FULLY OPTIMAL BASES IN LINEAR AND PSEUDOLINEAR PROGRAMMING, THE ACTIVE BIJECTION, AND APPLICATIONS

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A classical bijection

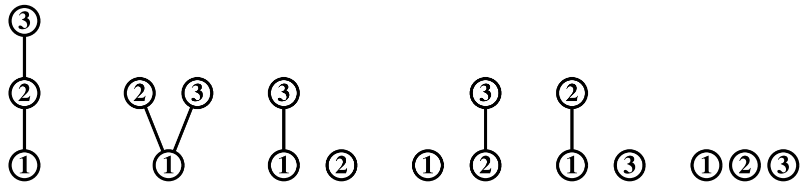
We recall that an *increasing forest* on n vertices is a forest on vertices labelled $1, 2, \ldots, n$ such that labels increase along paths from the vertex of smallest label in each component.

Let $p = a_1 a_2 \ldots a_n$ be a permutation of the integers $1, 2, \ldots, n$. For $i = 2, 3, \ldots, n$, if we have $a_j < a_i$ for some integer $1 \leq j < i$, then we define the edge $a_{i'} a_i$, where i' is the greatest such integer.

Then the graph defined by these edges is an increasing forest on n vertices. Furthermore, the construction can be reversed. The above algorithm defines a bijection between n-permutations and increasing forests on n vertices.

This bijection is well-known in enumerative combinatorics (see for instance R.Stanley, Enumerative combinatorics I (1986), p. 25). It has been introduced independently by W.H. Burge (1972), J. Françon (1976), X. Viennot (1976).

Example $n = 3$



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LP from a combinatorial point of view

A *linear program* in R^d is defined by a polytopal region - the *feasible region*, intersection of closed half-spaces defined by affine hyperplanes - the *program hyperplanes*, and by a linear form on R^d - the *objective function*

The problem is to determine, if it exists, the maximum of the objective function on the feasible region. When the feasible region is non empty and bounded, the maximum always exists.

For a combinatorial interpretation, we forget the scalar values, and only keep in mind the sign properties.

More precisely, to describe the program hyperplanes $h_1, h_2, \ldots, h_n$, we associate with every point of R^d the vector in $\{+, -, 0\}^n$ whose i-th component is $+$ resp. $-$ if the point is in the open half-space h_i^+ resp. h_i^-, and 0 if the point is on the hyperplane h_i. We obtain a finite collection of sign-vectors, called *covectors*. The sign-vectors associated with the vertices determined by $h_1, h_2, \ldots, h_n$ - called *cocircuits* - are sufficient to determine combinatorially all covectors.

The objective function f is represented by directing in its increasing direction all edges defined by $h_1, h_2, \ldots, h_n$ not parallel to f. We note that edges can be computed combinatorially from cocircuits. A vertex v of the feasible region is a maximum of the objective function if and only if *no edge of the feasible region incident to v is outgoing from v.*

Pseudolinear programming

Pseudolinear programming are other words for programming in oriented matroids.

The combinatorial definitions of real linear programming extend with small adaptations to pseudohyperplane arrangements, that is to oriented matroids.

Parallelism is defined by a special pseudohyperplane, called the *plane at infinity*. Two pseudohyperplanes, or a pseudohyperplane and a pseudoline, are *parallel* if their intersection is contained in the plane at infinity. A *pseudoline* of the arrangement is a dimension 1 intersection of pseudohyperplanes.

Edges are directed along pseudolines not parallel to the objective function in the direction from its negative side towards its positive side.

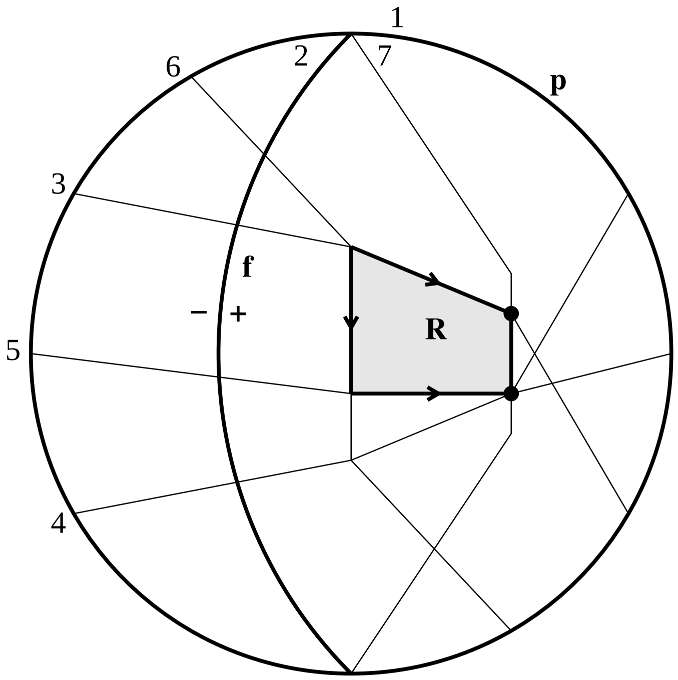
A region is *bounded* if has no vertex in the plane at infinity.

The main difference between the general versus real case is that the program graph may contain directed cycles.

Nevertheless, the main theorem remains valid:

the program graph restricted to a bounded feasible region contains at least one vertex with no outgoing edge.

Any such vertex is a *solution* to the pseudolinear program.

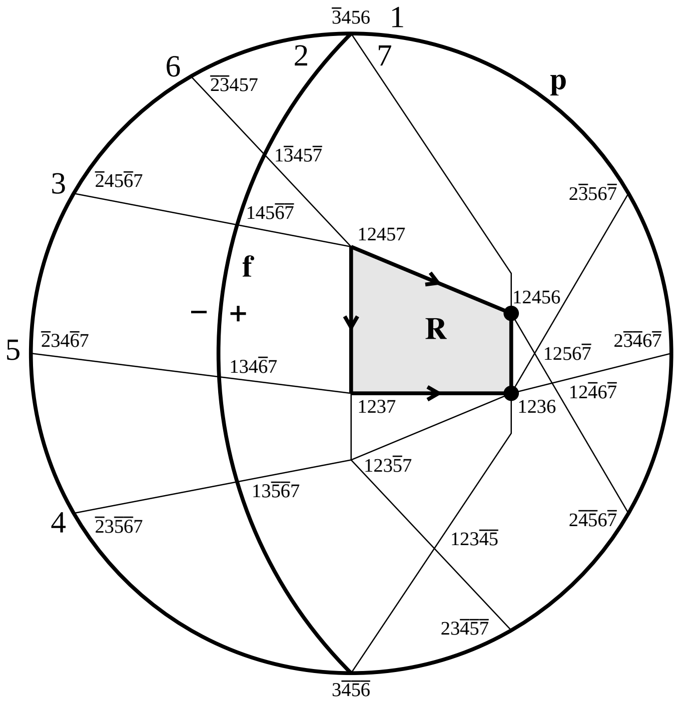


Combinatorial representation of a (pseudo)linear program

The plane at infinity, represented (projectively) by a sphere, is the (pseudo)hyperplane $p = h_1$. A vertex of the arrangement on the plane at infinity is represented by two opposite cocircuits.

The objective function is the (pseudo)hyperplane $f = h_2$.

The feasible region R is the fundamental region, defining the positive side of all (pseudo)hyperplanes.



The notation $2\overline{3}\overline{4}6\overline{7}$ abbreviates the sign-vector $0 + - - 0 + -$.

An hyperplane not containing a vertex v has a $+$ resp. $-$ sign in the cocircuit determined by v if and only if h separates resp. does not separate v and the fundamental region. In dimension 2, these signs can be easily read off from the figure.

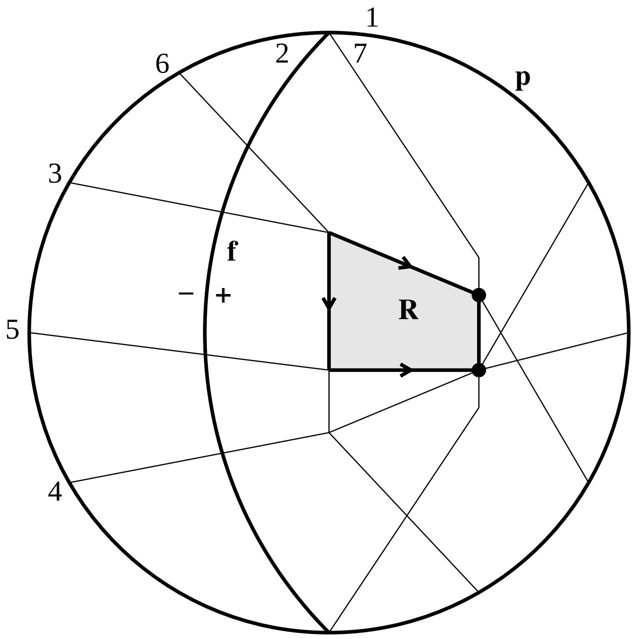
Tableau of a basis

A *basis* of a linear program in R^d is a set of $d+1$ affinely independent program hyperplanes. Geometrically, the hyperplanes of a basis constitute the facets of a d-simplex.

Let B be a basis of a linear program, and $h \in B$. The fundamental *cocircuit* of h with respect to B, denoted by $C^*(B;h)$, is the cocircuit determined by the vertex intersection of the hyperplanes of $B \setminus \{h\}$ such the sign of h is $+$.

The *tableau* of a basis B is a square matrix with rows and columns indexed by the program hyperplanes, and coefficients $+, -$ and 0 (omitted), such that the column for $h \in B$ is the fundamental cocircuit $C^*(B;h)$, and the column for $h \notin B$ consists of a $-$ in (h,h).

Then, for $h \notin B$ the coefficients in row h are up to a factor -1 the signs of the affine dependency expressing h in terms of the hyperplanes in B. This row is the opposite of the fundamental *circuit* $C(B;h)$. For $h \in B$ the row h consists of a $+$ in (h,h).



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6	+		−			−	−
7							+

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	1	2	3	4	5	6	7
1	+						
2	+	−		−	+		
3	+		−	+	−		
4				+			
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6	+			−	+	−	
7				+	−		−

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	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−	−			+
4				+			
5				+	−		−
6	+			+		−	−
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157

	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−		−		+
4				−	+		+
5					+		
6	+				+	−	−
7							+

Optimal bases. The Simplex Criterion

Let us consider the linear program defined by the hyperplanes $h_1, h_2, \ldots, h_n$, with plane at infinity $p = h_1$ and objective function $f = h_2$.

A basis B of this program is said *optimal* if

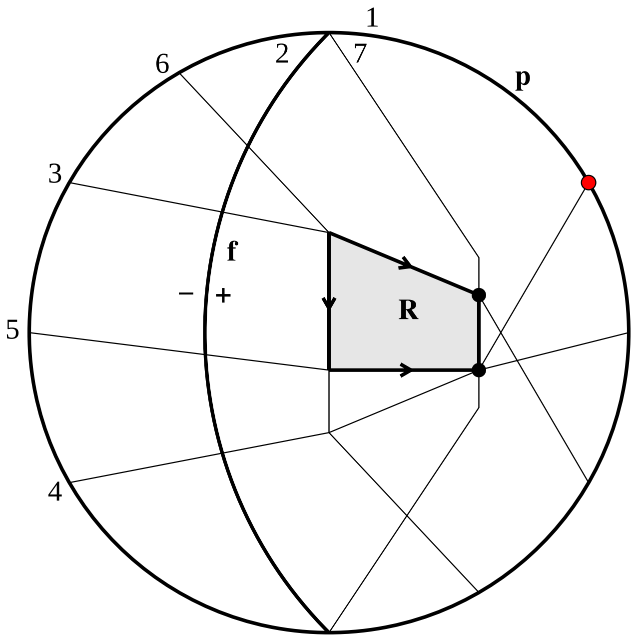
• $p \in B$
• $f \notin B$
• the fundamental cocircuit $C^*(B; p)$ is positive
• the fundamental circuit $C(B; f)$ has f as unique positive element

Proposition (The Simplex Criterion)

A vertex v of the feasible region R maximizes the objective function on R if and only if there is an optimal basis $B = \{b_1 = p, b_2, \ldots, b_r\}$ such that $v = b_2 \cap b_3 \cap \ldots \cap b_r$.

nb. The condition $C^*(B; p)$ positive means that v is a vertex of R. The condition $C^+(B; f) = \{f\}$ means that R does not meet the positive side of the hyperplane parallel to f through v [see A. Schrijver, Theory of Linear and Integer Programming, p. 93].

Optimal bases are generally not unique.



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6	+		−			−	−
7							+

145

	1	2	3	4	5	6	7
1	+						
2	+	−		−	+		
3	+		−	+	−		
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	1	2	3	4	5	6	7
1	+						
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5				+	−		−
6	+			+		−	−
7							+

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	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−		−		+
4				−	+		+
5					+		
6	+				+	−	−
7							+

Fully optimal bases

Let us consider the linear program defined by the indexed set of hyperplanes $h_1, h_2, \ldots, h_n$, with plane at infinity $p = h_1$ and objective function $f = h_2$.

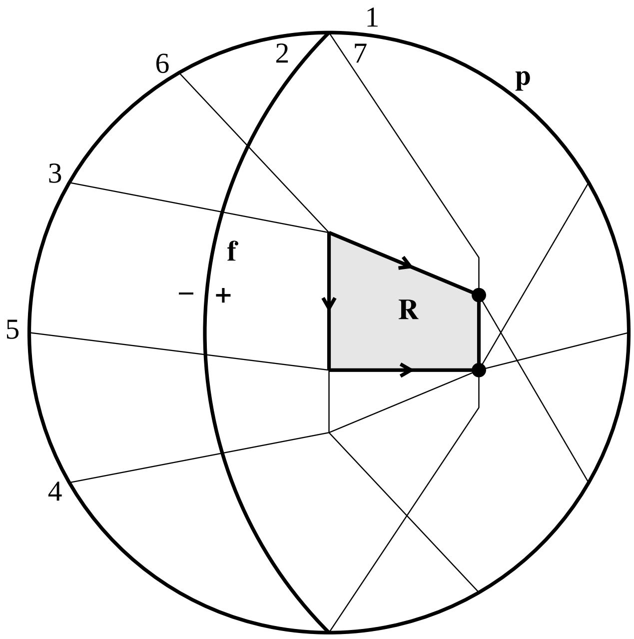
We say that a basis B of the program is *fully optimal* si

• $h_1 \in B$
• $h_2 \notin B$
• the first non zero sign in every row of the tableau of B is a $+$
• the first non zero sign in every column except the first is a $-$

A fully optimal basis is in particular optimal for the program defined on the fundamental region by the plane at infinity $p = h_1$ and the objective function $f = h_2$.

In particular the intersection of the hyperplanes $\neq p$ of a fully optimal basis is a vertex of the feasible region maximizing f, i.e. a solution of the program.

Whereas the property of being an optimal basis does not depend on the ordering, a fully optimal basis depends on the ordering.



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	1	2	3	4	5	6	7
1	+						
2	+	−					−
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6	+		−			−	−
7							+

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	1	2	3	4	5	6	7
1	+						
2	+	−		−	+		
3	+		−	+	−		
4				+			
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6	+			−	+	−	
7				+	−		−

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	1	2	3	4	5	6	7
1	+						
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	1	2	3	4	5	6	7
1	+						
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Main theorem

A bounded region has exactly one fully optimal basis.

The existence is established by means of algorithms.

The first algorithm (or, rather, two dual algorithms) is indirect. Given a basis satisfying certain properties, we construct a bounded region for which this basis is fully optimal.

We show directly that two different bases produce two different regions. A numerical equality establishes that this mapping is a bijection.

A second construction is direct. We construct the fully optimal basis of a bounded region by means of a sequence of ordered multi(pseudo)linear programs.

Activities of bases

Let B be basis of an ordered hyperplane arrangement $h_1, h_2, \ldots, h_n$.

We say that a hyperplanes $h \in B$ which is the smallest element of $C^*(B; h)$ is *internally active* (with respect to B).

We denote by $\iota(B)$ the number of internally active hyperplanes in B.

The external activity $\epsilon(B)$ is defined similarly from the fundamental circuits (these definitions are due to W.T. Tutte circa 1950).

The number b_{ij} of bases with $\iota(B) = i$ and $\epsilon(B) = j$ does not depend on the ordering of the hyperplanes.

A basis with external activity zero is *internal*. A basis with total activity - internal plus external - 1 is *uniactive*.

Proposition

A fully optimal basis is internal and uniactive.

(exercise).

From uniactive internal bases to bounded regions

The definition of a fully optimal basis in terms of tableau prompts an algorithm to redefine the feasible region by reversing the signs of certain hyperplanes so that a given uniactive internal basis become fully optimal.

Algorithm BASORI

Let $B = \{b_1 = h_1, b_2, \ldots, b_r\}$ be a uniactive internal basis.

(1) resign the hyperplanes in $C^*(B; b_1)$ so that all signs become positive

(2) for $i = 2, \ldots, r$ resign the hyperplanes in $C^*(B; b_i) \setminus \bigcup_{j<i} C^*(B; b_j)$ so that all signs become opposite to the sign after resigning of the smallest element of $C^*(B; b_i)$.

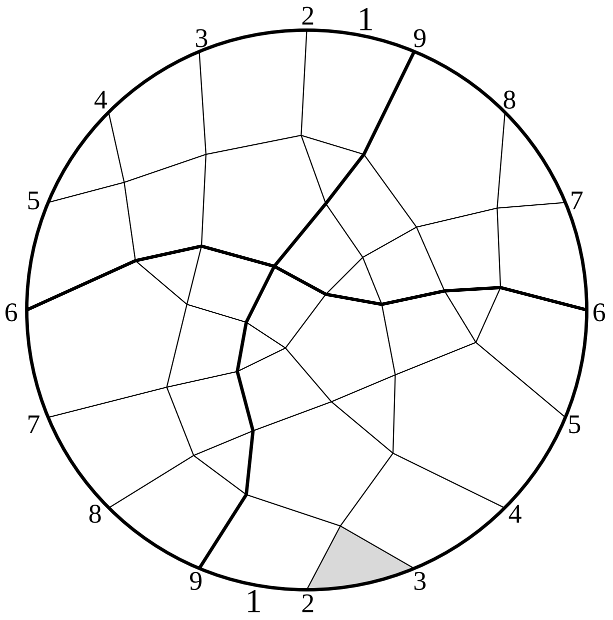
Note that by the properties of B the smallest element in (2) is in $\bigcup_{j<i} C^*(B; b_j))$, hence it has already be resigned. It is easily checked that we get a bounded region, and that B is fully optimal with respect to this region.

There is a dual algorithm using circuits instead of cocircuits.

The mapping from uniactive internal bases to bounded regions defined by BASORI is injective (proof 2/3 pages). Since the number of uniactive internal bases is equal to the number of bounded regions (T. Zaslavsky 1975, for real arrangements; M. Las Vergnas 1977 for o.m.), we have

Theorem

The mapping defined by BASORI, from uniactive internal bases to bounded regions, is a bijection.



	1	2	3	4	5	6	7	8	9
1	$+$								
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6						$+$			
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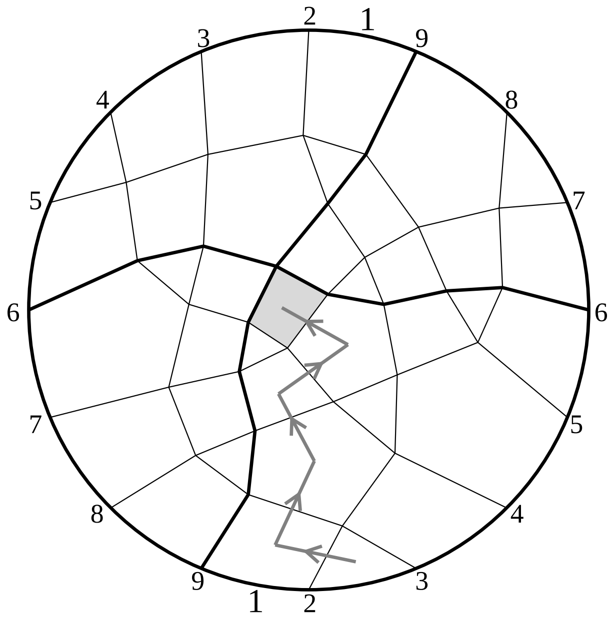
8	$-$					$+$		$-$	$+$
9									$+$

tableau de 169 dans R_9

$A_1 = 23478$



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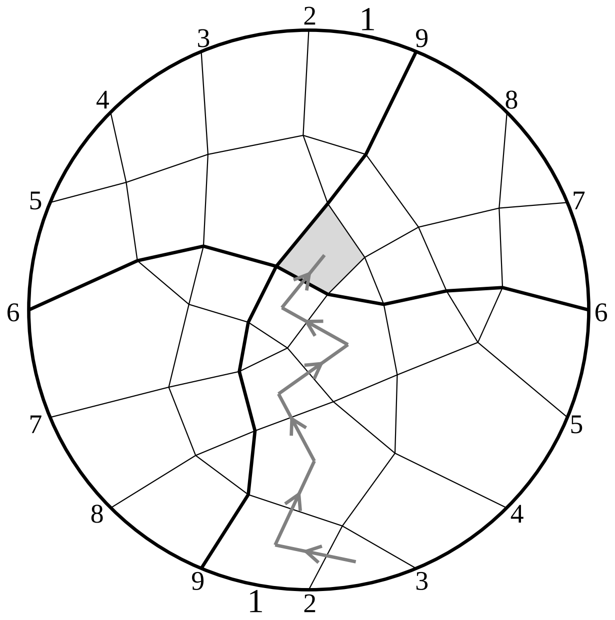
9									$+$

tableau de 169 dans $-_{23478}R_9$

$A_6 = 6$



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4	$+$			$-$		$+$			$+$
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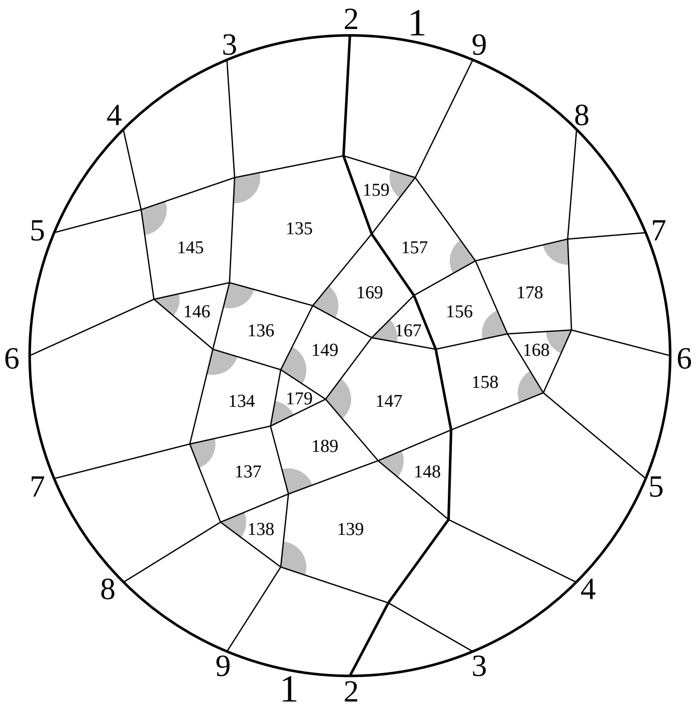
6						$+$			
7	$+$					$-$	$-$		$-$
8	$+$					$-$		$-$	$-$
9									$+$

tableau of 169 in $-_{234678}R_9$

$A_9 = \emptyset$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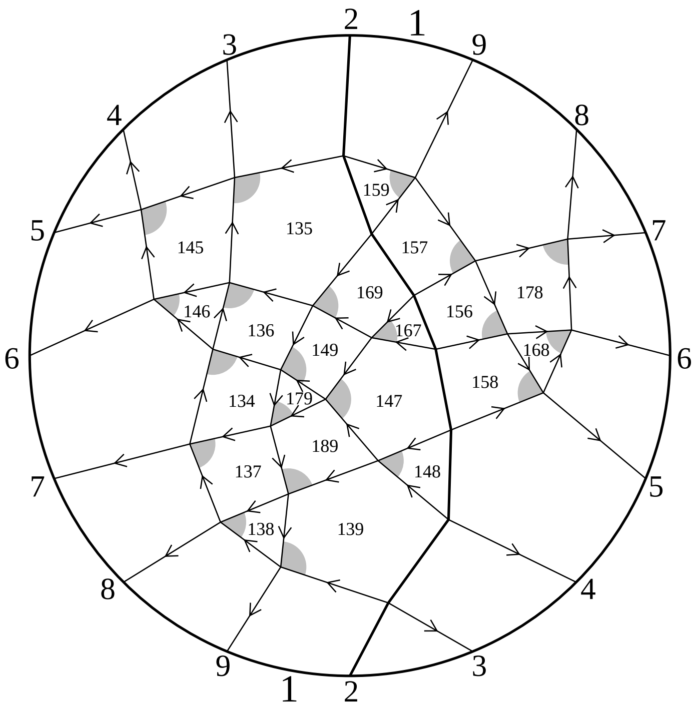
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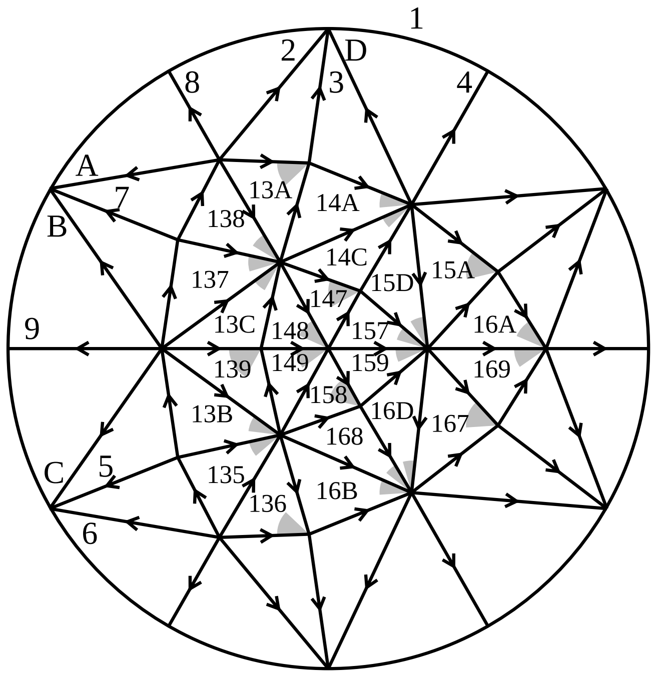
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16A
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169
158
13B
16D
167
168
C
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16B
136
6

From bounded regions to bases

For determining the fully optimal basis of a given bounded region by means of BASORI, we would have to apply the algorithm to successive internal uniactive bases until the desired region is obtained, clearly a not very efficient process.

The following algorithm ORIBASE constructs directly the fully optimal basis of a bounded region.

Step 1

Let R be a bounded region of an ordered hyperplane arrangement. By the simplex criterion, the desired fully optimal basis $B = \{b_1 = e_1, b_2, \ldots, b_r\}_<$ is such that the vertex $v_1 = v = b_2 \cap b_3 \cap \ldots \cap b_r$ is a solution of the linear program defined on R, with plane at infinity $p = b_1 = h_1$ and objective function $f = b_2$.

This property may not suffice to determine v when the program is degenerate. In general v will be the unique solution of a linear multiprogram defined by the ordering of the sequence of hyperplanes, or *ordered multiprogram*.

Let $B^{\min} = \{f_1 = h_1, f_2 = h_2, \ldots, f_r\}_<$ be the lexicographically minimal basis (for simplicity, we suppose that h_2 is not parallel to h_1). The ordered multiprogram on the fundamental region R is the problem of determining, for the plane at infinity $p = f_1$, the set of vertices of R maximizing f_2, then among this set those vertices maximizing f_3, etc. until a unique vertex is obtained.

The multiprogram graph

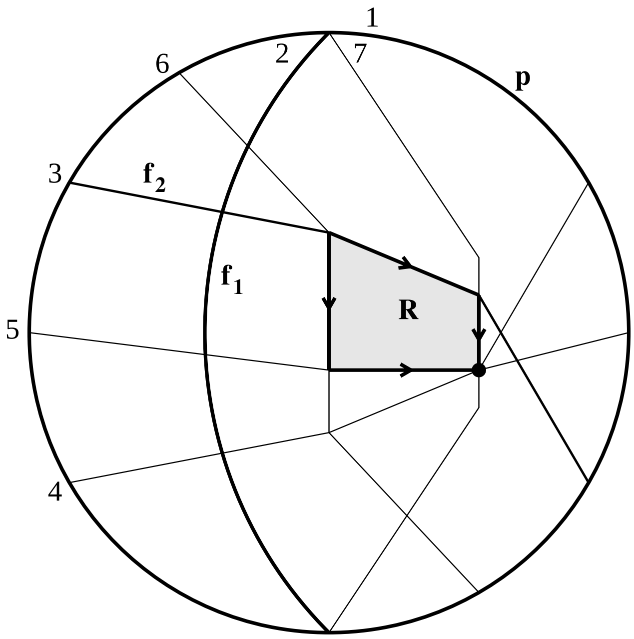
This first step of ORIBASE - ordered multiprogramming - can be expressed as a problem of graph.

Let vv' be an edge, not contained in the plane at infinity p, of the graph defined by the arrangement. The edge vv' is contained in a line d of the arrangement. Let f be the smallest objective function not parallel to d. We direct vv' away from $f \cap d$.

Observe that, classically, vv' would have been directed according to the direction of d going from f^- towards f^+. With the classical orientation, the problem to solve, mixing minima and maxima, would depend on the location of the region with respect to the objective functions. Or, we would have to redefine the orientation of the multiprogram graph for each region, taken as fundamental.

For the present choice, the directed graph remains the same for any region chosen as fundamental. The first step of ORIBASE is to determine a vertex maximal in terms of the multiprogram graph, i.e. a sink of the graph restricted to the region. This problem is purely graphical. We know that when the region is bounded there is unique vertex solution to this problem. We call it the *active vertex* $v = v_1$ of the region.

The hyperplane $b = b_2$ is the smallest hyperplane of the arrangement containing v_1.



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	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−	−			+
4				+			
5				+	−		−
6	+			+		−	−
7							+

157

	1	2	3	4	5	6	7
1	+						
2	+	−					−
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4				−	+		+
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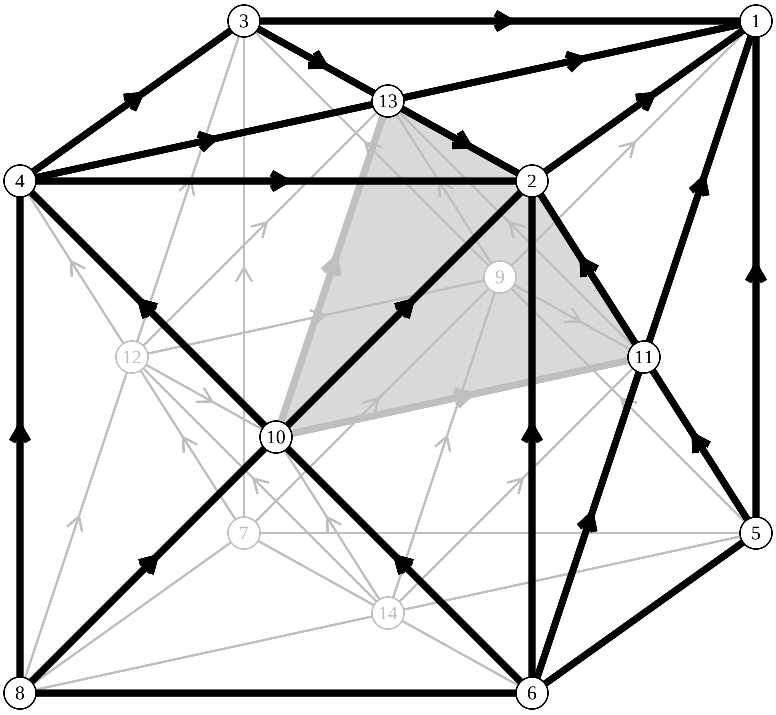
6	+				+	−	−
7							+



the 15 planes of C15

1	:	plane at infinity	6	:	2468	B	:	167
2	:	5678	7	:	1357	C	:	238
3	:	1234	8	:	467	D	:	146
4	:	3478	9	:	258	E	:	147
5	:	1256	A	:	358	F	:	235

215 regions, 51 bounded regions

71 vertices, 9 on the plane at infinity

C15 - Fully optimal basis of the tetrahedron 2ABD

(1) $B^{\min} = 1246$ $v_1 = 2$ $b_2 = 3$

The derived multiprogram

Step 2

We define a new ordered multiprogram with one dimension less, the program *derived at* v from the initial problem.

The *derived arrangement* is the sequence of traces on $p = b_1 = e_1$ of the hyperplanes containing $v = v_1$ The hyperplane $b = b_2$ of Step 1 is now the plane at infinity $p' = b \cap p = b_2 \cap h_1$ of the derived arrangement. The objective function are the traces on $p = h_1$ of the objective fonctions containing v.

The vertices of the derived graph are the traces on the hyperplane $p = h_1$ of the half-lines issued from v on the same side of b_2 that the region R. Let ww' be an edge of the derived graph not contained in b_2. This edge is supported by a line d' of the derived arrangement. The line d' and the vertex v define a plane cutting the hyperplane $b = b_2$ in a line d of the initial arrangement. The direction of the edge ww' of the derived graph depends on the orientation of d in v. There are two cases.

(i) The two edges supported by d incident to v_1 are directed *in-out* in the multiprogram graph.

This direction defines a direction on the line d, hence also on d' parallel to d. We direct all edges supported by d' in this direction, in particular ww'.

(ii) The two edges supported by d incident to v_1 are directed *out-out* in the multiprogram graph. By construction these directions proceed from a smallest objective function containing v. The trace of f on p is an objective function of the derived arrangement, and this the smallest objective function non parallel to d'. This objective function defines the direction of ww' as in Step 1.

The feasible region R' in the derived program is the projection of R from v.

Let v' be the solution of the ordered multiprogram on R', unique sink of the derived graph restricted to R'. Then $v_2 = v'$ is such that $< v_1, v_2 >= b_3 \cap b_4 \cap \ldots \cap b_r$, and b_3 is the smallest hyperplane containing v_1 and v_2.

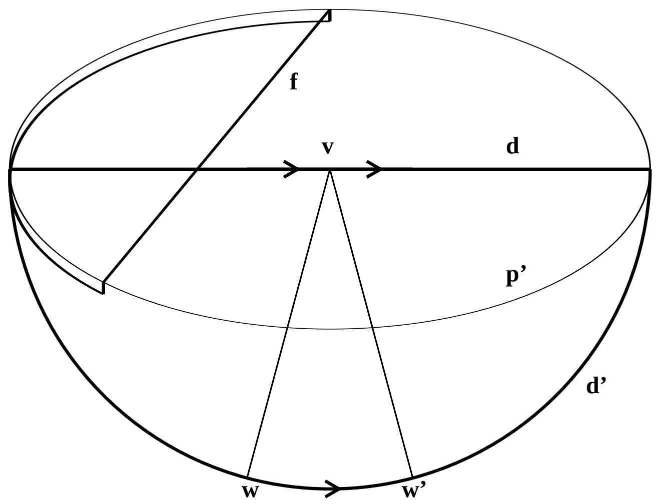
Remark

The construction in Step 2 is different from the construction in Step 1. In case (i) directions do not proceed from objective functions, but from directions in the graph of Step 1. Furthermore the feasible region R' is not necessarily bounded. (see the example in C15).

It is possible to unify the constructions by introducing 'virtual' derived objective functions in case (i).

In case (i) the local orientation of d at v is defined by a smallest objective function f not containing v (and not parallel to b). We add to the derived arrangement a hyperplane f' contained in p infinitely close of p', on the side of f not containing v and parallel to $f \cap p$.

Then directions given by the graphic method of Step 2 can now also be obtained from objective functions as in Step 1. In the new derived arrangement, the feasible region is bounded. The sink v' remains unchanged.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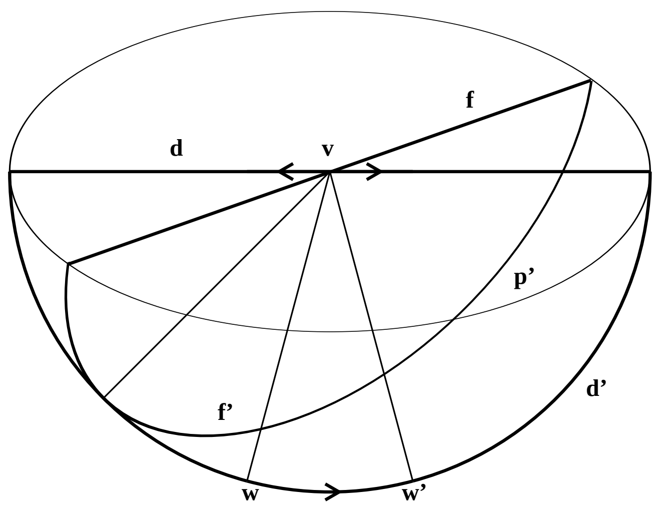
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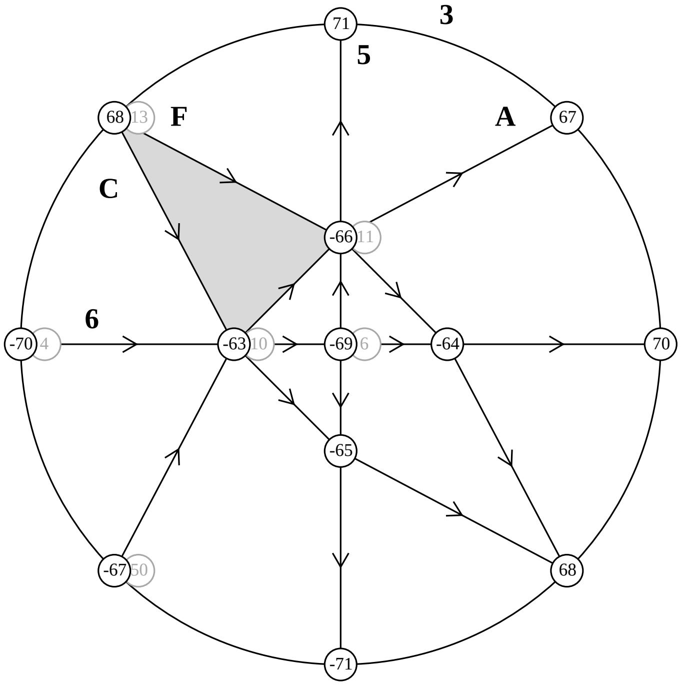
3. Other steps

We iterate the construction of Step 2.

The algorithm ORIBASE yields a sequence of vertices $v_1, v_2, \dots$ of the feasible region R such that $v_1, v_2, \dots, v_i$ span a face of dimension $i-1$ of R. Then, the hyperplane b_{i+1} of the fully optimal basis of R is the smallest hyperplane of the arrangement containing $v_1, v_2, \dots, v_i$.

We note that b_r supports a facet of R. However, $b_2, \dots, b_{r-1}$ may not support any face (see example).



projection of C15 from vertex 2

C15 - region 2ABD (continued)

(2) $v_2 = 11$ $b_3 = 5$

(3) $v_3 = 13$ $b_4 = \mathrm{F}$

fully optimal basis of 2ABD = 135F

	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
1	+														
2	+	−	−												
3			+												
4	+			−	−										
5					+										
6					+	−									−
7	+		−		−		−								+
8	+							−							−
9					+				−						−
A	+		−		−					−					+
B	+		−								−				+
C			+									−			−
D	+		−		−								−		+
E	+				−									−	+
F															+

tableau of 135F in $-_D C15$

Activities of bases and Tutte polynomial

The Tutte polynomial of a matroid M on a set E can be defined by the formula

$$t(M;x,y) = \sum_{A\subseteq E} (x-1)^{r(M)-r_M(A)}(y-1)^{|A|-r_M(A)}$$

It satisfies the following *deletion/contraction* inductive relations.

(i) if $e \in E$ is neither an isthmus nor a loop of M, then

$$t(M;x,y) = t(M\setminus e;x,y) + t(M/e;x,y)$$

(ii) if $e \in E$ is an isthmus of M, then

$$t(M;x,y) = xt(M\setminus e;x,y)$$

(iii) if $e \in E$ is a loop of M, then

$$t(M;x,y) = yt(M\setminus e;x,y)$$

(iv) $t(\emptyset;x,y) = 1$

Theorem [W.T. Tutte 1954]

Suppose the set E linearly ordered. Then

$$t(M;x,y) = \sum_{B \text{ basis of } M} x^{\iota_M(B)}y^{\epsilon_M(B)}$$

where $\iota(B)$ resp. $\epsilon(B)$ denotes the internal resp. external activity of B.

Activities of orientations and Tutte polynomial

Let M be an oriented matroid on a linearly ordered set E.

An element $e \in E$ is *orientation active* - or, *O-active* - if it is the smallest element of some positive circuit of M, it is *dual-orientation active* - or, O^*-*active* - if it is the smallest element of some positive cocircuit of M.

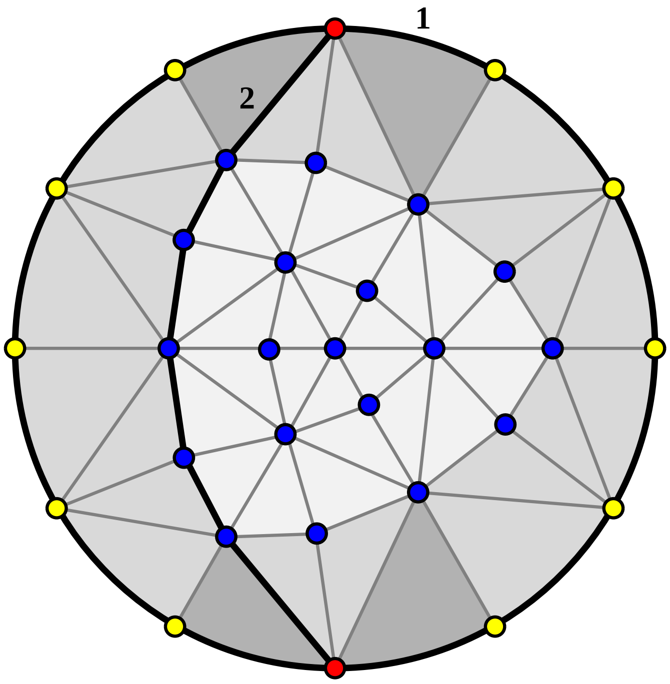
Let $o(M)$ resp. $o^*(M)$ denote the number of O-active resp. O^*-active elements of M. (definitions introduced by M. Las Vergnas circa 1980). The orientation activities do not depend on the ordering.

Theorem [M. Las Vergnas 1982]

$$t(M;x,y) = \sum_{A \subseteq E} (\frac{x}{2})^{o^*(-_A M)} (\frac{y}{2})^{o(-_A M)}$$

This theorem generalizes results of R. Stanley (1973), R.O Winder (1966), T. Zaslavsky (1975), M. Las Vergnas (1975) on counting acyclic orientations of graphs, regions of (pseudo)hyperplane arrangements, acyclic reorientations of oriented matroids.

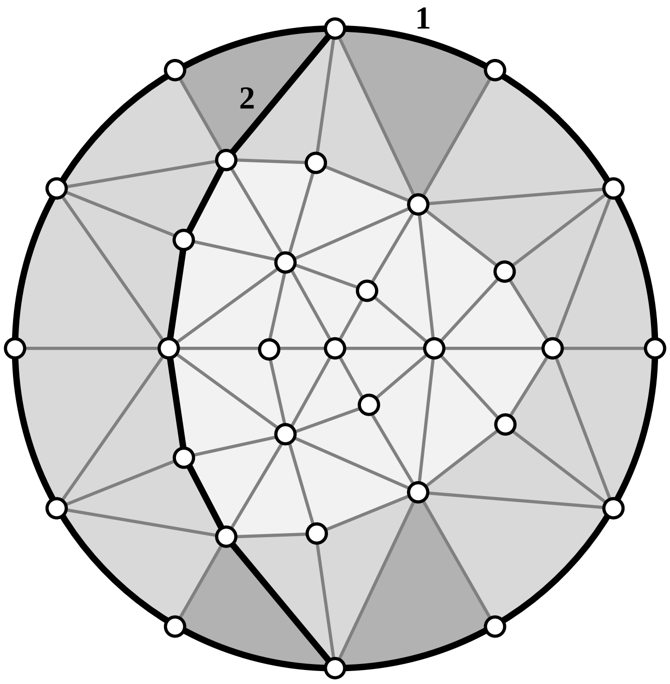
In each case, quoted in increasing order of generality, the number is the evaluation $t(2,0)$ of the Tutte polynomial of a suitable matroid.



minimal basis : 124

in blue, cocircuits beginning by 1
(vertices at finite distance, i.e. not in 1)
in green, cocircuits beginning by 2
(vertices at simple infinity: in 1, but not in 2)
in red, cocircuits beginning by 4
(vertices at double infinity: in $1 \cap 2$)

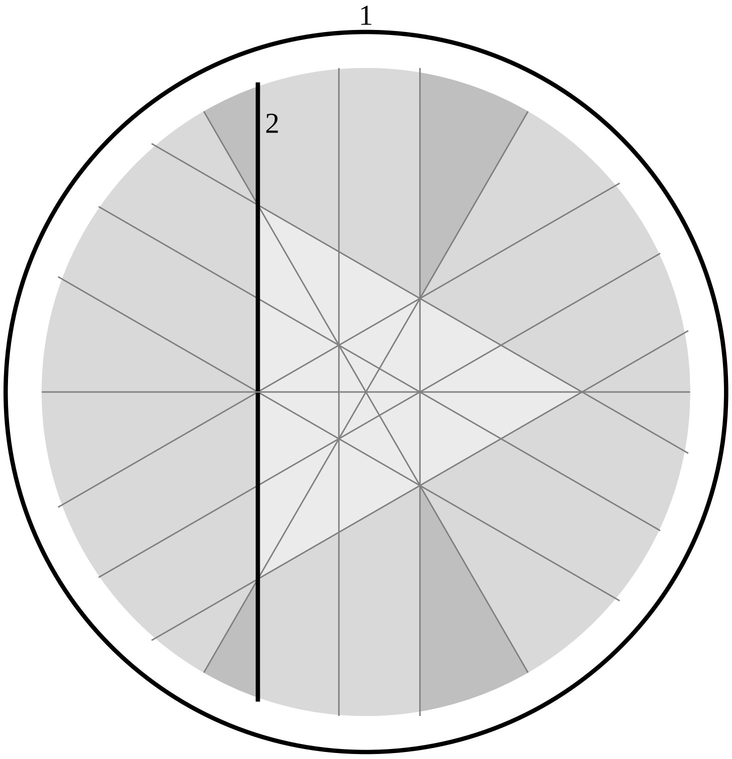
The activity of a region is the number of different types its vertices with respect to infinity.



minimal basis : 124

in blue, cocircuits beginning by 1
(vertices at finite distance, i.e. not in 1)
in green, cocircuits beginning by 2
(vertices at simple infinity: in 1, but not in 2)
in red, cocircuits beginning by 4
(vertices at double infinity: in $1 \cap 2$)

The activity of a region is the number of different types its vertices with respect to infinity.



linear version of the previous figure

simple infinity= non vertical points at infinity

double infinity = vertical points at infinity

The active reorientation-to-basis mapping

Comparing the two expressions of the Tutte polynomial, we get the relation

$$o_{ij} = 2^{i+j} b_{ij}$$

valid for all integers $i, j \geq 0$, between the number o_{ij} of reorientations with O^*-activity i and O-activity j, and the number b_{ij} of bases with internal activity i and external activity j.

Fully optimal bases provide a bijective interpretation of the relation

$$o_{1,0} = 2b_{1,0}$$

between the number of bounded regions $\frac{1}{2}o_{1,0}$ and the number of uniactive internal bases (real case T. Zaslavsky 1975, o.m. M. Las Vergnas 1977). The number $b_{1,0}$, also the coefficient of x in $t(M; x, y)$, is the β invariant of M.

A question arises to generalizes this bijection (actually a 1-2 mapping) to a natural $1 - 2^{i+j}$ mapping from all reorientations of an oriented matroid to its bases, compatible with the above relations.

Theorem

A mapping with these properties can be constructed.

One construction of the *active* mapping is by reducing to the case of $(1, 0)$ activities by means of decompositions of activities, both for orientations and for bases in internal or external uniactive parts. Matroid duality reduces the external case to the internal case.

Decompositions of activities

We sketch roughly this rather technical development.

Decomposing orientation activities is a simple matter.

Farkás Lemma for oriented matroids - *an element is either in a positive circuit or in a positive cocircuit, but not in both* - provides a bipartition of the set of elements, reducing the general case to the *acyclic case* (no positive circuit, or, equivalently, the set of elements is a union of positive cocircuits).

Let a be the greatest O^*-active element. We denote by A the union of all cocircuits *activated* by a - i.e. of smallest element - a. Set $M' = M \setminus A$, and proceed inductively. We get in this way, the *active partition* $E = A_1 + A_2 + \ldots + A_{\iota(M)}$. The restriction of M to each part is a uniactive acyclic oriented matroid.

Decomposing basis activities is more involved. An algorithm in terms of fundamental circuits and cocircuits has to be used (see next slide). We also get active partitions in terms of bases, each part being activated by its smallest element.

Theorem

- *A basis, image of a reorientation by the active mapping, has the same active partition*

- *The* 2^{i+j} *reorientations mapped on a same basis by the active mapping are obtained from anyone of them by reorienting an arbitrary union of parts of the active partition.*

Decomposition of basis activities (details)

Let M be a matroid on a linearly ordered set E, and B be a basis of M. For $X \subseteq E$ we set

$$f^1(X) = f(X) = X \cup \bigcup_{\substack{e \in E \setminus B \\ C_<(B;e) \subseteq X}} C(B;e)$$

$f^{i+1}(X) = f(f^i(X))$, et $\hat{f}(X) = \bigcup_{i \geq 1} f^i(X)$

Then, we define $F = \hat{f}(\emptyset)$. The set F^* is defined dually.. We set $B' = B \cup F$ $M' = M(F)$ $B'' = B \setminus F$ $M'' = M/F$

Proposition 1 [MLV - G. Etienne 1998)

We have $E = F + F^*$

$\iota_{M'}(B') = 0$, $\epsilon_{M'}(B') = \epsilon_M(B)$

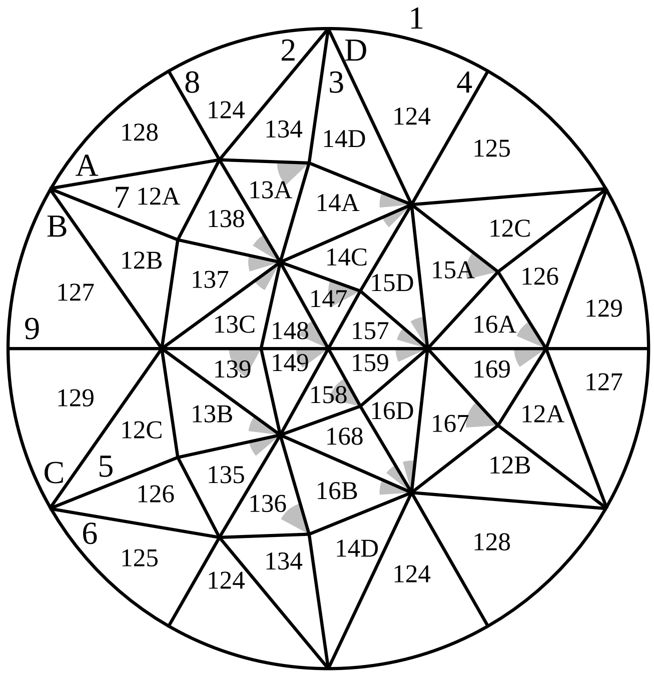
et $\iota_{M''}(B'') = \iota_M(B)$, $\epsilon_{M''}(B'') = 0$.

Let B be an *external* basis of M, i.e. such that $\iota_M(B) = 0$, and $a_1 < a_2 < \ldots < a_\ell$ be the active elements of B. For $1 \leq k \leq \ell$, we define $\hat{F}_k$ as $\hat{F}$, just replacing '$e \in E \setminus B$' by '$e \in E \setminus B$ and $e \geq a_k$' in the definition of $f^1(X)$.

We have $F_1 = E \supset F_2 \supset \ldots \supset F_\ell$. Writing $A_k = F_k \setminus F_{k+1}$, we obtain $E = A_1 + A_2 + \ldots + A_\ell$. Set $B_k = B \cap A_k$ and $M_k = M / \sum_{i<k} A_i \setminus \sum_{i>k} A_i$

Proposition 2

B_k is a uniactive external basis of the matroid M_k on A_k.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124
134
14D
125
128
12A
13A
14A
138
12C
12B
137
14C
147
15D
15A
126
127
13C
148
157
16A
129
139
149
159
169
127
129
12C
13B
158
16D
167
12A
168
126
135
136
16B
12B
125
134
14D
124
128
124
124

The braid arrangement and the complete graph

The *braid arrangement* $\mathcal{B}_n$ can be defined in R^n by the $\binom{n}{2}$ hyperplanes h_{ij} with equation $-x_i + x_j = 0$ for $1 \leq i < j \leq n$. The intersection of these hyperplanes is the line $x_1 = x_2 = \ldots = x_n$.

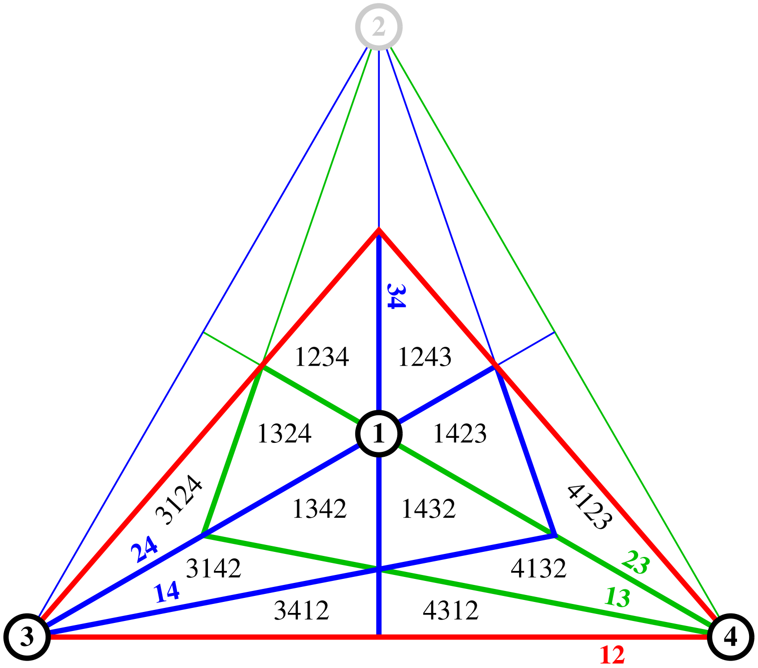
The *arrangement* $\mathcal{B}_n$ is related to the complete graph K_n: with the hyperlane h_{ij} is associated the directed edge $i \to j$. The regions of $\mathcal{B}_n$ correspond then bijectively to the acyclic orientations of K_n, themselves in bijection with permutations of the vertex-set. Actually, an acyclic orientation of K_n defines a linear ordering of the vertices, associated with a unique path through all vertices $a_1 \to a_2 \to \ldots \to a_n$, i.e. to the permutation $a_1 a_2 \ldots a_n$, and conversely.

A geometrical point of view is obtained by considering the first barycentric subdivision of the regular n-simplex of R^{n-1}. The hyerplanes of the arrangement are the mirrors of symmetry of theedges of the simplex. The permutation $a_1 a_2 \ldots a_n$ is associated with the face a_1, $a_1 a_2$, $\ldots$, $a_1 a_2 \ldots a_n$, where the vertex $a_1 a_2 \ldots a_i$ is the barycenter of the vertices a_1 a_2 $\ldots a_i$ of the simplex.

A standard ordering of the hyperplanes of $\mathcal{B}_n$ is the colexicographic ordering:

$$h_{12} < h_{13} < h_{23} < h_{14} < h_{24} < h_{34} < \ldots$$

and, generally, $h_{ij} < h_{k\ell}$ if and only if either $j < \ell$ or $j = \ell$ and $i < k$.


2
34
1234
1243
1324
1
1423
3124
1342
1432
4123
24
3142
4132
23
14
13
3412
4312
3
4
12

The active mapping for the complete graph

Let $p = a_1 a_2 \dots a_n$ be a permutation of $12 \dots n$, $n \geq 2$. If n is firs or last in p, set $t(p) = h_{1n}$. Otherwise, let m be the letter next to n on the opposite side of 1, i.e. such that $p = \dots 1 \dots nm \dots$ or $p = \dots mn \dots 1 \dots$, and set $t(p) = h_{mn}$.

Proposition 1

The active mapping for the braid arrangement with respect to the colexicographic ordering is given by $\alpha(p) = \{t_2, t_3, \dots, t_n\}$*, where for* $2 \leq i \leq n$*, we have* $t_i = t(p')$ *for the permutation* p' *obtained from* p *by removing all letters* $> i$*.*

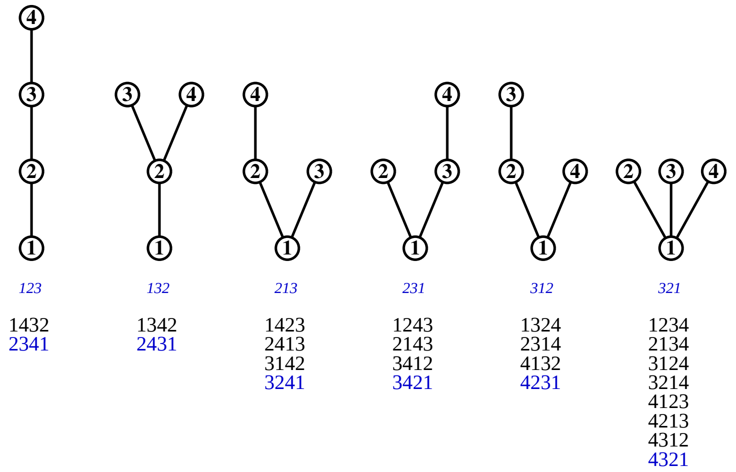
As easily checked, the bases $\alpha(p)$ internal and uniactive are associated with the increasing trees of K_n.

Let $p[a]$ denote the smallest interval of p containing all letters $\leq a$. We say that a letter a is *active* in p if all letters smaller than a are either all to the left or all to the right of a, i.e. if and only if $p[a] = a \dots p[a-1]$ or $p[a] = p[a-1] \dots a$.

Proposition 2

Two permutations have a same image under α *if and only if one is obtained from the other by reversing of intervals* $p[a]$ *for active letters* a*.*

It follows that there is exactly one permutation beginning resp. ending by 1 in each class of α^{-1}*.*



By Proposition 2, the active mapping restricted to the regions of $\mathcal{B}_n$ is equivalent to the classical bijection between $(n-1)$-permutations and increasing forests on $n-1$ vertices by the correspondence

$$a_1 a_2 \dots a_{n-1} 1 \longleftrightarrow a'_1 a'_2 \dots a'_{n-1}$$

avec $a'_i = a_i - 1$ pour $i = 1, 2, \dots, n-1$.

Other applications

- the hyperoctahedral arrangement and signed permutations

- activity preserving bijection between the internal spanning trees of a graph and the acyclic orientations with a unique sink at a given vertex