quad \{\mathbf{C} \subseteq E \mid \mathbf{C}$ is a negative cycle $\}$

Theorem (Boros, Borys, Elbassioni, Gurvich and Khachiyan, 2005)

Given a directed graph G with real weights on its arcs, generating all negative cycles of G is ***NP-hard****. Even if $w_{ij} \in \{\pm 1\}$ for all arcs*

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Open $\quad \leftrightsquigarrow \quad$??? **blocking short paths** ???

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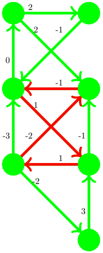

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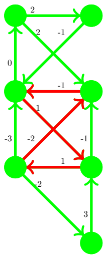

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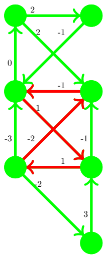

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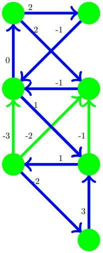

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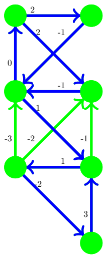

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Outline

The hunt is over ...

For a directed graph $G = (V, E)$ and edge weights $w_{ij} \in \{-1, +1\}$ for all arcs $(i, j) \in E$, define

$$\mathcal{S}_{G,w} = \{ x_i - x_j \leq w_{ij} \quad \forall\ (i,j) \in E \}$$

$$P_{G,w} = \left\{ y \in \mathbb{R}^E \;\middle|\; \begin{array}{rcl} \displaystyle\sum_{i:(i,j)\in E} y_{ij} - \sum_{k:(j,k)\in E} y_{jk} & = & 0 \ \forall\ j \in V \\ \displaystyle\sum_{(i,j)\in E} w_{ij} y_{ij} & = & -1 \end{array} \right\}$$

The hunt is over ...

Corollary (BBEGK, 2005)

(i) *The problem of generating all minimal inconsistent subsystems of linear inequalities is* ***NP-hard****, already for the family* $\{S_{G,w}\}$.

(ii) *The problem of generating vertices of polyhedra is* ***NP-hard****, already for the family* $\{P_{G,w}\}$.

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Further consequences

Theorem (Fukuda, Liebling and Margot, 1997)

*Given a polyhedron P and an open half-space H, deciding if $V(P) \cap H \neq \emptyset$ is **NP-hard**.*

*The same problem for a polytope P is **polynomial** (LP).*

Corollary (BBEGK, 2005)

(iii) *Given a polytope P and an open half-space H, generating $V(P) \cap H$ is **NP-hard**.*

(iv) *Generating extremal rays of polyhedra is **NP-hard**.*

Generating both vertices and extreme rays of polyhedra is equivalent with generating vertices of polytopes, and the complexity of this is still **open**.

Further consequences

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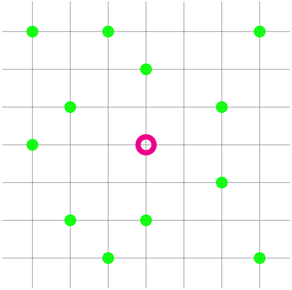
Outline

Simplices



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Simplices: $\operatorname{Min}(\mathbf{\Pi})$

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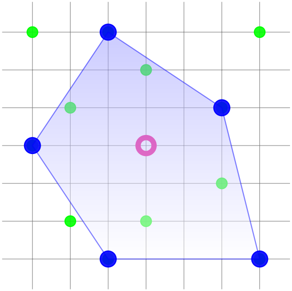

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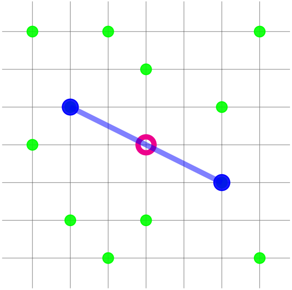

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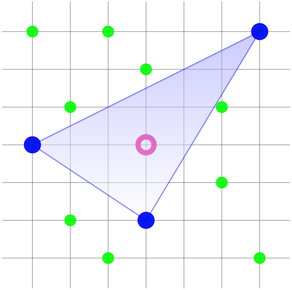

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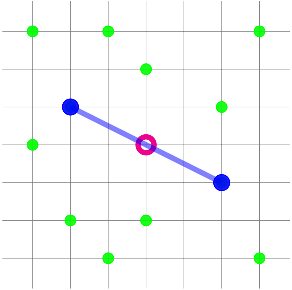

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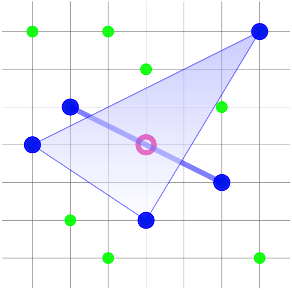

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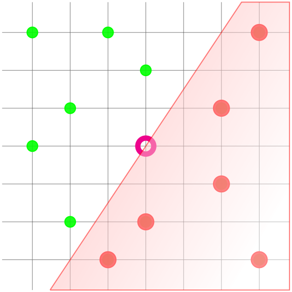

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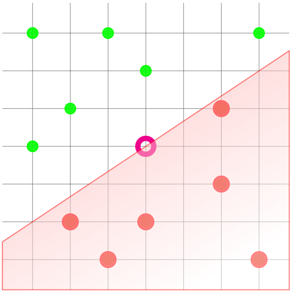

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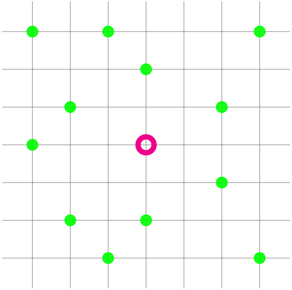

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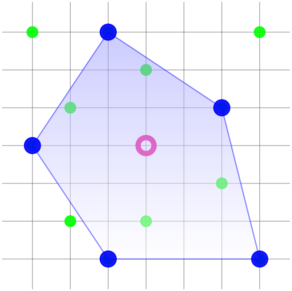

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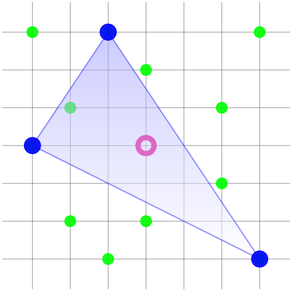

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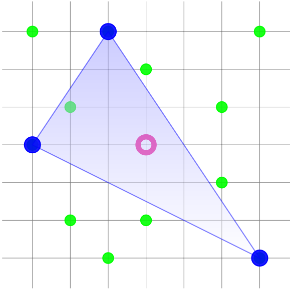

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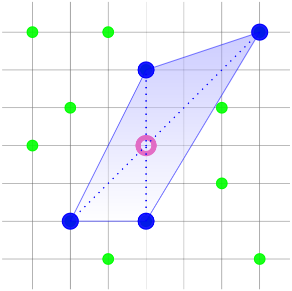

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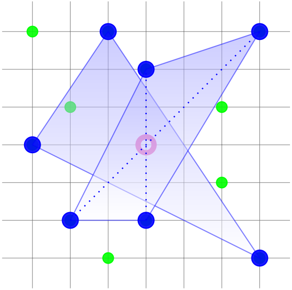

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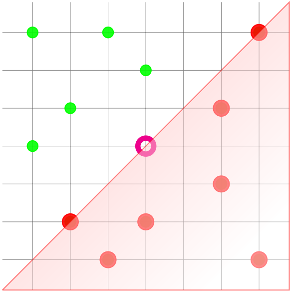

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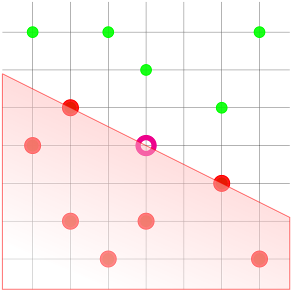

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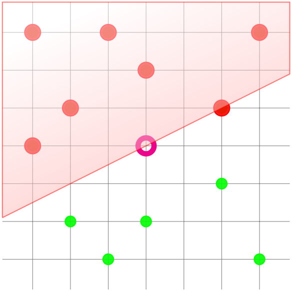

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