# The Active Bijection between Regions and Simplices in Supersolvable Arrangements of Hyperplanes

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*Dedicated to R. Stanley on the occasion of his 60th birthday*

**Abstract.** Comparing two expressions of the Tutte polynomial of an ordered oriented matroid yields a remarkable numerical relation between the numbers of reorientations and bases with given activities. A natural activity preserving reorientation-to-basis mapping compatible with this relation is described in a series of papers by the present authors. This mapping, equivalent to a bijection between regions and no broken circuit subsets, provides a bijective version of several enumerative results due to Stanley, Winder, Zaslavsky, and Las Vergnas, expressing the number of acyclic orientations in graphs, or the number of regions in real arrangements of hyperplanes or pseudohyperplanes (i.e. oriented matroids), as evaluations of the Tutte polynomial. In the present paper, we consider in detail the supersolvable case - a notion introduced by Stanley - in the context of arrangements of hyperplanes. For linear orderings compatible with the supersolvable structure, special properties are available, yielding constructions significantly simpler than those in the general case. As an application, we completely carry out the computation of the active bijection for the Coxeter arrangements $A_n$ and $B_n$. It turns out that in both cases the active bijection is closely related to a classical bijection between permutations and increasing trees.

**Keywords.** Hyperplane arrangement, matroid, oriented matroid, supersolvable, Tutte polynomial, basis, reorientation, region, activity, no broken circuit, Coxeter arrangement, braid arrangement, hyperoctahedral arrangement, bijection, permutation, increasing tree.

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## 1. Introduction

The Tutte polynomial of a matroid is a variant of the generating function for the cardinality and rank of subsets of elements. When the set of elements is ordered linearly, the Tutte polynomial coefficients can be combinatorially interpreted in terms of two parameters associated with bases, called activities [8][24]. If the matroid is oriented, another combinatorial interpretation of these coefficients can be given in terms of two parameters associated with reorientations, also called activities [17]. Comparing these two expressions of the Tutte polynomial of an ordered oriented matroid, we get the relation $o_{i,j} = 2^{i+j}b_{i,j}$ between the number $o_{i,j}$ of reorientations and the number of bases $b_{i,j}$ with the same activities $i, j$.

The above relation is a strengthening of several results of the literature on counting acyclic orientations in graphs (Stanley 1973), regions in arrangements of hyperplanes (Winder 1966, Zaslavsky 1975) and pseudohyperplanes, or acyclic reorientations of oriented matroids (Las Vergnas 1975) [14][22][24][28] (see also [5][13][15][16]).

The natural question arises whether there exists a bijective version of this relation [17]. More precisely, the problem is to define a natural reorientation-to-basis mapping that associates an $(i, j)$-active basis with every $(i, j)$-active reorientation, in such a way that each $(i, j)$-active basis is the image of exactly $2^{i+j}$ $(i, j)$-active reorientations.

A construction of a mapping with the requested properties for general oriented matroids is given in [12]. This mapping has several interesting additional properties, implying in particular its natural equivalence with a bijection, and its relationship with linear programming [12a] and decomposition of activities [12b]. We have made a detailed study of some particular classes in separate papers: uniform and rank-3 oriented matroids in [10], graphs in [11]. In the present paper, we consider active mappings in the case of supersolvability, a notion introduced by R. Stanley in [20][21]. Here, the existence of fibers allows us to simplify the construction significantly.

The paper is written in terms of arrangements of hyperplanes in $R^d$. Regions correspond to acyclic reorientations of matroids and simplices to matroid bases. The generalization of the results of the present paper to oriented matroids - i.e. from hyperplane to pseudohyperplane arrangements - is straightforward.

The paper is organized as follows. Section 2 recalls the main features of the *active reorientation-to-basis bijection* for general oriented matroids [12]. In Section 3, we recall the definition and basic properties of supersolvable hyperplane arrangements. We derive in a simple way from the existence of fibers the *weakly active mapping* from the set of regions onto the set of internal simplices. In Section 4, we show how the general construction by deletion/contraction of the *active mapping* [12c] can be simplified in the supersolvable case. The weakly active mapping is simpler to construct, the active mapping has more interesting properties. In particular, the set of regions having a same image under the active mapping has a natural characterization in terms of sign reversals on arbitrary parts of the active partition. As a consequence, the active mapping restricted to the set of regions on positive sides of their active elements (minimal elements in the active partition) is a bijection onto the set of internal simplices, and this restriction generates the entire active mapping by sign reversals. Actually, the active mapping can be refined into an activity preserving bijection between the set of regions and the set of simplices containing no broken circuits, a basis of the Orlik-Solomon algebra [1][19][27].

In the remainder the paper, we apply the previous results to the computation of the active mapping in two important particular cases. In Section 5, we compute the active mapping for the braid arrangement, a well-known arrangement related to acyclic orientations of complete graphs, permutations of $n$ letters, and the Coxeter group $A_n$. For the braid arrangement, the weakly active mapping and the active mapping are equal. They constitute a variant of a classical bijection between permutations and increasing spanning trees [7][9][25] (see [23] p. 25), and also another construction of the bijection of [11] between trees and acyclic orientations with a fixed unique sink in the complete graph. In Section 6 we compute the active mappings for the hyperoctahedral arrangement, related to signed permutations, and the Coxeter group $B_n$. In this case also, the two active mappings are equal. They constitute another variant of the same classical bijection.

## 2. The active bijection for general oriented matroids

Oriented matroid terminology is used throughout the paper. Basic definitions and properties of matroids and oriented matroids can be found in [3][18].

The Tutte polynomial of a matroid $M$ on a set of elements $E$ can be defined by the formula

$$t(M;x,y) = \sum_{A \subseteq E} (x-1)^{r(M)-r_M(A)}(y-1)^{|A|-r_M(A)}$$

where $r_M$ is the rank function of $M$.

Activities have been introduced by W.T. Tutte for spanning trees in graphs [24], and extended to matroid bases by H.H. Crapo [8]. Let $B$ be a basis of a matroid $M$ on a linearly ordered set $E$, or *ordered* matroid. An element $e \in B$ is *internally active* if $e$ is the smallest element of its fundamental cocircuit $C^*(B;e)$ with respect to $B$. Dually, an element $e \in E \setminus B$ is *externally active* if $e$ is the smallest element of its fundamental circuit $C(B;e)$ with respect to $B$. We denote by $AI(B)$ the set of internally active elements of $B$, and by $AE(B)$ the set of externally active non elements of $B$. We set $\iota(B) = |AI(B)|$ and $\epsilon(B) = |AE(B)|$. The non-negative integers $\iota(B)$ and $\epsilon(B)$ are called the *internal* respectively *external activity* of $B$.

Let $B_M^{\min} = \{f_1, f_2 \dots, f_r\}_<$ be the basis of $M$ minimal for the lexicographic order with respect to the ordering of $E$, or *minimal basis* of $M$ for short. It can be easily shown that every element of the minimal basis is internally active, and that any element internally active in some basis is an element of the minimal basis.

We say here that a basis $B$ with $\iota(B) = i$ and $\epsilon(B) = j$ is an $(i,j)$-*basis.* Denoting by $b_{i,j} = b_{i,j}(M)$ the number of $(i,j)$-bases of $M$, the Tutte polynomial has the following expression in terms of basis activities [8][24]

$$t(M;x,y)=\sum_{i,j\geq 0} b_{i,j}x^iy^j$$

Let $M$ be an ordered oriented matroid on $E$. An element $e\in E$ is *orientation active*, or *$\mathcal{O}$-active*, if $e$ is the smallest element of some positive circuit of $M$. An element $e\in E$ is *orientation dually-active*, or *$\mathcal{O}^*$-active*, if $e$ is the smallest element of some positive cocircuit. We denote by $A\mathcal{O}(M)$ respectively $A\mathcal{O}^*(M)$ the set of $\mathcal{O}$- respectively $\mathcal{O}^*$-active elements of $M$, and we set $o(M)=|A\mathcal{O}(M)|$, $o^*(M)=|A\mathcal{O}^*(M)|$. The non-negative integer $o(M)$ respectively $o^*(M)$ is called the *orientation activity*, or *$\mathcal{O}$-activity*, respectively *orientation dual-activity*, or *$\mathcal{O}^*$-activity*, of $M$.

For $A\subseteq E$, we denote by $-_AM$ the *reorientation* of $M$ obtained by reversing signs on $A$ (this notation differs slightly from the notation $\overline{_A}M$ used in [3]). If no confusion results, we occasionally say that the set $A$ itself is a *reorientation*. We denote by $o_{i,j}(M)$ the number of subsets $A\subseteq E$ such that $o^*(-_AM)=i$ and $o(-_AM)=j$. We say that a reorientation $A$ such that $o^*(-_AM)=i$ and $o(-_AM)=j$ is an *$(i,j)$-reorientation.*

The notions of $O$- and $O^*$-activities have been introduced in [17] in relation to the following expression of the Tutte polynomial in terms of orientation activities

$$t(M;x,y)=\sum_{i,j} o_{i,j}2^{-i-j}x^iy^j$$

From this formula, it immediately follows that $\sum_i o_{i,0}=t(2,0)$ is the number of acyclic reorientations of $M$. Hence, the above formula generalizes results of [5][14][22][26] [28].

Since the Tutte polynomial does not depend on any ordering, as a consequence of this formula, $o_{i,j}$ does not depend on the ordering of $E$. Comparing with the expression of the Tutte polynomial in terms of basis activities, we get the following relation between the numbers of reorientations and bases with the same activities

$$o_{i,j}=2^{i+j}b_{i,j}$$

This relation is at the origin of our work on active bijections [10][11][12].

The *active reorientation-to-basis mapping* $\alpha$ introduced by the authors in [12a] has several definitions. One way is to use a reduction to $(1,0)$ activities. Let $B$ be a basis with activities $(1,0)$ of an ordered oriented matroid $M$ on $E$. There exists $A\subseteq E$, unique up to complementation, such that, after reorienting on $A$, the covector $C^*(B;b_1)\circ C^*(B;b_2)\circ\ldots\circ C^*(B;b_r)$ is positive, and the vector $C(B;c_1)\circ C(B;c_2)\circ\ldots\circ C(B;c_r)$ has only $b_1=e_1$ negative, where $B=\{b_1,b_2,\ldots,b_r\}_<$ and $E\setminus B=\{c_1,c_2,\ldots,c_{n-r}\}_<$, and $C(B;e)$ respectively $C^*(B;e)$ is chosen in the pair of signed fundamental circuits respectively cocircuits such that $e$ is positive. We recall that the operation $\circ$ is the *composition* of signed sets defined by $(X \circ Y)^+ = X^+ \cup (Y^+ \setminus X^-)$ and $(X \circ Y)^- = X^- \cup (Y^- \setminus X^+)$ [3]. Then, $-_A M$ is orientation $(1,0)$-active, and the correspondence between $B$ and $A$ is a bijection up to opposites. We set $\alpha(-_A M) = B$. A simple algorithm computes $A$ knowing $B$ [12b].

The general case is obtained by decomposing activities into $(1,0)$-activities, both for bases and for orientations, and then by glueing the bijections of the $(1,0)$ case. We obtain in this way $\alpha$ for any reorientation, as the inverse of a construction using bases.

A direct construction of $\alpha$ from a given reorientation can be given, but is more elaborate. The computation of the unique basis satisfying the above properties, *the fully optimal basis*, of an ordered $(1,0)$-active oriented matroid $M$, can be made by using oriented matroid programming [12a].

The decomposition of activities in $(1,0)$-activities uses minors associated with active partitions both for bases and orientations. The active partition associated with a basis is too technical to be described here. We will use in the paper the *orientation active partition*. For our purpose, it suffices to describe the acyclic case (which implies the general case by matroid duality [12b]).

Let $A\mathcal{O}^* = \{a_1, a_2, \ldots, a_k\}_<$ be the (orientation dually-)active elements of $M$. For $i = 1, 2, \ldots, k$, let $X_i$ be the union of all positive cocircuits of $M$ with smallest element $\geq a_i$. The sets $X_i$ $i = 1, 2, \ldots, k$ are the *active covectors* of $M$, and the sequence $\mathcal{X} = X_k \subset \ldots \subset X_1$ is the *active (covector) flag.* The *active partition* $E = A_1 + A_2 + \ldots + A_k$ of $M$ is defined by $A_i = X_i \setminus X_{i+1}$ for $i = 1, 2, \ldots, k-1$, and $A_k = X_k$. The active partition is naturally ordered by the order of the smallest elements in its parts.

The active mapping preserves active partitions. It turns out that the $2^{i+j}$ $(i,j)$-active reorientations associated with a given $(i,j)$-active basis are obtained from any one of them by reversing signs on arbitrary unions of parts of the active partition.

Another way to define the active mapping is by means of inductive relations using deleting/contraction of the greatest element. We will use this approach in the proofs of Section 4. Here, also, we restrict ourselves to the acyclic case.

Let $M$ be an acyclic ordered oriented matroid on $E$, and $\omega$ be the greatest element of $E$. We denote by $A\mathcal{O}^*_\omega(\mathrm{M})$ the set of smallest elements of positive cocircuits of $M$ containing $\omega$. Note that by definition $\max A\mathcal{O}^*_\omega$ is the smallest element of the part containing $\omega$ in the active partition. As usual, $M\backslash e$ respectively $M/e$ denotes the oriented matroid obtained from $M$ by deletion respectively contraction of an element $e$. An *isthmus* of $M$ is an element $e$ such that $M\backslash e = M/e$, or, equivalently, $r(M\backslash e) = r(M) - 1$.

**Theorem 2.1.**[12c] *Let $M$ be an acyclic ordered oriented matroid with greatest element $\omega$. The active mapping $\alpha$ associating a basis with $M$ is determined by the following inductive relations.*

*(1) If $-_\omega M$ is acyclic, and if $\omega$ is not an isthmus of $M$, then*

*(1.1) if $\max A\mathcal{O}^*_\omega(M) > \max A\mathcal{O}^*_\omega(-_\omega M)$, we have $\alpha(M) = \alpha(M\backslash\omega)$,*

*(1.2) if $\max A\mathcal{O}^*_\omega(M) < \max A\mathcal{O}^*_\omega(-_\omega M)$, we have $\alpha(M) = \alpha(M/\omega) \cup \{\omega\}$,*

*(1.3) if $\max A\mathcal{O}^*_\omega(M) = \max A\mathcal{O}^*_\omega(-_\omega M)$, let $B = \alpha(M/\omega)$, $C = C^*(B \cup \{\omega\}; \omega)$, and $e = \min\big(C \setminus \bigcup D\big)$, where the union is over all positive cocircuits $D$ of $M$ such that $\min D > \max A\mathcal{O}^*_\omega(M)$, then*

*(1.3.1) if $e$ and $\omega$ have a same sign in $C$, we have $\alpha(M) = \alpha(M\backslash\omega)$,*

*(1.3.2) if $e$ and $\omega$ have opposite signs in $C$, we have $\alpha(M) = \alpha(M/\omega) \cup \{\omega\}$.*

*(2) If $-_\omega M$ is not acyclic, we have $\alpha(M) = \alpha(M\backslash\omega)$.*

*(3) If $\omega$ is an isthmus of $M$, we have $\alpha(M) = \alpha(M/\omega) \cup \{\omega\}$.*

It follows from Theorem 2.1 that, when both $M$ and $-_\omega M$ are acyclic, we have $\{\alpha(M), \alpha(-_\omega M)\} = \{\alpha(M/\omega) \cup \{\omega\}, \alpha(M\backslash\omega)\}$. This equality expresses a symmetry between $M$ and $-_\omega M$.

A simple interpretation of Theorem 2.1 in terms of linear programming in the uniform case is given in [10].

The paper is mainly written in terms of hyperplane arrangements, a language well-suited for the geometric intuition of a fiber, our main tool in the sequel. When convenient, we will nevertheless occasionally use the language of matroids. We briefly survey the relationship between matroids and hyperplane arrangements.

To associate an oriented matroid with a central arrangement of hyperplanes $H$ of $\mathbf{R}^d$, we need that signs be associated with the half-spaces defined by the hyperplanes of $H$. When the hyperplanes are defined by linear forms, the oriented matroid $M = M(H)$ of $H$ is the oriented matroid of linear dependencies over $\mathbf{R}$ of the linear forms defining the arrangement. Otherwise, signs can be attributed arbitrarily, and a standard construction can be given [3]. The oriented matroid $M$ is acyclic if and only if the (unique) region on the positive sides of all hyperplanes of $H$, called the *fundamental region*, is non-empty. More generally, a region $R$ of $H$ is determined by its *signature* (called maximal covector in oriented matroid terminology), that is signs relative to the hyperplanes of $H$ of any of the interior points of $R$. A signature determines a (non-empty) region $R$ of the arrangement if and only if, by reorienting the matroid $M$ on the subset $A$ of hyperplanes with negative signs, we get an acyclic oriented matroid. The region $R$ is the fundamental region of $-_A M$. Thus, we have a bijection between regions and subsets $A$ such that $-_A M$ is acyclic.

The vertices of the fundamental region $R$ of an acyclic oriented matroid $M$ correspond bijectively to the positive cocircuits of $M$. Actually, we should have more accurately said extremal ray instead of vertex, since the regions of $H$ are polyhedral cones. However, if no confusion results, we will use the terminology of polyhedra, as usual in the theory of oriented matroids. The positive cocircuit $C_v$ associated with a vertex $v$ of $R$ is the set of hyperplanes of $H$ not containing $v$. A hyperplane $h$ of $H$

supports a facet $F$ of the fundamental region $R$ if and only if $-_hM$ is acyclic. The fundamental region of $-_hM$ is the region opposite to $R$ with respect to $F$.

When the arrangement is ordered, we usually represent geometrically the smallest hyperplane as the *plane at infinity.* Then, orientation $(1,0)$-active regions, having no vertex in the plane at infinity, are *bounded* regions. More generally, the minimal basis can be seen as the standard coordinate basis, yielding a hierarchy of directions at infinity, namely, the ordered partition of the vertex set defined by vertices not in $f_1$, vertices in $f_1$ but not in $f_2, \ldots,$ and in general vertices in $(f_1 \cap f_2 \cap \ldots \cap f_i) \setminus f_{i+1}$, for $1 \leq i \leq r-1$. Then, the orientation dual-activity of a region is the number of different sorts of vertices it contains. In other words, it is also the number of non-null intersections of the frontier of the region with successive differences of the minimal flag $f_1 \cap f_2 \cap \ldots \cap f_r \subset \ldots \subset f_1 \cap f_2 \subset \ f_1 \subset R^d$.

Theorem 2.2 sums up the main properties of the active mapping from regions onto the set of simplices (more accurately simplicial cones) with zero external activity, or *internal* simplices, sufficient for our purpose in the present paper.

**Theorem 2.2.** [12] *The active mapping $\alpha$ maps the regions of an ordered hyperplane arrangement onto the set of internal simplices of the arrangement. It not only preserves activities, but also the active partition.*

*A $(k,0)$-active simplex is the image of $2^k$ $(k,0)$-active regions. The signatures of these regions are related by reversing signs on arbitrary unions of parts of the active partition.*

The active mapping is naturally equivalent to several bijections involving regions and simplices. The bijection (iii) below is the *active region-to-simplex bijection* mentioned in the title of the paper.

(i) *Bijection between activity classes of regions and internal simplices.*
We call *activity class* of a region with activities $(k,0)$ the set of $2^k$ regions obtained by reversing arbitrary parts of its active partition. By Theorem 2.2, the active mapping, defined in Theorem 2.1, satisfies: $\alpha(-_AR) = \alpha(R)$, where $R$ is any region and $A$ is a union of parts of the active partition of $R$. Note that $-_AR$ has the same active partition as $R$. This $2^k$ to 1 correspondence between regions and internal simplices is a bijection between activity classes of regions and internal simplices. This bijection is invariant under reorientation. In other words, it does not depend on the signature of the arrangement or on a fundamental region. It depends only on the unsigned arrangement, i.e., on the unique reorientation class of oriented matroids defined by any oriented matroid associated with the geometric hyperplane arrangement.

(ii) *Bijection between regions and the set $\mathcal{NBC}$ of no broken circuit subsets.*
We recall that a *no broken circuit subset* is a subset of elements containing no circuit with its smallest element deleted. When a signature or a fundamental region is fixed,

the bijection (i) can be refined in the following way: let $\alpha_{\mathcal{NBC}}(R) = \alpha(R) \setminus \{a_{i_1}, \ldots, a_{i_j}\}$, where $R$ is a region, and $\{a_{i_1}, \ldots, a_{i_j}\}$ the set of its orientation dually-active elements signed negatively in the signature of $R$. This mapping $\alpha_{\mathcal{NBC}}$ is a bijection between regions and $\mathcal{NBC}$, since $\mathcal{NBC} = \uplus_{B \text{ basis}}[B \setminus AI(B), B]$ as well-known [1]. This bijection preserves activities generalized to subsets accordingly with this partition of $\mathcal{NBC}$.

(iii) *Bijection between regions with positive active elements and internal simplices.*
When a signature, or a fundamental region is fixed, the common restriction of the mappings $\alpha$ or $\alpha_{\mathcal{NC}}$ on regions with active elements signed positively is a bijection with the set of internal simplices.
Bijection (ii) can also be obtained from bijection (iii). We have $\alpha_{\mathcal{NBC}}(-_A R) = \alpha(R) \setminus \{a_{i_1}, \ldots, a_{i_j}\}$, where $R$ is a region with positive active elements, and $A$ is a union of parts of the active partition of $R$ with smallest elements $\{a_{i_1}, \ldots, a_{i_j}\}$.

(iv) *Bijection between (pairs of opposite) bounded regions and* $(1,0)$*-simplices.*
This bijection, a restriction of any of the bijections (i), (ii) or (iii), and for which a direct definition has been given above, does not depend on a signature, like (i).

We mention that in the case of graphs, assuming that the lexicographically minimal spanning tree is edge-increasing with respect to some given vertex, there is also a *bijection between acyclic orientations having this given vertex as unique sink and internal spanning trees* [11] (see also Section 5 below, in the case of $K_n$).

Finally, we point out that definitions and results presented here in terms of hyperplane arrangements generalize in a straightforward way to oriented matroids, equivalently, to arrangements of pseudohyperplanes. A definition of supersolvable oriented matroids can be found in [2].

## 3. Supersolvable hyperplane arrangements

The notion of supersolvable lattice has been introduced by R. Stanley in connection with the factorization of Poincaré polynomials [20][21]. By definition a lattice is *supersolvable* if it contains a maximal chain of modular elements. Accordingly, a hyperplane arrangement is *supersolvable* if and only if its lattice of intersections ordered by reverse inclusion is supersolvable.

We will use in the sequel the following definition of supersolvability of a hyperplane arrangement by induction on its rank [2]. We recall that the *rank* of a hyperplane arrangement is equal to the dimension of the ambient space minus the dimension of the intersection of all hyperplanes, plus 1 (i.e., equal to the rank of its matroid).

- Every hyperplane arrangement of rank at most 2 is *supersolvable.*
- A hyperplane arrangement $H$ of rank $r \geq 3$ is *supersolvable* if and only it contains a supersolvable sub-arrangement $H'$ of rank $r-1$ such that for all $h_1 \neq h_2 \in H \setminus H'$ there is $h' \in H'$ such that $h_1 \cap h_2 \subseteq h'$. In this situation, we write $H' \lhd H$.

Classical examples of supersolvable real arrangements are the braid arrangement, related to the Coxeter group $A_n$ (see Section 5 below), and the hyperoctahedral arrangement, related to the Coxeter goup $B_n$ (see Section 6 below), and also arrangements associated with chordal graphs (see Example 3.2 below).

Let $H' \lhd H$. We denote by $\Pi(R)$ the region of $H'$ containing a region $R$ of $H$. The *fiber* of a region $R$ in $H$ is the set $\Pi^{-1}(\Pi(R))$ of regions of $H$ contained in the region of $H'$ containing $R$.

The *adjacency graph* of a hyperplane arrangement is the graph having regions as vertices, such that two vertices are joined by an edge if and only if the corresponding regions have a common facet, equivalently, if one region can be obtained from the other in the oriented matroid of the arrangement by reversing the sign of the hyperplane supporting the common facet.

**Proposition 3.1.** [2] *Let $H$ be a supersolvable arrangement, and $H' \lhd H$. The restriction of the adjacency graph to a fiber is a path of length $|H \setminus H'|$.*

We say that a region is *extreme* in its fiber if the corresponding vertex is at an end of the fiber path in Proposition 3.1.

Let $H$ be a supersolvable hyperplane arrangement of rank $r$. We call a *resolution* of $H$ a sequence $H_i$, $i = 1, 2, \ldots, r$, of supersolvable sub-arrangements of $H$ such that $H_i$ is of rank $i$ for $i = 1, 2, \ldots, r$ and $H_1 \lhd H_2 \lhd \ldots \lhd H_r = H$.

When $H$ is supersolvable and linearly ordered, we say that a resolution $H_1 \lhd H_2 \lhd \ldots \lhd H_r = H$ is *ordered* if $H_1 < H_2 \setminus H_1 < \ldots < H_r \setminus H_{r-1}$, where $H_1 < H_2 \setminus H_1$ means that elements in $H_1$ are smaller than elements in $H_2 \setminus H_1$.

In an ordered resolution, we have $\min(H \setminus H_{i-1}) \in H_i$ for all $1 \leq i \leq r$. Hence, the minimal basis is $B^{\min} = \{f_1, f_2, \ldots, f_r\}_<$ with $f_i = \min(H_i \setminus H_{i-1})$ for all $1 \leq i \leq r$.

In the remainder of this section, $H_1 \lhd H_2 \lhd \ldots \lhd H_r = H$ is an ordered resolution of a supersolvable arrangement.

**Example.** Figure 1 shows an ordered resolution $1 \lhd 1234 \lhd 123456789$ of the supersolvable arrangement associated with the Coxeter group $B_3$.

Activities of regions and simplices have simple characterizations in the supersolvable case. We will use them, together with the adjacency graph, to build an activity preserving mapping from regions to simplices, called the *weakly active mapping*.

**Proposition 3.2.** *A basis $B = \{b_1, b_2, \ldots, b_r\}_<$ of $H$ is internal if and only if $b_i \in H_i \setminus H_{i-1}$ for all $1 \leq i \leq r$. In this case, $AI(B) = B \cap B^{\min}$.*

*Proof.* We prove Proposition 3.2 by induction on $r$. If $r = 1$ we have $b_1 = f_1$. Let $B = \{b_1, b_2, \ldots, b_{i-1}\}$ be an internal basis of $H_{i-1}$, i.e. a basis with zero external activity. If

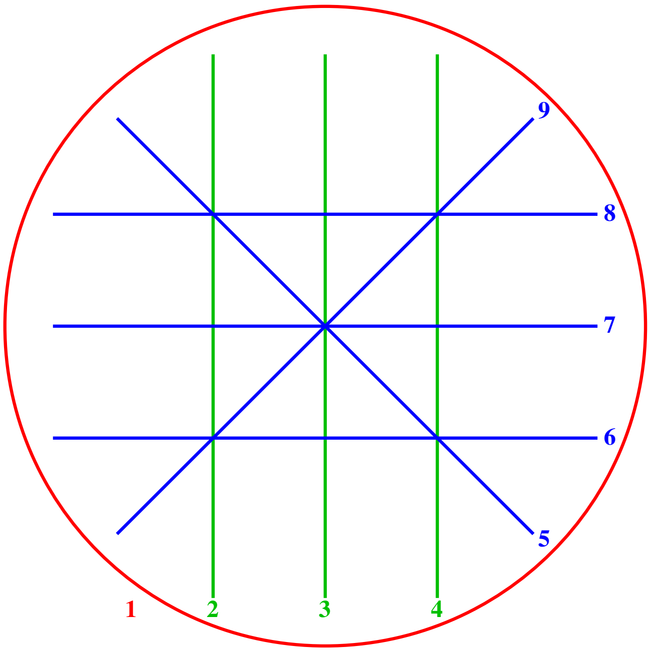



Figure 1. Ordered resolution of a supersolvable hyperplane arrangement

$b_i \in H_i \setminus H_{i-1}$, then $B \cup b_i$ is a basis of $H_i$, which is internal since $H_{i-1} < H_i \setminus H_{i-1}$ and the intersections of hyperplanes in $H_i \setminus H_{i-1}$ are in $H_{i-1}$.

Conversely, if a basis $B = \{b_1, b_2, \ldots, b_r\}_<$ is not of this form, then there exist $i$, $j$ and $k$ such that $\{b_i, b_j\} \subseteq H_k \setminus H_{k-1}$. Since the intersection of $b_i$ and $b_j$ is contained in a hyperplane of $H_{k-1}$, there exists a circuit containing $b_i$, $b_j$, and an element $e \in H_{k-1} \setminus B$. Note that $e$ is smaller than $b_i$ and $b_j$ since $H_{k-1} < H_k$. Hence the basis $B$ is not internal.

The inclusion $AI(B) \subseteq B \cap B^{\min}$ is true in general. In the supersolvable case, if $b_i \in B \cap B^{\min}$ then the flat generated by $\{b_j, j < i\}$ is $H_{i-1}$, and $b_i \in H_i \setminus H_{i-1}$. So $b_i = f_i = \min(H_i \setminus H_{i-1}) = \min(E \setminus \mathit{closure}(B - b_i))$. Hence $b \in AI(B)$. $\square$

**Proposition 3.3.** *Let $R$ be a region of $H = H_r$, with fiber $\Pi(R)$ in $H_{r-1}$. If $R$ is not extreme in its fiber, then $A\mathcal{O}^*(R) = A\mathcal{O}^*(\Pi(R))$. If $R$ is extreme in its fiber, then $A\mathcal{O}^*(R) = A\mathcal{O}^*(\Pi(R)) \cup \{f_r\}$.*

*Proof.* The element $f_{i+1}$, $i < r-1$, is dually active in the region $\Pi(R)$ of $H_{r-1}$ if this region is adjacent to the flat $H_{i-1}$ of $H_{r-1}$ (geometrical interpretation of activities of reorientations). If $\Pi(R)$ is adjacent to the flat $H_i$, and if $\Pi(R)$ is cut in $H = H_r$ by a hyperplane $e$, then $e$ cuts $H_i$. According to Proposition 3.1, the region $R$ has at most two facet hyperplanes in $H_r$. The intersection of these hyperplanes is included in the frontier of $R$, and is included in a hyperplane of $H_{r-1}$, by definition of a supersolvable arrangement. Hence, for all $i < r-1$, $R$ is adjacent to $H_i$ in $H_r$ if and only if $\Pi(R)$ is adjacent to $H_i$ in $H_{r-1}$. Hence $\Pi(R)$ and $R$ have the same dual-active elements, except maybe $f_r$.

The extreme regions of the fiber in $H$ are those touching the flat $H_{r-1}$ of $H$. Geometrically, this means that they touch the line of intersection of the elements of $H_{r-1}$ in $H$, and this means that $f_r$ is dually-active. Conversely, non-extreme regions do not touch this line, and $f_r$ is not dually-active. □

**Definition-Algorithm 3.4.** *Inductive construction of the weakly active mapping $\alpha_1$*

We define a mapping $\alpha_1$ from regions to simplices of a supersolvable ordered arrangement $H$ with an ordered resolution by induction on the rank. In rank 1, the arrangement is reduced to one hyperplane $h_1$, there are two regions $R_1$ and $R_2$. We set $\alpha_1(R_1) = \alpha_1(R_2) = \{h_1\}$.

Suppose the rank $\geq 2$, and let $R$ be a region of $H$. By induction, we know that $\alpha_1(\Pi(R))$ is equal to a simplex $\{b_1, b_2, \ldots, b_{r-1}\}_<$ of $H_{r-1}$. By Proposition 3.1 the adjacency graph of $H$ restricted to the fiber of $R$ is a path $\lambda$ joining the two extreme regions of the fiber.

- If $R$ is extreme in its fiber, set $b_r = f_r$, where $f_r$ is the $r$-th element of the minimal basis, the smallest hyperplane in $H_r \setminus H_{r-1}$, i.e. the smallest edge of $\lambda$.
- If $R$ is not extreme in its fiber, then $R$ has two facets in $H_r \setminus H_{r-1}$, corresponding to the two edges of $\lambda$ incident to $R$. One of these two facets separates $R$ from at least one of the two regions of the fiber adjacent to $f_r$. Let $b_r$ be the other facet. Graphically, if we direct the edges of $\lambda$ different from $f_r$ away from $f_r$, then $b_r$ is the edge of $\lambda$ directed away from $R$.

We set $\alpha_1(R) = \{b_1, b_2, \ldots, b_r\}_<$.

**Theorem 3.5.** *The mapping $\alpha_1$ is an activity preserving (surjective) mapping from regions to internal simplices of an ordered supersolvable hyperplane arrangement.*

*The number of regions associated with a basis with internal activity $i$ is $2^i$.*

*Proof.* For each fiber associated with an internal basis $B$ of $H_{r-1}$, the two extreme regions of the fiber are associated with $B \cup f_r$. If $H_r \setminus H_{r-1}$ is not reduced to $f_r$, then the mapping built from the adjacency graph of regions in the fiber, preserves activities by Propositions 3.1 and 3.2. Since the two extreme regions in each fiber (and only they in each fiber) have the same image, we get the last result by induction on the rank. □

**Example 3.1.** Figure 2 shows the weakly active mapping $\alpha_1$ for the arrangement of Figure 1. We show the construction for two fibers associated with the bases 12 (the left one) and 14 of $H_2$.

**Example 3.2.** The hyperplane arrangement $H(G)$ associated with a graph $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, is the arrangement of $\mathbf{R}^n$ having a hyperplane of equation $x_i = x_j$ for each edge $v_i v_j \in E$. A graph is said to be *chordal*, or *triangulated*, if every cycle of length at least 4 has a *chord*, i.e. if there exists an edge of the graph joining two non-consecutive vertices of the cycle. As well-known, the arrangement $H(G)$ is supersolvable if and only if $G$ is chordal [21]. The following classical alternate definition of chordal graphs is the graphic form of the inductive definition of supersolvable arrangement of

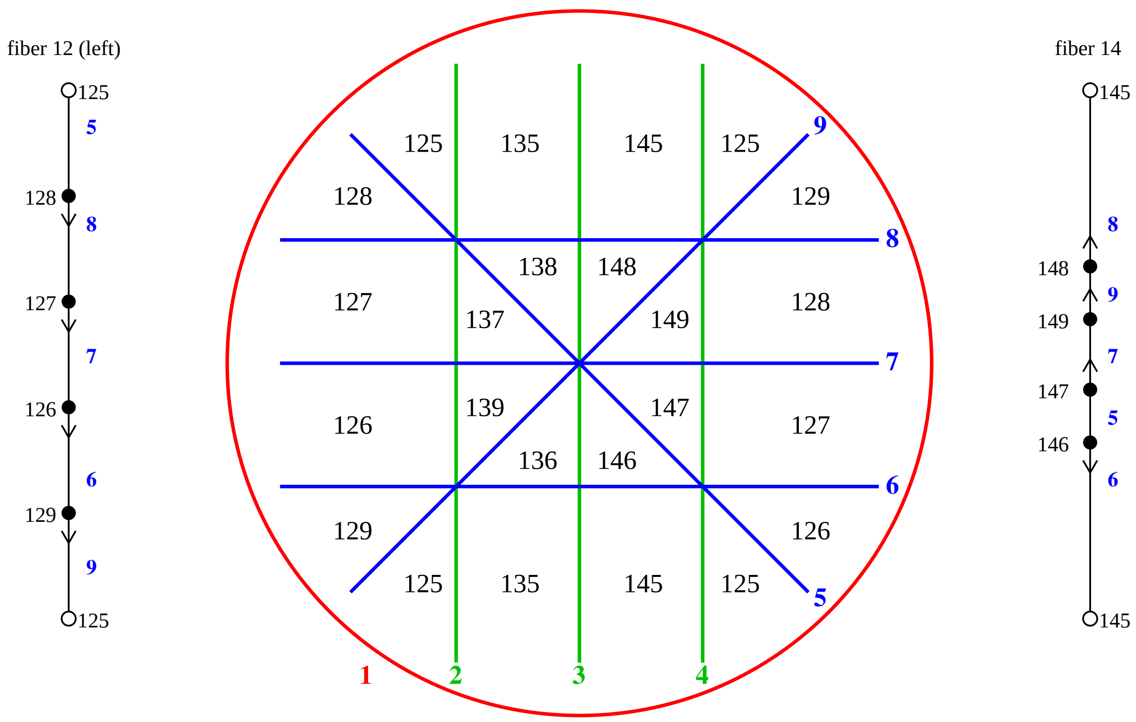



Figure 2. The weakly active mapping for the arrangement of Figure 1

[2]. The graph $G = (V, E)$ is triangulated if and only if there exists a reindexing of the vertices such that, for all $2 \leq i \leq n$, the vertices $v_j$ with $j < i$ adjacent to $v_i$ constitute a clique of $G$.

For $2 \leq i \leq n$, let $E_{i-1}$ be the set of edges $v_j v_k \in E$ such that $j, k \leq i$. Then, with $r = n - 1$, $E_1 \lhd E_2 \lhd \ldots \lhd E_r = E$ is a resolution of $H(G)$. Assume the edge-set of $G$ is linearly ordered, such that the above resolution is ordered. The mapping $\alpha_1$ from acyclic orientations of $G$ to spanning trees is constructed by applying inductively Definition-Algorithm 3.4 as follows.

Let $\overrightarrow{G}$ be an acyclic orientation of $G$, and $T' = \alpha_1(\overrightarrow{G} \setminus v)$ with $v = v_n$. Let $N$ be the set of neighbours of $v$. Since $\overrightarrow{G}[N]$ is a complete acyclic directed graph, there is a unique directed path $u_1 \to u_2 \to \ldots \to u_k$ containing all vertices of $N$. The orientation of $\overrightarrow{G}$ being acyclic, there is $0 \leq j \leq k$ such that the edges joining $v$ and $N$ are directed from $u_i$ to $v$ for $1 \leq i \leq j$ and from $v$ to $u_i$ for $j + 1 \leq i \leq k$. Set $\overrightarrow{G}_j = \overrightarrow{G}$. Then, as is easily seen, the fiber path of $\overrightarrow{G}$ is $\overrightarrow{G}_0$ —— $\overrightarrow{G}_1$ —— $\ldots$ —— $\overrightarrow{G}_k$. Two consecutive acyclic orientations $\overrightarrow{G}_{i-1}$ and $\overrightarrow{G}_i$, $1 \leq i \leq k$, of this path are related by reversing the direction of the edge $u_i v$. Therefore the corresponding regions of the fiber are separated by the hyperplane associated with $u_i v$.

Suppose $u_\ell v$, $1 \leq \ell \leq k$, is the smallest edge of $E \setminus E_{r-1}$ in the ordering of $E$.

Then, applying Definition-Algorithm 3.4, we have
- $\alpha_1(\overrightarrow{G}_j) = T' \cup \{u_\ell v\}$ if $j = 0$ or $j = k$,
- $\alpha_1(\overrightarrow{G}_j) = T' \cup \{u_j v\}$ if $1 \leq j \leq \ell - 1$,
- $\alpha_1(\overrightarrow{G}_j) = T' \cup \{u_{j+1} v\}$ if $\ell \leq j \leq n - 1$.

The case when $G$ is a complete graph is studied more completely in Section 5.

**Remark**. The construction of $\alpha_1$ in each fiber only uses the adjacency graph, the element of the minimal basis cutting this fiber, and the compatibility of the ordering with the resolution of $H$. Hence, the image of a region under $\alpha_1$ is not affected by changing the linear order provided it is compatible with the given resolution and has the same minimal basis (i.e. the smallest element in each $H_j \setminus H_{j-1}$ is not changed).

## 4. The active mapping for supersolvable hyperplane arrangements

The weakly active mapping having a simple construction in the supersolvable case may seem more natural than the active mapping considered in this section. However, the active mapping has many interesting structural properties. The regions associated with a given basis have a natural characterization, related to the fact that the active mapping not only preserves active elements, but also active partitions. In the general case, the two active mappings coincide for $(1, 0)$ activities, i.e. for bounded regions of arrangements [12c].

We point out that, in the bounded case, the active mapping has a natural interpretation in terms of optimization and linear programming [12a]. Finally, particularly in the supersolvable case, the active mapping can be seen as a refinement of the weakly active mapping. The same construction is used in each path of a sequence of nested paths representing the fiber, when it has a dual activity superior to 1, instead of a single path representing the fiber as in the previous Section.

In the whole section $H_1 \lhd H_2 \lhd \ldots \lhd H_r = H$ is an ordered resolution of a supersolvable hyperplane arrangement.

Let $R$ be a region of $H$ with $\mathcal{AO}^*(R) = \{a_1, \ldots, a_k\}_<$. By Proposition 3.3, every active flag of a non-extreme region of the fiber of $R$ is of the form $X_k \subset X_{k-1} \subset \ldots \subset X_1 = H$ with associated active partition $A_k = X_k$, $A_i = X_i \setminus X_{i+1}$ for $1 \leq i \leq k-1$, and with $\min(A_i) = \min(X_i) = a_i$. We order this set of active partitions of non-extreme regions in the fiber $\Pi(R)$ by lexicographic inclusion: the partition $A_1 + \ldots + A_{k-1} + A_k$ is smaller than the partition $A'_1 + \ldots + A'_{k-1} + A'_k$ if and only if there exists an index $i$ with $1 \leq i \leq k$, such that $A_i \subset A'_i$ and $A_{i'} = A'_{i'}$ for all $i'$, $i < i' \leq k$. Active flags are ordered consistently.

**Proposition 4.1.** *Let $\mathcal{A}$ be the active partition of a region in a fiber $\Pi$. The set of regions in $\Pi$ with active partition smaller than $\mathcal{A}$ constitute a connected subpath of the path defined by all regions in $\Pi$ in the adjacency graph.*

*Let $\mathcal{X} = X_k \subset \ldots \subset X_1$ be the active flag corresponding to $\mathcal{A}$. Let $1 \leq j \leq k+1$ be the smallest index such that the frontier of every region with active flag smaller than $\mathcal{X}$ contains the intersection of hyperplanes in $H_r \setminus X_j$ (we make the convention $X_{k+1} = \emptyset$). This intersection is a face $F$, and, when $X_j \neq \emptyset$, $X_j$ is the support of the covector corresponding to $F$.*

*Then, the set of hyperplanes in $H_r \setminus H_{r-1}$ which are facets of regions with active flag smaller than $\mathcal{X}$ is $H_r \setminus (H_{r-1} \cup X_j)$. Furthermore, a hyperplane belongs to this set if and only if it contains the face $F$.*

*Proof.* First, consider a given face in the arrangement. The set of regions, in the fiber $\Pi$, of which frontier contains this face form a path. Indeed each region in this set is obtained from any other region in this set by successive reorientations of elements, one by one, such that the intermediate regions remain in the set, and every element is used at most once. Now, consider $i$ fixed subsets $X_k \subset \ldots \subset X_{k-i+1}$. With the geometrical interpretation of active flags, and the above observation, we deduce that the set of regions for which these $i$ fixed subsets are the $i$ first subsets in the active flag form a path, since it is an intersection of subpaths of a path. The inclusion relation of faces corresponding to active flags corresponds exactly to the lexicographic inclusion of the subsets that form the active flags. Hence, the set of regions whose active sequence is smaller than a given one is exactly a set of regions having $i$ fixed smallest subsets $X_k \subset \ldots \subset X_{k-i+1}$, and thus forms a path in the fiber. By definition of the ordering of active flags, the set $X_j$ is maximal belonging to every active flag smaller than $\mathcal{X}$, if it exists. The face $F$ corresponds to a covector with support $X_j$ if $j < k+1$, and to the intersection of all hyperplanes (null vector) if $j = k+1$. The hyperplanes which are facets of regions in the path are exactly hyperplanes containing the corresponding face $F$. So they form the set $(H_r \setminus H_{r-1}) \cap (H_r \setminus X_j)$. $\square$

When $\mathcal{A}$ is the active partition of a region in $\Pi$, we define $P(\mathcal{A})$ as the path of Proposition 4.1 included in $\Pi$, together with the two regions in $\Pi$ adjacent to the extremity regions of this path. We also define $F(\mathcal{A})$ as the intersection of the set of hyperplanes separating regions of this path, i.e. edges of $P(\mathcal{A})$. In the notations of Proposition 4.1, this face is $F$ and corresponds to the covector with support $X_j$ when $X_j \neq \emptyset$.

We have four isomorphic ordered sets, relative to the set of non-extreme regions of a given fiber:
(1) the set of active partitions $\mathcal{A}$ ordered lexicographically by (set) inclusion,
(2) the set of active flags $\mathcal{X}$ (successive unions in $\mathcal{A}$) ordered consistently,
(3) the set of paths $P(\mathcal{A})$ ordered by (graphical) inclusion,
(4) the set of faces $F(\mathcal{A})$ ordered by (geometrical) reverse inclusion.

Let $\mathcal{A}$ be the active partition of a region in $\Pi$. We associate with $\mathcal{A}$ a minor of the path $\Pi$ as follows: for every path $P(\mathcal{A}')$, strictly contained in $P(\mathcal{A})$, all vertices (regions) of this path are deleted, except the extreme ones, and all edges are deleted, except the smallest. The remaining path is called the *reduced path* of $\mathcal{A}$. By construction, every non extreme region in the fiber corresponds to a non extreme region of one and only one reduced path in the fiber.

The following definition-algorithm gives a direct definition of the active mapping in the supersolvable case. We will then establish that the general active mapping of Section 2 and the present one are equal in this special case. To distinguish them until the equality is proved, the active mapping of Section 2 will be denoted by $\underline{\alpha}$.

**Definition-Algorithm 4.2.** *Inductive construction of the active mapping $\alpha$.*

We define the mapping $\alpha$ from the regions of $H$ to its internal simplices by induction on the rank. Let $R$ be a region of $H$. By induction, we know that $\alpha(\Pi(R))$ is equal to the simplex $\{b_1, b_2, \dots, b_{r-1}\}_<$ of $H_{r-1}$.

The input for the computation is the path of the fiber $\Pi(R)$, and the active partition - or active flag - of each region in $\Pi(R)$.

• If $R$ is extreme in $\Pi(R)$, set $b_r = f_r$, where $f_r$ is the $r$-th element of the minimal basis, hence the smallest hyperplane in $H_r \setminus H_{r-1}$, and the smallest edge of the fiber.

• If $R$ is not extreme in $\Pi(R)$, then let $\mathcal{A}$ be its active partition, let $\lambda$ be the reduced path of $\mathcal{A}$, and let $e$ be the smallest edge (hyperplane) of $\lambda$. Then $R$ is adjacent to two edges (hyperplanes) in $\lambda$. One of these two hyperplanes separates $R$ from at least one of the two regions of the fiber adjacent to $e$. Let $b_r$ be the other hyperplane. Graphically, if we direct the edges of $\lambda$ different from $e$ away from $e$, then $b_r$ is the edge of $\lambda$ directed away from $R$.

We set $\alpha(R) = \{b_1, b_2, \dots, b_r\}_<$.

Note that the above construction is very similar to the construction of $\alpha_1$, except that the path which has to be considered is the reduced path associated with the region, instead of its whole fiber.

Note also that a direct computation, not using the reduction to reduced paths, is obtained by replacing the second point with the following:

• If $R$ is not extreme in $\Pi(R)$, let $\mathcal{A}$ be its active partition. By convention, we set the active partitions of extreme regions of the fiber to be strictly greater than the others. Let $R_1$, $R_2$ be the first vertices (regions) with active partitions greater than $\mathcal{A}$ in both sides of $R$ on the path $\Pi$ associated with the fiber. Let $e$ be the smallest edge (hyperplane) of the subpath $[R_1, R_2]$ of $\Pi$. Reversing if necessary, we adapt the notation such that $e$ is in $[R_1, R]$. Let $R'$ be the first vertex with active partition greater than or equal to $\mathcal{A}$ when going from $R$ on the subpath $]R, R_2]$ (we may have $R' = R_2$, but by definition $R' \neq R$). Then $b_r$ is the smallest edge of the subpath $[R, R']$.

We mention briefly that, in fact, the reduction to reduced paths is related to a more general definition of $\alpha$ by decomposition of activites [12b]. The basis associated with $\alpha$ is calculated in a minor where the induced region is bounded with respect to the smallest element. Here, this minor is the arrangement of hyperplanes containing the face $F(\mathcal{A})$ where the smaller faces, in the ordered set (4) mentioned above, are contracted. As we shall see in next Proposition 4.4, the mappings $\alpha$ and $\alpha_1$ coincide for bounded regions.

Thus, it is not surprising that the construction applied to reduced paths for $\alpha$ is the same as the construction applied to the whole fiber path for $\alpha_1$.

**Theorem 4.3.** *The mapping $\alpha$ is an activity preserving (surjective) mapping from the set of regions to the set of internal simplices of an ordered supersolvable arrangement.*

*Two regions have the same image under $\alpha$ if and only if they have the same active partition, and one can be obtained from the other by reorienting parts of the active partition. The number of regions in the inverse image of a simplex with internal activity $i$ is $2^i$.*

*Proof.* The mapping $\alpha$ is an activity preserving mapping in exactly the same way as $\alpha_1$ in Theorem 3.5. The reorientation property is available for the general construction [12b]. In the present case of a supersolvable arrangement, this property has an easy proof by induction on the rank, since reorienting a subset in the active flag amounts to reversing a path in the fiber, that is to reversing several reduced paths. Furthermore, the construction of the maximal element of the basis associated with a region is invariant under the reversal of the relevant reduced path. □

**Proposition 4.4** *The mappings $\alpha$ and $\alpha_1$ coincide on regions with activities* $(1, 0)$.

*Proof.* This property is obvious since the inductive definitions of $\alpha$ and $\alpha_1$ coincide for regions not touching $f_1$ (except at the null vector). Indeed, all these regions have $A\mathcal{O}^* = \{f_1\}$ as their set of orientation dually-active elements. □

**Example 4.1.** Figure 3 shows the active mapping for the example of Figures 1 and 2. The active paths for the two fibers associated with 12 are shown. In these two fibers, for regions associated with 127 or 128, the active partition is $1678 + 23459$, since the hyperplanes 6, 7, 8 and 1 meet at one point, which means that the intersection of the frontiers of these two regions is this intersection point. For regions associated with 126 or 129, the active partition is $1 + 23456789$, since 1 is a facet of these regions. For regions associated with 125, the active partition is $1+234+56789$, which is the minimal flag. We observe that the paths formed by regions associated with 125, 126, and 129 are reversed in the two fibers associated with 12, due to the reorientation of 23456789 to pass from one region to the other, whereas the paths formed by regions associated with bases 127 and 128 have same direction, due to the reorientation of 23459 to pass from one region to the other. Moreover, in the fiber on the left, we see that regions associated with bases 126, 127 and 128 are switched in Figures 2 and 3, showing that $\alpha_1$ and $\alpha$ may be different on regions with internal activity $> 1$.

**Example 4.2.** Figure 4 is a more involved example of a fiber in a rank-4 supersolvable arrangement, with incomparable active partitions. First consider three independent hyperplanes 1, 2 and 3 in the real affine space with rank 3, and a region delimited by these hyperplanes, that is a cone with apex $O = 1 \cap 2 \cap 3$. This cone is cut by hyperplanes

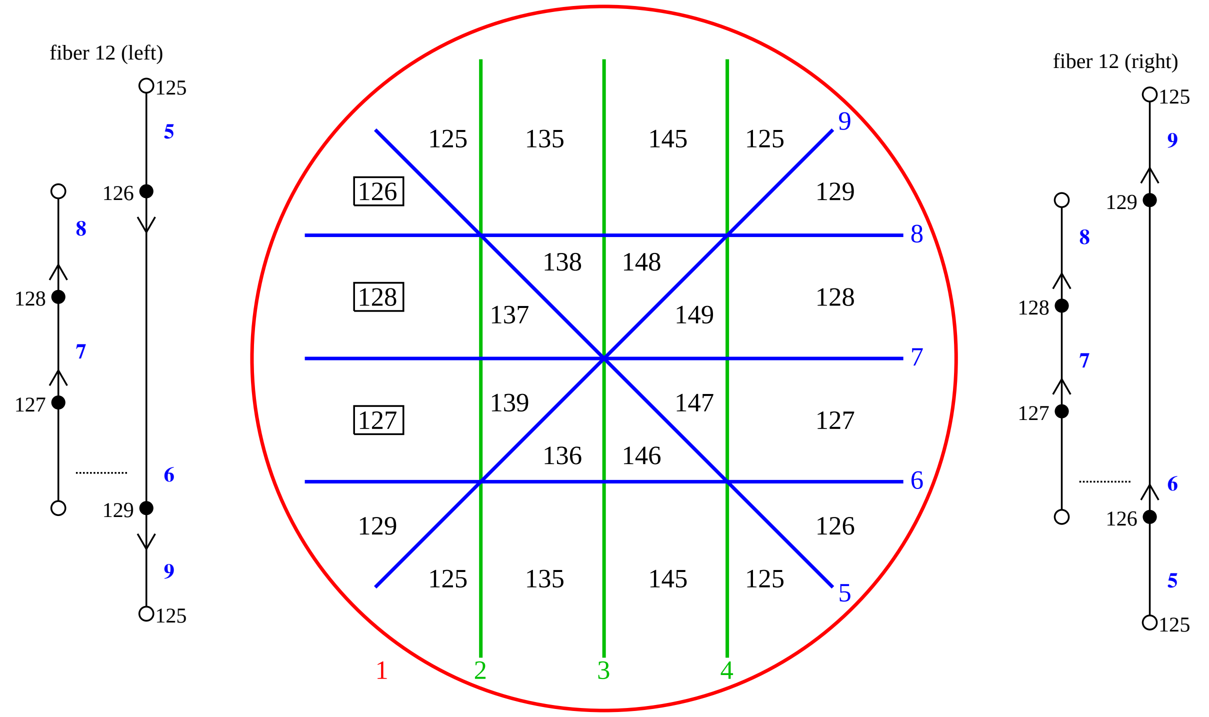
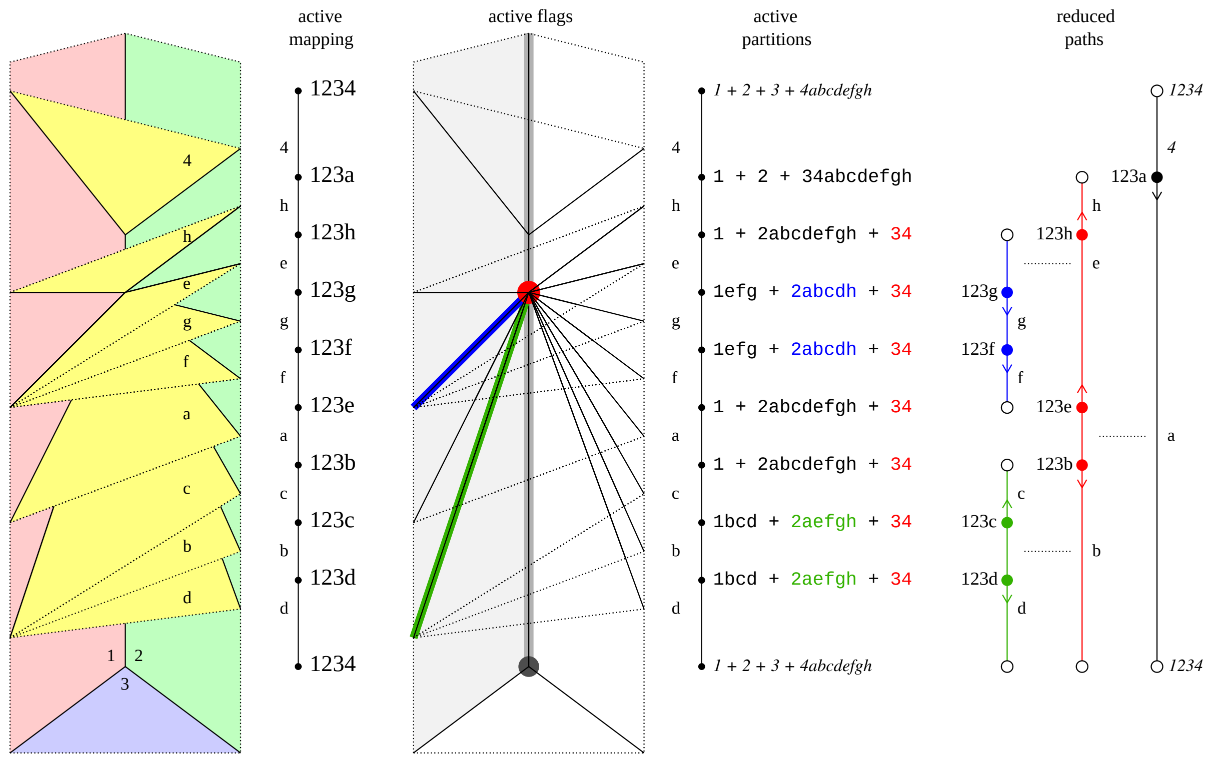



Figure 3. The active mapping for the arrangement of Figure 1

$4, a, b, c, d, e, f, g, h$ in such a way that two of these hyperplanes do not cut inside the cone, and the intersections with 1 and 2 are represented in Figure 4. In particular $a \cap b \cap c \cap d \cap e \cap f \cap g \cap h$ is a point $I$. Hence this figure is a partial representation of the cone, whose information is sufficient to build the mappings. We use the ordering $1 < 2 < 3 < 4 < a < b < c < d < e < f < g < h$.

We have to check that this arrangement can be completed into a supersolvable arrangement for which no other hyperplane cut the cone, and for which every other hyperplane contains $O$. For $i, j \in \{4, a, b, c, d, e, f, g, h\}$ and $i \neq j$, set $H_{ij}$ to be the hyperplane containing $i \cap j$ and $O$. For $i, j \in \{a, b, c, d, e, f, g, h\}$, the hyperplane $H_{ij}$ contains the line $(OI)$. Moreover, for $i, j \in \{a, b, c, d, e, f, g, h\}$, we have $H_{4i} \cap H_{4j} \subseteq H_{ij}$. For $i \in \{a, b, c, d, e, f, g, h\}$, set $H_{3i}$ to be the hyperplane containing $O$, $I$ and the point $3 \cap 4 \cap i$. Then for $i \in \{a, b, c, d, e, f, g, h\}$ we have $3 \cap H_{4i} \subseteq H_{3i}$. Finally, we get a supersolvable arrangement with resolution $H_1 \lhd H_2 \lhd H_3 \lhd H_4$ equal to $\{1\} \lhd H_1 \cup \{2\} \cup \big\{H_{ij} \mid i, j \in \{a, b, c, d, e, f, g, h\}\big\} \cup \big\{H_{3i} \mid i \in \{a, b, c, d, e, f, g, h\}\big\} \lhd H_2 \cup \{3\} \cup \big\{H_{4i} \mid t \in \{a, b, c, d, e, f, g, h\}\big\} \lhd H_3 \cup \{4, a, b, c, d, e, f, g, h\}$. For a compatible ordering, this arrangement fits the setting of the previous results, which we apply below. By construction, the chosen cone defines a fiber delimited by 1, 2 3 and cut only by $4, a, b, c, d, e, f, g, h$. Hence we omit on the figure and in the active partitions the other hyperplanes that are useless for the construction.

Thus, the (partial) ordered resolution of this supersolvable arrangement is $1 \lhd 12 \lhd 123 \lhd 1234abcdefgh = H$. The minimal basis is 1234, and the minimal flag $1 \supset 1 \cap 2 \supset$




4. The active mapping in a rank-4 fiber

$1 \cap 2 \cap 3$. The fiber has orientation dually-active elements $1, 2, 3$. Hence it is associated with 123 in $H_3$. Since the two extreme regions in the fiber have orientation dually-active elements $1, 2, 3, 4$, they are associated with 1234 by $\alpha$.

A perspective view of the arrangement and of the active mapping is shown in the left part of Figure 4. The median part of Figure 4 shows the sequences of nested faces, representing geometrically the active flags, followed by the (partial) active partitions of regions. The partially directed reduced paths used in the Definition-Algorithm 4.2 are represented in the right part of Figure 4. For the non-extreme regions, the corresponding active flags are $1bcd \subset 1bcd2aefgh \subset H$ and $2aefg \subset 1bcd2aefgh \subset H$ which are minimal, and both strictly smaller than $1 \subset 1bcd2aefgh \subset H$.

The isomorphism of ordered sets mentioned previously appears in the right part of Figure 4 (in colors). Precisely, the active partition $1bcd + 2aefgh + 34$ corresponds to the 2-dimensional face $1 \cap b \cap c \cap d$, and to the path delimited by $c$ and $d$ (in green). The active partition $1efg + 2abcdh + 34$ corresponds to the 2-dimensional face $1 \cap e \cap f \cap g$, and to the path delimited by $e$ and $f$ (in blue). These two intervals being minimal, they are equal to their associated active path. The active partition $1 + 2abcdefgh + 34$ corresponds to the 1-dimensional face $1 \cap 2 \cap a \cap b \cap c \cap d \cap e \cap f \cap g \cap h$, and to the path delimited by $d$ and $h$ (in red). The corresponding active path has edges $a$, $b$, $e$ and $h$. Finally, the active partition $1 + 2 + 34abcdefgh$ corresponds to the 0-dimensional intersection of all hyperplanes - not represented in this affine representation - and to the path delimited by 4 and $d$. The corresponding active path has two edges 4 and $a$.

The construction of $\alpha$ can be done by first considering the path induced by $d, b, c$, since the flag $1bcd \subset 1bcd2aefgh \subset E$ is minimal. Then the edges $d$ and $c$ are directed away from $b$, yielding the mapping for $123d$ and $123c$. Independently, the path $g, f, e$ yields the mapping for $123f$ and $123g$. Then $c, d, f, g$ are deleted, and we consider the path induced by $b, a, e$, and, lastly, the paths induced by $a$ and 4.

An equivalent definition of the active mapping, closer to the general inductive definition by deletion/contraction of Theorem 2.1 [12c], is given by Lemma 4.5.2 below.

For $h \in H$, we denote by $R \setminus h$ the region of $H \setminus h$ containing $R$. Note that if $h \in H_r \setminus H_{r-1}$, then $H \setminus h$ is supersolvable with resolution $H_1 \lhd \ldots \lhd H_{r-1} \lhd H_r \setminus h$.

**Lemma 4.5.1.** *Let $\omega \in H_r \setminus H_{r-1}$ be a facet of $R$. We assume that $R$ and $-_\omega R$ are not extreme. Let $a_i = \max(A\mathcal{O}^*_\omega(R))$, and let $A_1 + \ldots + A_k$ be the active partition of $R$. Let $a_{i'} = \max(A\mathcal{O}^*_\omega(-_\omega R))$, and let $A'_1 + \ldots + A'_k$ be the active partition of $-_\omega R$.*

*We have $a_i < a_{i'}$ if and only if $A_{i'} \subset A'_{i'}$ and $A_j = A'_j$ for all $j$ such that $i' < j \leq k$.*

*We have $a_i = a_{i'}$ if and only if $A_j = A'_j$ for all $j$ such that $1 \leq j \leq k$.*

*Moreover, in these two cases, the active partition of $R \setminus \omega$ equals $A'_1 \setminus \omega + \ldots + A'_k \setminus \omega$.*

*Proof.* First, every positive cocircuit of $R$, resp. ${}_\omega R$, with smallest element $a_j > \max(a_i, a_{i'})$ does not contain $\omega$, and so is also a positive cocircuit of $-_\omega R$, resp. $R$. Hence $A_j = A'_j$ for all $j$ such that $\max(i, i') < j \leq k$.

Secondly, we assume that $a_i < a_{i'}$. Every positive cocircuit of $R$ with smallest element $a_{i'}$ does not contain $\omega$ and so $A_{i'} \subseteq A'_{i'}$. But $\omega \in A'_{i'} \setminus A_{i'}$. Hence $A_{i'} \subset A'_{i'}$.

Thirdly, we assume that $a_i = a_{i'}$. Let $e \in H$ such that the smallest element of its part in the active partition of $R$, resp.$-_\omega R$, is $a_j$, resp. $a'_j$. Assume that $a_j < a'_j \leq a_i$. By definition, there exists a cocircuit $C'$ with smallest element $a'_j$, positive in $-_\omega R$. If $C'$ does not contain $\omega$, then $C'$ is also a positive cocircuit of $R$, which is a contradiction with $a'_j > a_j$ and the definition of $a_j$. Hence $C'$ has only one negative element $\omega$ in $R$. By definition of $a_i$, there exists a positive cocircuit $C$ of $R$ containing $\omega$ with smallest element $a_i$. Let $C''$ be a cocircuit of $R$ containing $e$, obtained by matroid elimination of $\omega$ from $C$ and $C'$. Then $C''$ is a positive cocircuit of $R$ containing $e$ with smallest element $\geq a'_j$, which is a contradiction with $a'_j > a_j$ and the definition of $a_j$. Hence $a_j = a'_j$, and so the active partitions or $R$ and $-_\omega R$ are equal. The two implications above prove the two equivalences in the lemma.

Finally, we assume that $a_i \leq a_{i'}$. Let $e \in H$, such that the smallest element of its part in the active partition of $-_\omega R$, resp. $R \setminus \omega$, is $a'_j$, resp. $a_j$. Every positive cocircuit of $-_\omega R$ with smallest element $a'_j$ and containing $e$ contains a positive cocircuit of $R \setminus \omega$, such that this cocircuit contains $e$ and has its smallest element greater than or equal to $a'_j$. Hence $a'_j \leq a_j$. Conversely, let $C$ be a positive cocircuit of $R \setminus \omega$ with smallest element $a_j$ and containing $e$. If $C$ is a positive cocircuit of $-_\omega R$, then $a_j \leq a'_j$ by definition of $a'_j$. Indeed, otherwise, it can be written $C = (D \setminus \omega) \cup (D' \setminus \omega)$ where $D$, resp. $D'$, is a positive cocircuit containing $\omega$ of $R$, resp. $-_\omega R$. So $a_j = \min(\min(D), \min(D'))$. If $D'$ contains $e$ then $a_j \leq \min(D') \leq a'_j$. If $D$ contains $e$ then $a_j \leq \min(D) \leq a_i \leq a_{i'}$.

Let $D''$ be a positive cocircuit of $-_\omega R$ containing $\omega$ with $\min(D'') = a_{i'}$. By matroid elimination of $\omega$ from $D$ and $D''$, there is a positive cocircuit $D'''$ of $-_\omega R$ containing $e$ with $\min(D''') \geq \min(D)$. Hence $a_j \leq \min(D) \leq \min(D''') \leq a'_j$. Therefore, the active partition of $R \setminus \omega$ conforms to the description given in the lemma. □

**Lemma 4.5.2.** *The mapping $\alpha$ is constructed by the following algorithm.*
*Let $R$ be a region of $H$, and $\omega$ be the greatest hyperplane in $H$.*

*(1) If $\omega > f_r$ is a facet of $R$, then*
*(1.1) if $\max \mathcal{AO}^*_\omega(R) > \max \mathcal{AO}^*_\omega(-_\omega R)$, then $\alpha(R) = \alpha(R \setminus \omega)$,*
*(1.2) if $\max \mathcal{AO}^*_\omega(R) < \max \mathcal{AO}^*_\omega(-_\omega R)$, then $\alpha(R) = \alpha(\Pi(R)) \cup \{\omega\}$,*
*(1.3) if $\max \mathcal{AO}^*_\omega(R) = \max \mathcal{AO}^*_\omega(-_\omega R)$, then, set $s = \max(\alpha(R \setminus \omega))$,*
*(1.3.1) if $s$ is a facet of $R$, then $\alpha(R) = \alpha(R \setminus \omega)$,*
*(1.3.2) otherwise, $\alpha(R) = \alpha(\Pi(R)) \cup \{\omega\}$.*
*(2) If $\omega > f_r$ is not a facet of $R$, then $\alpha(R) = \alpha(R \setminus \omega)$.*
*(3) If $\omega = f_r$ then $\alpha(R) = \alpha(\Pi(R)) \cup \{\omega\}$.*

Note that, when $\omega > f_r$ is a facet of $R$, this algorithm builds at the same time the image of $R$ and $-_\omega R$ under $\alpha$, one being equal to $\alpha(\Pi(R)) \cup \{\omega\}$, and the other to $\alpha(R \setminus \omega) = \alpha(\Pi(R)) \cup \{s\}$.

*Proof.* First, if $R$ is extreme, then $\max \mathcal{AO}^*_\omega(R) > \max \mathcal{AO}^*_\omega(-_\omega R)$. Secondly, if $\omega$ is not a facet of a region $R$ then the active partition of $R \setminus \omega$ is obtained by removing $\omega$ from its part in the active partition of $R$. Moreover, $\max \mathcal{AO}^*_\omega(R) = \max \mathcal{AO}^*_\omega(-_\omega R)$ if and only if $R$ and $-_\omega R$ are non extreme regions of the same reduced path, thanks to Lemma 4.5.1. Thus, the equivalence of this construction with the definition of $\alpha$ is easy to check. We omit the details. □

**Lemma 4.5.3.** *For all regions $R$ of a supersolvable arrangement of hyperplanes $H$ with an ordered resolution, we have $\underline{\alpha}(R) \setminus \max(\underline{\alpha}(R)) = \underline{\alpha}(\Pi(R))$.*

*Proof.* Let $\omega$ be the greatest element of $H$. By definition of $\underline{\alpha}$ (Section 2): if $\omega \in \underline{\alpha}(R)$ then $\underline{\alpha}(R) = \underline{\alpha}(R/\omega) \cup \omega$, and if $\omega \notin \underline{\alpha}(R)$ then $\underline{\alpha}(R) = \underline{\alpha}(R \setminus \omega)$. Moreover, if $\omega = f_r$, then $\omega$ is an isthmus and the result is obvious. We assume now that $\omega > f_r$.

Clearly, $H \setminus \omega$ is supersolvable, and the fibers of $H \setminus \omega$ are obtained by removing $\omega$ in the fibers of $H$. Hence, all elements superior to $\max(\underline{\alpha}(R))$ can be deleted, so that we may assume, for the sequel, $\omega = \max(\underline{\alpha}(R))$. Thus $\underline{\alpha}(R) \setminus \max(\underline{\alpha}(R)) = \underline{\alpha}(R/\omega)$.

Let $e \in H$ with $f_r \leq e < \omega$. By definition of a supersolvable arrangement of hyperplanes, the intersection of $e$ and $\omega$ is included in a hyperplane of $H_{r-1}$. Hence the face $(R/\omega) \setminus e$ of $H \setminus e$ cannot be cut by $e$. In other words, $e$ does not belong to a positive cocircuit of $-_e R/\omega$. Hence, by definition of $\underline{\alpha}$, we have $\underline{\alpha}(R/\omega) = \underline{\alpha}(R/\omega \setminus e)$. Applying this successively to all $e \in ((H_r \setminus \omega) \setminus H_{r-1})$, we then get $\underline{\alpha}(R/\omega) = \underline{\alpha}(R/\omega \setminus ((H_r \setminus \omega) \setminus H_{r-1}))$. But $\omega$ is an isthmus of $R \setminus ((H_r \setminus \omega) \setminus H_{r-1})$, hence $\underline{\alpha}(R/\omega \setminus ((H_r \setminus \omega) \setminus H_{r-1})) = \underline{\alpha}(R \setminus (H_r \setminus H_{r-1})) = \underline{\alpha}(\Pi(R))$. □

**Theorem 4.5.** *The mapping $\alpha$ from regions of an ordered supersolvable arrangement to internal simplices is equal to the mapping $\underline{\alpha}$ (restricted to regions).*

*Proof.* We prove this by induction on the rank of $H$. We have to prove that the definition given in Lemma 4.5.2 coincides with the definition of $\underline{\alpha}$. In view of Lemma 4.5.3, we just have to check that the two definitions coincide in the case where $\max A\mathcal{O}^*_\omega(R) = \max A\mathcal{O}^*_\omega(-_\omega R)$. This case corresponds to the case 1.3 of the definition of $\underline{\alpha}$.

In that case, let $B = \underline{\alpha}(R/\omega)$. By Lemma 4.5.3, we have $B = \underline{\alpha}(\Pi(R))$, and by the induction hypothesis, $B = \alpha(\Pi(R))$. Let $C = C^*(B \cup \omega; \omega)$. Since $B$ is included in $H_{r-1}$, the flat of $M$ generated by $B$ is $H_{r-1}$. Hence the support of $C$ is $H_r \setminus H_{r-1}$.

Let $e = \min\big(C \setminus \bigcup D\big)$, where the union is over all positive cocircuits $D$ of $M$ such that $\min D > \max A\mathcal{O}^*_\omega(M)$. Let $a_1 < \ldots < a_k$ be the set of active elements of $R$, and $\mathcal{X} = X_k \subset \ldots \subset X_1$ be the active flag of $R$, with corresponding active partition $\mathcal{A}$. Let $a_i = \max(A\mathcal{O}_\omega(M))$. We get $e = \min(C \setminus X_{i+1}) = \min(H_r \setminus (H_{r-1} \cup X_{i+1}))$. Let $F_\omega$ be the face corresponding to the positive covector of $R$ with support $X_{i+1}$.

The hyperplane $\omega$ contains the face $F(\mathcal{A})$ by Proposition 4.1, since it is a facet of the path $P(\mathcal{A})$ for which $R$ is a non-extreme vertex. Hence $F(\mathcal{A}) \subseteq F_\omega$. If $F(\mathcal{A}) \subset F_\omega$ then there would be a region $R'$ with active flag $\mathcal{X}' = X'_k \subset \ldots \subset X'_1$ and $\omega \in X'_{i+1}$, which would be a contradiction with $\mathcal{X}'$ being smaller than $\mathcal{X}$. Hence $F(\mathcal{A}) = F_\omega$.

So $X_{i+1} \cap (H_r \setminus H_{r-1})$ is the set of edges of the path $P(\mathcal{A})$, and $e$ is the minimal edge of this path. Hence $e$ is the minimal edge of the reduced path of $\mathcal{A}$. By definition of $\alpha$, we have $b_r = \omega$ if and only if $\omega$ separates $R$ and $e$, that is if and only if $\omega$ and $e$ have opposite signs in $C$. Hence the two definitions are the same. $\square$

## 5. The active mappings for the braid arrangement

We apply in this section the results of Section 3 and 4 to the braid arrangement. The two active mappings $\alpha$ and $\alpha_1$ are equal, and equivalent to a known bijection between permutations and increasing trees, in a simple and explicit way. The active mappings are constructed here from Definition-Algorithm 3.4 and Definition-Algorithm 4.2 for supersolvable arrangements. Another way could be by applying the results of [11] Sections 6-7 for graphs, since the braid arrangement is graphic.

The braid arrangement, denoted here by $\mathcal{B}_n$, is a real arrangement consisting of $n(n-1)/2$ hyperplanes. In $R^n$, a realization of $\mathcal{B}_n$ is given by the equations $h_{i,j} \equiv -x_i + x_j = 0$ for $1 \le i < j \le n$. This arrangement is of rank $n-1$: all hyperplanes contain the line $x_1 = x_2 = \ldots = x_n$. Projecting along this line, we get an alternate description of $\mathcal{B}_n$ as the arrangement of full rank comprised by the mirrors of symmetry of the regular simplex $S_n$ of $R^{n-1}$.

As is well-known, the braid arrangement $\mathcal{B}_n$, the complete graph $K_n$ with vertices indexed by $\{1, 2, \ldots, n\}$, the permutation group $\mathcal{S}_n$, and the Coxeter group $\mathbf{A}_{n-1}$ are closely related combinatorial objects.

• $\mathcal{B}_n$ and $K_n$. If the hyperplane $h_{i,j}$ is associated with the directed edge $ij$ of $K_n$, then the regions of $\mathcal{B}_n$ are in bijection with the acyclic orientations of $K_n$. The fundamental region with all $h_{i,j} > 0$ corresponds to the acyclic orientation of $K_n$ with all edges directed from $i$ to $j$ for $1 \leq i < j \leq n$.

• $K_n$ and $\mathcal{S}_n$. An acyclic orientation of $K_n$ defines a linear ordering of its vertices, that is a permutation of $\{1, 2, \ldots, n\}$, and conversely. An edge $ij$ is directed from $i$ to $j$ when $i < j$. Hence, the source respectively sink of the orientation is the minimal respectively maximal element of the associated permutation. The fundamental region is associated with the identity permutation.

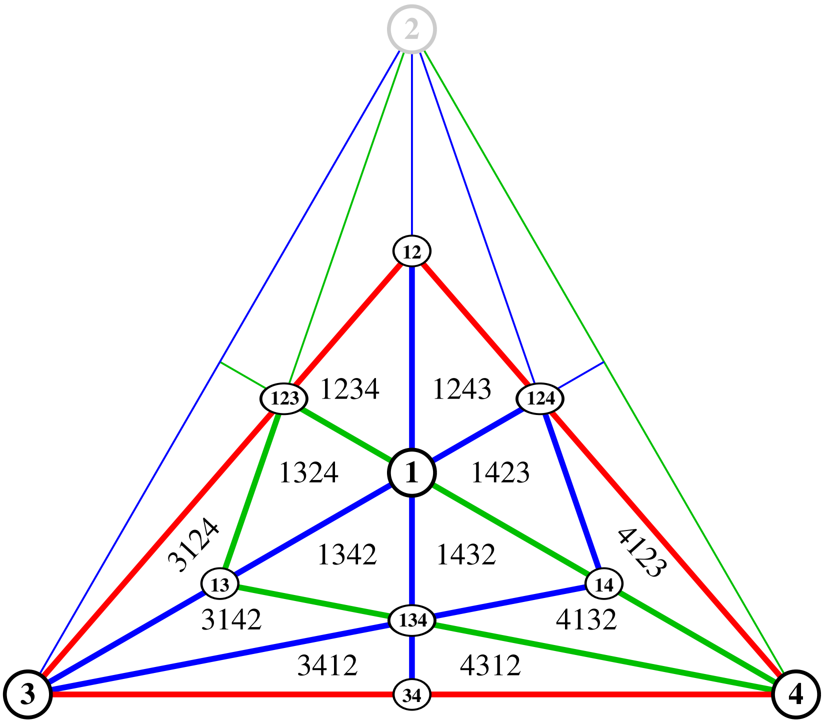



Figure 5. 4-permutations and the barycentric subdivision of the 4-simplex

• $\mathcal{S}_n$ and $\mathbf{A}_{n-1}$ The transpositions $s_i = (i, i+1)$, $i = 1, 2, \ldots, n-1$, a standard set of generators of $\mathcal{S}_n$, constitute $n-1$ involutions. They satisfy the relations $(s_i s_{i+1})^3 = 1$ for $1 \leq i \leq n-1$ and $(s_i s_j)^2 = 1$ if $1 \leq i, j \leq n-1$ with $j \geq i+2$, hence these involutions define the Coxeter group $\mathbf{A}_{n-1}$

• $\mathcal{B}_n$, $\mathcal{S}_n$ and $\mathbf{A}_{n-1}$. In the interpretation of the Coxeter group $\mathbf{A}_{n-1}$ as the symmetry group of the regular simplex $S_n$ of $R^{n-1}$, the reflections of $\mathbf{A}_{n-1}$, conjugates of the generators $s_1, s_2, \ldots, s_{n-1}$ in the group, are geometrically the *mirrors of symmetry* of the edges of $S_n$, i.e. the hyperplanes orthogonal to the edges at their middles. These reflections define the first barycentric subdivision $BS_n$ of $S_n$, dividing the polytope $S_n$ into $n!$ simplicial cells. The elements of $\mathbf{A}_{n-1}$ corresponds bijectively to the permutations of $12\ldots n$, and also to the simplices of $BS_n$. With the permutation $i_1 i_2 \ldots i_n$

is associated the simplex of $BS_n$ with vertices $i_1$, $i_1 i_2$,..., $i_1 i_2 \dots i_n$, where $i_1 i_2 \dots i_k$ denotes the barycenter of the vertices $i_1, i_2, \dots, i_k$ of $S_n$. See Figure 5 and Figure 6.

In the sequel, we will use whichever language is more convenient.

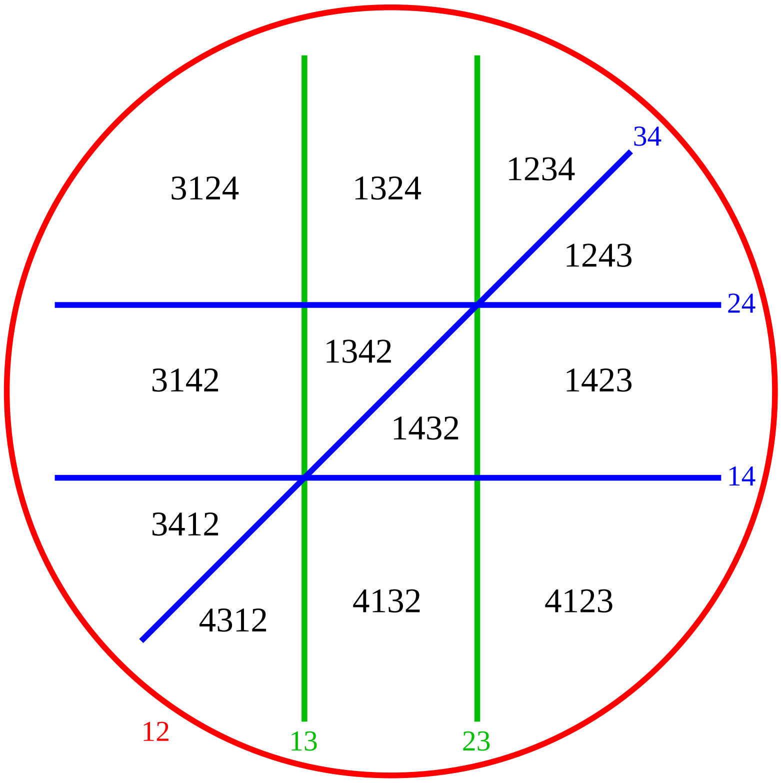



Figure 6. The braid arrangement $\mathcal{B}_3$ and 4-permutations

The braid arrangement is supersolvable as pointed out by Stanley [21] Prop. 2.8. The *standard resolution* of $\mathcal{B}_n$ is $\mathcal{B}_2 \lhd \mathcal{B}_3 \lhd \dots \lhd \mathcal{B}_n$. It follows immediately from the equations that $h_{i,n} \cap h_{j,n} \subset h_{i,j}$.

The *colexicographical ordering* of the hyperplanes $ij = h_{i,j}$ is a standard linear ordering

$$12 < 13 < 23 < 14 < 24 < 34 < \dots$$

of $\mathcal{B}_n$, defined by $ij < k\ell$ if either $j < \ell$, or $j = \ell$ and $i < k$. Actually, the colexicographic ordering is only one among many linear orderings of $\mathcal{B}_n$ yielding the desired properties for active mappings. We say that a linear ordering of $\mathcal{B}_n$ is *admissible* if it is an ordering compatible with the standard resolution and such that $1i$ is the smallest hyperplane of $\mathcal{B}_i \setminus \mathcal{B}_{i-1}$ for $2 \leq i \leq n$. In Section 5, we suppose $\mathcal{B}_n$ ordered by an admissible linear ordering.

Any ordering of the hyperplanes of $\mathcal{B}_n$ induces corresponding orderings of the edges of $K_n$, of the transpositions of $\mathcal{S}_n$ and of the reflections of $\mathbf{A}_{n-1}$.

The fiber of a permutation $p$ of $12\dots n$ is the set of $n$ permutations obtained by putting the letter $n$ at each of the $n$ possible places defined by the permutation $p'$ obtained from $p$ by removing $n$. Let $p' = i_1 i_2 \dots i_{n-1}$. The fiber path is

$$p_1 = n i_1 i_2 \dots i_{n-1} \;\text{——}\; p_2 = i_1 n i_2 \dots i_{n-1} \;\text{——}\; \dots \;\text{——}\; p_n = i_1 i_2 \dots i_{n-1} n$$

**Lemma 5.1.1.** *For any admissible ordering of $\mathcal{B}_n$, the smallest hyperplane separating two regions in the above fiber is $1n$.* □

**Proposition 5.1.** *Let $p = i_1 i_2 \dots i_n$ be a permutation of $12\dots n$, $n \geq 2$, and $2 \leq k \leq n$. The letter $k$ determines two subpermutations $q_1 = p[i_1 \dots k]$ and $q_2 = p[k \dots i_n]$ of $p$. If one of these two subpermutations, say $q$, does not contain 1 and contains a letter smaller than $k$, set $t_k = jk$, where $j$ is the letter $< k$ closest to $k$ in $q$. Otherwise, set $t_k = 1k$.*

*Then, the weakly active mapping for an admissible linear ordering of $\mathcal{B}_n$ is given by $\alpha_1(p) = \{t_2, t_3, \dots, t_n\}$.*

*Proof.* To determine $t_k$, we have to apply Algorithm 3.3 to the greatest letter $k$ of $p'$, where $p'$ is obtained from $p$ by deleting all letters $> k$. If the subpermutation $q_1$ or $q_2$ of $p$ not containing 1 does not contain a letter smaller than $k$, then $k$ is extreme in $p'$, and we have $t_k = 1k$ by Algorithm 3.3 and Lemma 5.1.1. Otherwise $k$ is not extreme in $p'$, and we have $p' = \dots 1 \dots kj \dots$ or $p' = \dots jk \dots 1 \dots$ with $j < k$. By Algorithm 3.3 applied to $p'$, we have $t_k = jk$. In this second case, we observe that in $p$ all letters between $j$ and $k$ are $> k$, achieving the proof. □

Proposition 5.1 and its proof implicitly use the following definition. We say that a letter $a$ is *active* in a permutation $p$ if $a$ does not separate the letters $< a$.

By the properties of $\alpha_1$, $\{t_2, t_3, \dots, t_n\}$ is a spanning tree of $K_n$, and this spanning tree is internal for the colexicographic ordering. As easily seen, a spanning tree $T$ of $K_n$ with vertices labelled by $12\dots n$ is internal for the colexicographic ordering if and only if vertex labels increase along each of its paths beginning at 1. We say that a tree with this property is *increasing*.

Note that by [12], for any ordering we have $\alpha = \alpha_1$ on bounded regions, i.e. regions having no vertex in the smallest hyperplane. For the braid arrangement, we may have $\alpha \neq \alpha_1$ on an unbounded region if the order is not admissible (reverse the ordering of 14 and 34 in $K_4$, for instance).

**Theorem 5.2.** *For any admissible linear ordering of the braid arrangement, we have $\alpha = \alpha_1$.*

To prove Theorem 5.2 we need a description of active partitions in order to be able to apply Algorithm 4.2.

**Lemma 5.2.1.** *Consider an acyclic orientation of $K_n$, associated with the directed path $i_1 i_2 \dots i_n$. The positive cocircuits of $K_n$ are determined by partitions of this path into two subpaths. A positive cocircuit of $K_n$ consists of all edges joining the two sets of vertices $i_1 i_2 \dots i_j$ and $i_{j+1} \dots i_n$ for some integer $1 \leq j \leq n-1$.* □

Lemma 5.2.1 is immediate.

An alternate point of view, in relation to the group structure, applies also to the non-graphic hyperoctahedral arrangement of Section 6. A positive cocircuit $C$ of $\mathcal{B}_n$ is the set of hyperplanes not containing some vertex $v$ of the fundamental region $R$. Let $p = i_1 i_2 \dots i_n$ be the permutation associated with $R$. The hyperplanes supporting the facets of $R$ are the $n-1$ transpositions $i_j i_{j+1}$ for $j = 1, 2, \dots, n-1$. Since $R$ is a simplex, a vertex $v$ of $R$ is determined by the unique facet opposite to it. It follows from the group structure that the hyperplanes of $\mathcal{B}_n$ containing the vertex $v$ opposite to the facet $i_j i_{j+1}$ are the transpositions of the subgroup of $S_n$ generated by the facet hyperplanes of $R$ containing $v$. These facets, namely the transpositions $i_1 i_2$, $i_2 i_3$, …, $i_{j-1} i_j$, $i_{j+1} i_{j+2}$, …, $i_{n-1} i_n$, generate the permutation groups $S_j[i_1, i_2, \dots, i_j]$ and $S_{n-j}[i_{j+1}, \dots, i_n]$. The cocircuit $C$ consists of all transpositions of $S_n$ not in these two subgroups: we recover in the language of groups the characterization of Lemma 5.2.1 stated in terms of graphs.

The smallest letters of $i_1 i_2 \dots i_j$ and $i_{j+1} \dots i_n$ are 1 and $a \neq 1$ up to the order. Then the smallest element of $C$ is the transposition $1a$, by definition of an orientation dually-active element. Since $a$ is smallest in its part, we observe that there is no letter $i < a$ such that $p = \dots 1 \dots a \dots i \dots$ or $p = \dots i \dots a \dots 1 \dots$. Conversely, if this property holds, then $1a$ is smallest in at least one positive cocircuit, namely $i_1 \dots 1 \dots | a \dots i_n$ or $i_1 \dots a | \dots 1 \dots i_n$.

For a letter $a$ active in $p$, let $p[a]$ be the smallest interval of $p$ containing all letters $\leq a$. The intervals $p[a]$ are inclusion comparable. Let $2 = a_1 < a_2 < \dots < a_k$ be the active letters of $p$. Clearly, we have $p[a_i] = a_i \dots p[a_{i-1}]$ or $p[a_i] = p[a_{i-1}] \dots a_i$.

If a letter $a$ of $p$ is active then the positive cocircuits with smallest element $1a$ are exactly those defined by the cuts separating $a$ from all letters $< a$. We say that an edge of a positive cocycle with smallest element $1a$ is *activated* by $1a$, or more briefly, by $a$. The active partition of a region of $BS_n$ associated with a permutation $p$ of $n$ letters is a partition of the set of pairs of integers, i.e. edges of $K_n$, $\binom{[1\dots n]}{2} = A_{a_1} + A_{a_2} + \dots + A_{a_k}$ indexed by the active letters of $p$ in increasing order. If $a$ is an active letter, the set $A_a$ is the set of edges activated by $a$ but by no active letter $> a$. It follows immediately from Lemma 5.2.1 that the set $A_a$ is exactly the set of edges of $p[a]$ separated by at least one of the cuts separating $a$ from all letters $< a$, i.e. separating $a = a_i$ from $p[a_{i-1}]$, with $a_0 = 1$. Therefore

**Lemma 5.2.2.** *Let $2 = a_1 < a_2 < \dots < a_k$ be the active letters of $p$. Set $a_0 = 1$. We have $A_{a_i} = \mathcal{B}(p[a_i]) \setminus \mathcal{B}(p[a_{i-1}])$* □

In order to apply the results of Section 4 for a proof of Theorem 5.2, we need a description of the order structure of the active partitions of the permutations in a fiber.

Let $p' = i_1 i_2 \dots i_{n-1}$ be a permutation of $12\dots n-1$. Its fiber in $\mathcal{B}_n$ is $p_1 = n i_1 i_2 \dots i_{n-1}, p_2 = i_1 n i_2 \dots i_{n-1}, \dots, p_n = i_1 i_2 \dots i_{n-1} n$. Then, if $i_k = 1$, we have $p_k = \dots n1 \dots$, $p_{k+1} = \dots 1n \dots$. The colexicographic ordering of the active partitions has been defined in Section 4. Let $\mathcal{A}_i$ denote the active partition of $p_i$.

**Lemma 5.2.3.** *With above notation we have*

$$\mathcal{A}_2 \geq \mathcal{A}_3 \geq \dots \geq \mathcal{A}_k < \mathcal{A}_{k+1} \leq \mathcal{A}_{k+2} \leq \dots \leq \mathcal{A}_{n-1}$$

*Proof.* Let $p = i_1 \dots na \dots i_{n-1}$, and $p' = i_1 \dots an \dots i_{n-1}$ obtained from $p$ by transposing $a$ and $n$. We denote by $\mathcal{A}$ and $\mathcal{A}'$ the corresponding active partitions. It follows from the discussion after Lemma 5.2.1 that, if $a$ is not active, we have $\mathcal{A} = \mathcal{A}'$. Suppose $a$ is active, and consider the case where $p = \dots a \dots 1 \dots$ Since $n$ is the greatest letter, and is not at an end of either one, $p$ and $p'$ have the same active letters. The letter $i_1$ on the left of $a$ is clearly active. Let $a'$ be the greatest active letter on the left of $a$. We have $a' > a$ since $a$ is active, and 1 is on its right. We have $p[a'] = a' \dots na \dots$, and $p'[a'] = a' \dots an \dots$ after transposing $a$ and $n$. It follows from Lemma 5.2.2 that $A_{a'} \supset A'_{a'}$, and the inclusion is strict since $an$ is in $A_{a'}$ but not in $A'_{a'}$. Let $b$ be an active letter $> a'$. Since $b$ is active, it is not between $a'$ and 1. Hence we have either $p = \dots b \dots a' \dots na \dots 1 \dots$ or $p = \dots a' \dots na \dots 1 \dots b \dots$. As easily seen by using Lemma 5.2.2, in both cases we have $A_b = A'_b$. Therefore by definition of an admissible linear ordering, we have $\mathcal{A} > \mathcal{A}'$.

The case $p = \dots a \dots 1 \dots$ is identical up to reversing inequalities. $\square$

We are now in position to prove Theorem 5.2.

*Proof of Theorem 5.2.* By Lemma 5.2.3, the smallest hyperplane cutting an active interval of a fiber is always $1n$. Hence, when applying Algorithm 4.2 to reduced active intervals, directions from regions to hyperplanes defined by this smallest hyperplane are identical for $\alpha$ and $\alpha_1$. Therefore $\alpha = \alpha_1$. $\square$

The above proof of Theorem 5.2 by applying Algorithm 4.2 follows from the inductive construction of active mappings by deletion/contraction [12c]. Another construction of the active mappings is by decomposition of activities [12a][12b], used in [11] for the general graphical case. The first step is to decompose the orientation dual-activity of the acyclic oriented matroid under consideration into uniactive components by means of the active partition, and then apply an algorithm valid in the uniactive case, or bounded case, to compute the active basis. The general algorithm is based on oriented matroid programming. Here, we will apply Algorithm 3.3. This construction by decomposition applies simply to the braid arrangement, yielding an alternate proof of Theorem 5.2.

More precisely, let $M$ be an acyclic oriented matroid on a linearly ordered set $E$ of elements. Let $k \geq 1$ be the orientation dual-activity of $M$, and $E = A_1 + A_2 + \ldots + A_k$ be its active partition. The active partition has a natural ordering induced by the order of elements, since $A_i$ contains a unique active element $a_i$, which is its smallest element. Notation is chosen such that $a_1 < a_2 < \ldots < a_k$. We form $M_i = M/(A_1 + A_2 + \ldots + A_{i-1}) \setminus (A_{i+1} + A_{i+2} + \ldots + A_k)$, where / and \ denote the usual contraction and deletion operations of matroid theory. Then, $M_i$ is uniactive, and we have $\alpha(M) = \sum_{i=1,2,\ldots,k} \alpha(M_i)$ [12b] (see also [11] for the graphic case).

*Alternate proof of Theorem 5.2.* Let $a$ be the $i$-th active letter of a permutation $p$. The main step is to verify that the corresponding $M_i$ is again associated with a permutation. As we have observed above $A_a$ can be computed in $p[a]$. By Lemma 5.2.2, reducing $p$ to $p[a]$ amounts to deleting the edges in $A_{i+1} + A_{i+2} + \ldots + A_k$. If $a$ is not the smallest active letter, let $a'$ be the active letter immediately smaller than $a$, otherwise set $a' = 1$. For convenience set $p[1] = 1$. We can write $p[a] = aqp[a']$ or $p[a] = p[a']qa$. By Lemma 5.2.2, contracting all edges in $A_1 + A_2 + \ldots + A_{i-1}$ amounts to identifying to 1 all vertices in the path of $K_n$ corresponding to $p[a']$, i.e. to reduce the permutation $p[a']$ to the letter 1. Finally $M_i$ is associated with the permutation $1qa$.

By construction all letters in $q$ are $> a$, showing that $1qa$ has indeed activity 1 (in terms of a hyperplane arrangement with plane at infinity 1, the region $1qa$ is bounded). In this case we know from [12c] that $\alpha(1qa) = \alpha_1(1qa)$.

To achieve the proof, we have to check that $\alpha_1(1qa)$ is the restriction of $\alpha_1(p)$ to the letters of $qa$. This follows immediately from the definition of $\alpha_1$ in Theorem 5.1, since the edge associated with a letter by $\alpha_1$ in $p$ is computed in the smallest $p[a]$ containing it, hence also in the final $1qa$, therefore in the same way in $p$ and in $1qa$. $\square$

Interestingly enough, it turns out that $\alpha$ is equivalent in a very simple way to a classical bijection between permutations and increasing trees.

We quote from [23] p. 25. "Given $p = i_1 i_2 \ldots i_n \in \mathcal{S}_n$, construct an (unordered) tree $T(p)$ with vertices $0, 1, 2, \ldots, n$ by defining vertex $i$ to be the successor of the rightmost element $j$ of $p$ which precedes $i$ and which is less than $i$. If there is no such element $j$, then let $i$ be the successor of the root 0. The correspondence $p \mapsto T(p)$ is a bijection between $\mathcal{S}_n$ and increasing trees on $n+1$ vertices." According to the Notes [23] p. 41, "The technique of representing [..] permutations by models such as words and trees has been extensively developed primarily by the French." More precisely, the bijection between permutations and increasing trees has been independently introduced in three papers [7][9][25].

We denote this bijection by $\underline{\alpha}$.

**Lemma 5.3.1.** *Two permutations have the same image under $\alpha$ if and only if they are related by a sequence of reversals of intervals of the form $p[a]$ for $a$ an active letter of $p$.*

Lemma 5.3.1 follows from the general theory of [12b] since by Lemma 5.2.2 reversing $p[a]$ amounts to redirecting all edges in the part with smallest element $a$ of the active partition of $p$. It can also easily be proved directly.

*Proof.* As observed in the alternate proof of Theorem 5.2, the computation of $\alpha_1(i)$ is made in the smallest $p[a]$ containing $i$. It follows immediately from its definition that it is not affected by reversals of intervals of the form $p[a]$. Hence the condition is sufficient. Let $p$ and $p'$ be two permutations such that $\alpha(p) = \alpha(p') = T$. The permutations $p$ and $p'$ have the same active letters, which are the neighbours of 1 in $T$. If $a > 1$ is active then $p[a] = aqp[b]$ or its reverse, $p'[a] = aq'p'[b]$ or its reverse, where $b$ is the active letter preceding $a$. We may suppose inductively that $p[b] = p'[b]$ or its reverse. By definition of $\alpha_1$, the letters of $q$ and $q'$ are the labels of vertices not equal to $a$ of the connected component of $T \setminus \{1\}$ containing $a$. Hence $q$ and $q'$ have the same letters. An easy induction using the increasing property of $T$ - which is omitted - would show that $q$ and $q'$ are equal or reverse depending on whether $p$ and $p'$ begin by the same letter or not. □

**Lemma 5.3.2.** *A reversal class of permutations contains exactly $2^k$ permutations, where $k$ is the common number of active letters of permutations in the class. In each reversal class, exactly one permutation ends respectively begins with* 1. □

**Proposition 5.3.** *Let $p$ be a permutation of $12\ldots n$, and $i_1i_2\ldots i_{n-1}1$ be the unique permutation ending by* 1 *in the reversal class of $p$. Define $p' = i'_1i'_2\ldots i'_{n-1}$ by $i'_j = i_j - 1$ for $1 \leq j \leq n-1$. Then the increasing tree $T = \alpha(p)$ is obtained from the increasing tree $T' = \underline{\alpha}(p')$ by adding 1 to all vertex labels.* □

Proposition 5.3 follows immediately from Proposition 5.1 by using Lemma 5.3.2. This proposition contains the property that the active mapping restricted to permutations ending or beginning with 1 is the active bijection onto the set of increasing trees. We point out that this bijection is actually a particular case of a bijection induced by $\alpha$ in the general graphical case between internal trees and acyclic orientations with unique given sink or source [11]. The admissible edge ordering of the present section satisifies the condition required in [11] for this bijection.

Figure 7 illustrates Proposition 5.3 in the case $n = 4$. Each column is associated with an increasing tree $T$ on 4 vertices. The 3-permutation $p'$ such that $\underline{\alpha}(p') = T'$ appears in italics. The 4-permutations in a column constitute the reversal class associated with $T$. The active bijection is defined by $\alpha(p) = T$, where $p$ is the unique 4-permutation ending with 1 in a reversal class - last in the corresponding column of Figure 7 and appearing in boldface.

Proofs of the following propositions are straightforward. We omit them. Properties (i) and (ii) of Proposition 5.4 follow from the results of this section, (iii) is restated from [23] Prop. 1.3.16. We recall that a *descent* in a permutation $i_1i_2\ldots i_n$ is a letter $i_k$ with $1 \leq k \leq n-1$ such that $i_k > i_{k+1}$.

**Proposition 5.4.** *Let $p = i_1i_2\ldots i_n$ be a permutation of $12\ldots n$, and $a_1 = 2, \ldots, a_k$ be the active letters of $p$.*

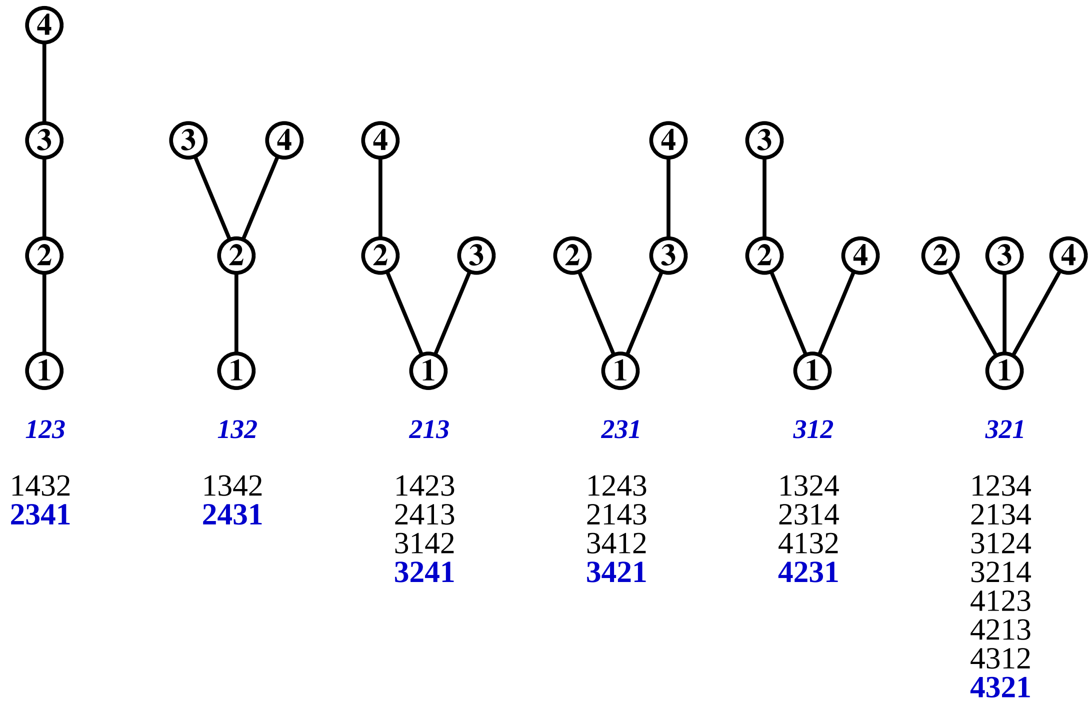



Figure 7. The active mapping from 4-permutations onto increasing trees on 4 vertices

*(i) The neighbours of the root (vertex labelled by 1) of the increasing tree on $n$ vertices $T = \alpha(p)$ are the active letters of $p$.*

*(ii) The vertices of the subtree of $T$ with root $a_j$, $1 \leq j \leq k$, are the letters of the interval $p[a_j] \setminus p[a_{j-1})$.*

*(iii) The leafs of $T$ are the descents of $p$ and the letter $i_n$.* □

Let $a' < a$ be two consecutive active letters. We have $p[a] = p[a']qa$ or the reverse. We observe that the active part $A_a$ is not changed if we permute $q$. The converse can be easily established.

**Proposition 5.5.** *Two permutations have the same active partition if and only they are related by a sequence of active reversals and permutations of $q$'s as above.* □

## 6. The active mappings for the hyperoctahedral arrangement

In this section, we apply results of Sections 3 and 4 to the hyperoctahedral arrangement. Again, the two active mappings $\alpha$ and $\alpha_1$ are equal, and equivalent to the classical bijection between permutations and increasing trees.

The hyperoctahedral arrangement, denoted here by $\mathcal{HO}_n$, is a real arrangement of $n^2$ hyperplanes. A realization in $R^n$ is given by the equations $h_{i,j} \equiv x_i - x_j = 0$ and

$\overline{h}_{i,j} \equiv x_i + x_j = 0$ for $1 \leq i < j \leq n$, $h_i \equiv x_i = 0$ for $1 \leq i \leq n$. These hyperplanes are the mirrors of symmetry of the regular $n$-dimensional hyperoctahedron $HO_n$, and also of its dual polytope, the regular $n$-dimensional hypercube.

The regions of $\mathcal{HO}_n$ - simplicial cones in the above representation - correspond bijectively to the cells of the first barycentric subdivision $BHO_n$ of the hyperoctahedron, or, equivalently, of the hypercube. Let the vertices of the hyperoctahedron be the unit coordinate vectors, positive and negative, denoted by the letters $1 = (1, 0, \ldots, 0)$, $\overline{1} = (-1, 0, \ldots, 0)$, $2 = (0, 1, \ldots, 0)$, $\overline{2} = (0, -1, \ldots, 0)$, etc. A simplex of $BHO_n$ has $n$ vertices of the form $i_1$, $i_1 i_2$, …, $i_1 i_2 \ldots i_n$, where we denote by $i_1 i_2 \ldots i_k$ the barycenter of the vertices labelled $i_1, i_2, \ldots, i_k$ of $HO_n$. As in the case of the braid arrangement, with these $n$ vertices labelled by words of increasing length, a simplex is naturally associated the permutation $i_1 i_2 \ldots i_n$. However, in the case of $BHO_n$ some letters may be endowed with a minus sign: we have a *signed permutation*. See Figure 8.

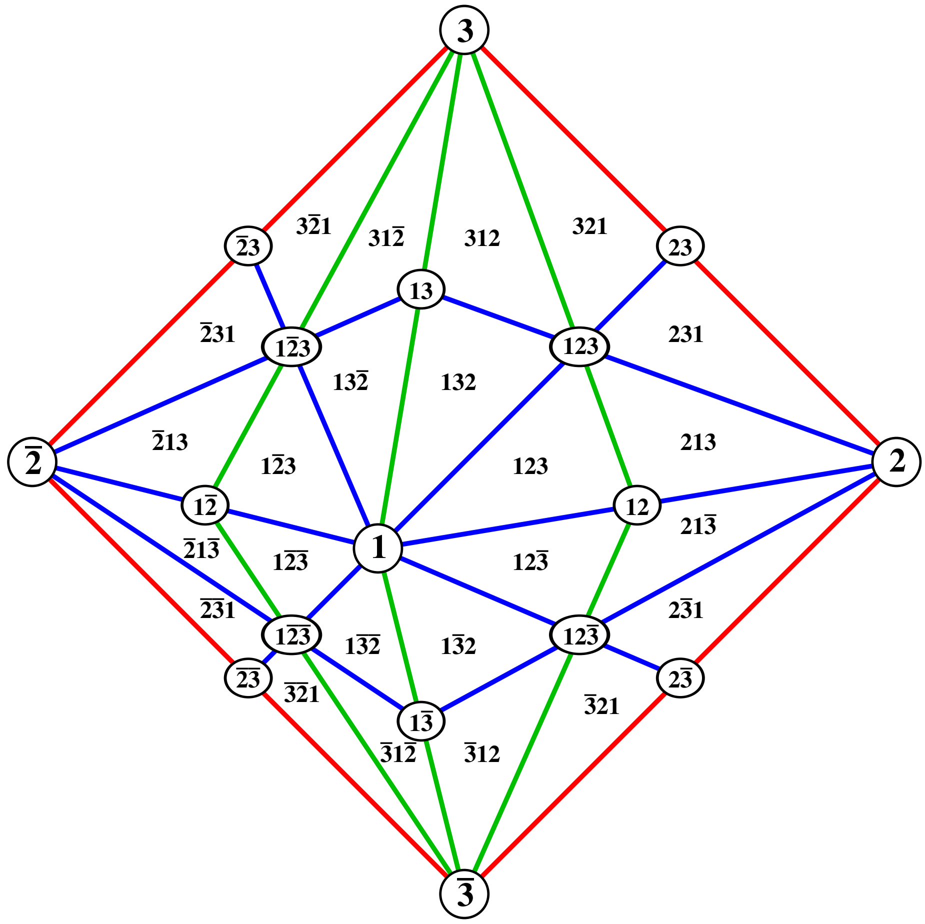



Figure 8. The barycentric subdivision of the octahedron and signed 3-permutations

We denote by $\overline{1}, \overline{2}, \ldots$ the letters $1, 2, \ldots$ endowed with a minus sign. In the literature, signed permutations associated with $\mathcal{HO}_n$ are more often described as ordinary permutations of the $2n$ letters $n^*, \ldots, 2^*, 1^*, 1, 2, \ldots, n$ commuting with the involution defined by the symbol $*$. The correspondence with signed permutations is straightforward: the signed permutation $2\overline{3}1$ is associated with the permutation $1^*32^*23^*1$, and conversely.

The reflections associated with the hyperplanes of $\mathcal{HO}_n$ are the reflections of the finite Coxeter group $B_n$. It is convenient to denote the hyperplanes of $\mathcal{HO}_n$ by the action of the corresponding reflection on the coordinates. We have $h_{i,j} = (i,j)$, $\overline{h}_{i,j} = (i, \overline{j})$, and $h_i = (i, \overline{i})$. We extend the notation to negative integers by setting $\overline{i} = -i$, and using the convention that if $i$ is negative, then $x_i = -x_{|i|}$. We use the shorthand notation $(i,j) = ij$. We have $ij = -ji = -\overline{i}\overline{j} = \overline{j}\overline{i}$ for any positive or negative integers $i, j$. If $|i| \neq |j|$, the equation of $ij$ is $x_i - x_j = 0$.

The position of a region $R$, i.e. its signs with respect to the hyperplanes of $\mathcal{HO}_n$, can be easily determined from the corresponding signed permutation $p$. By the choice of the equation signs, the identity permutation $12\ldots n$ corresponds to the fundamental region with all signs positive. The sign of a letter in $p$ gives the position of $R$ with respect to the corresponding variable. If a letter $i$ is positive respectively negative in $p$ then $R$ is on the positive respectively negative side of the hyperplane $i\overline{i}$. Let $p = \ldots i \ldots j \ldots$, where $i, j$ can be positive or negative. Then $R$ is on the positive side of the hyperplane $ij$, and on the positive side of the hyperplane $i\overline{j}$. The first rule follows, via reflections defined by hyperplanes $i\overline{i}$ for negative letters $i$ in $p$, from the same property in the positive hyperoctant, as a consequence of the all positive signs for the fundamental region. The second rule follows immediately from the equation $x_i + x_j = 0$ of the hyperplane $i\overline{j}$ since $x_{|i|}$ has the sign of $i$ on $R$. See Figure 9.

*Example.* Let $p = \overline{2}1\overline{3}$. We read

$2\overline{2} = -,\quad 1\overline{1} = +,\quad 3\overline{3} = -,\quad \overline{2}1 = +,\quad \overline{2}\overline{3} = +,\quad 1\overline{3} = +,\quad \overline{2}\overline{1} = +,\quad \overline{2}3 = +,\quad 13 = +.$

Hence, after reordering,

$1\overline{1} = +,\quad 2\overline{2} = -,\quad 1\overline{2} = -,\quad 12 = +,\quad 3\overline{3} = -,\quad 2\overline{3} = -,\quad 1\overline{3} = +,\quad 13 = +,\quad 23 = -.$

Here, we have $ij = +$ if and only if $R$ and the fundamental region 123 are on the same side of the hyperplane $ij$.

The regions of $HO_n$ have a group structure by composing the associated signed permutations. The composition of two signed permutations is obtained by composing the underlying ordinary (unsigned) permutations and applying a multiplicative rule of signs. Example: let $p = 31\overline{4}2$, $q = 2\overline{3}4\overline{1}$, then $pq = 421\overline{3}$ (composition from left to right: first $p$, then $q$)

The resulting group is the Coxeter group $B_n$. The reflections $s_i = i(i+1)$ for $1 \leq i \leq n-1$ and $s_n = n\overline{n}$ constitute a standard set of involutions generating $B_n$. The classical relations between generators of $B_n$ can easily be checked: $(s_i s_j)^2 = 1$ for $1 \leq i < j \leq n$, $j - i \geq 2$, $(s_i s_{i+1})^3 = 1$ for $1 \leq i \leq n-1$, $(s_{n-1} s_n)^4 = 1$.

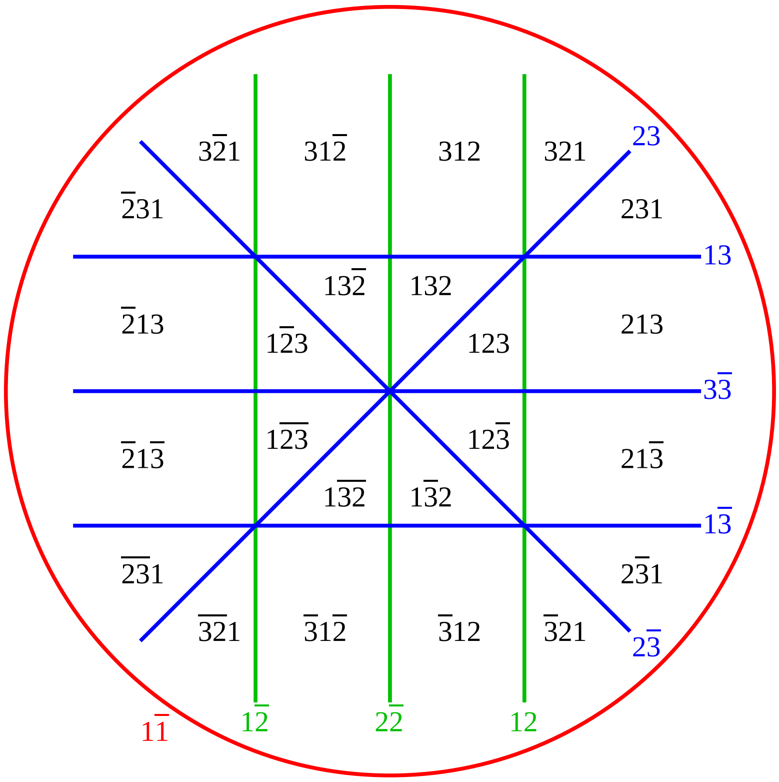



Figure 9. The hyperoctahedral arrangement $\mathcal{HO}_3$ and signed 3-permutations

The arrangement $\mathcal{HO}_n$ is supersolvable. The standard resolution of $\mathcal{HO}_n$ is

$$\mathcal{HO}_1 \triangleleft \mathcal{HO}_2 \triangleleft \ldots \triangleleft \mathcal{HO}_n$$

It is immediate to check that the intersection of two hyperplanes of $\mathcal{HO}_n \setminus \mathcal{HO}_{n-1}$ is contained in a hyperplane of $\mathcal{HO}_{n-1}$: the elimination of the variable $x_n$ from the corresponding equations obviously produces the equation of a hyperplane in $\mathcal{HO}_{n-1}$.

Unlike $\mathcal{B}_n$, no standard linear ordering of $\mathcal{HO}_n$ prevails in the literature. A natural choice is

$$1\bar{1} < 2\bar{2} < 1\bar{2} < 12 < 3\bar{3} < 2\bar{3} < 1\bar{3} < 13 < 23 < \ldots$$

However, as in Section 5, many other choices can be made for our purpose. We say that a linear ordering of $\mathcal{HO}_n$ is *admissible* if it is compatible with the standard resolution, and such that $h_i = i\bar{i}$ is the smallest hyperplane of $\mathcal{HO}_i \setminus \mathcal{HO}_{i-1}$ for all $2 \leq i \leq n$. The example of Figures 1-2-3 is $\mathcal{HO}_3$, but with a non-admissible linear ordering.

**Proposition 6.1.** *Let $p = i_1 i_2 \ldots i_n$ be a signed permutation of the integers $1, 2, \ldots, n$, $n \geq 1$. For $1 \leq k \leq n$, we set $t_k = i_j i_k$, where $j$ is the greatest index less than $k$ such that $|i_j| < |i_k|$, if such an index exists, and $t_k = i_k \overline{i_k}$ otherwise.*

*Then, for any admissible linear ordering of $\mathcal{HO}_n$, we have $\alpha_1(p) = \{t_1, t_2, \ldots, t_n\}$.*

Let $i_1, i_2, \ldots, i_{n-1}$ be the $(n-1)$-permutation obtained by deleting $n$ or $\overline{n}$ from the $n$-permutation associated with a region $R$ of $\mathcal{HO}_n$. The fiber of $\mathcal{HO}_n$ containing $R$ is

$p_1 = \overline{n} i_1 i_2 \ldots i_{n-1}$ — $p_2 = i_1 \overline{n} i_2 \ldots i_{n-1}$ — $\ldots$ — $p_n = i_1 i_2 \ldots i_{n-1} \overline{n}$ — $p_{n+1} = i_1 i_2 \ldots i_{n-1} n$ — $\ldots$ — $p_{2n-1} = i_1 i_2 \ldots n i_{n-1}$ — $p_{2n} = n i_1 i_2 \ldots i_{n-1}$

where the fiber path follows from Proposition 3.1.

**Lemma 6.1.1.** *The smallest hyperplane separating two adjacent regions in the fiber of a region of $\mathcal{HO}_n$ is $n\overline{n}$. This hyperplane is located at the middle of the fiber path. With above notation, it separates $p_n$ and $p_{n+1}$.* □

Using Lemma 6.1.1, the proof of Proposition 6.1 by applying Algorithm 3.3 is straightforward.

As in Section 5, we recognize $\alpha_1$ as a variant of the same classical bijection between permutations and increasing trees. We will discuss this point at the end of the section, after obtaining $\alpha$.

Proposition 6.1 implicitly uses the following definition. We say that a (signed) letter $a = i_k$ of a signed permutation $p = i_1 i_2 \ldots i_n$ of $12 \ldots n$ is *active in $p$* if there is no letter $i_j$ of $p$ with $j < k$ such that $|i_j| < |a|$. The first letter of $p$ and the signed letters $1$, $\overline{1}$ are always active.

**Theorem 6.2.** *For any admissible linear ordering of $\mathcal{HO}_n$, we have $\alpha = \alpha_1$.*

To prove Theorem 6.2 by means of Algorithm 4.2, we have to compare the active partitions of the regions in a fiber of $\mathcal{HO}_n$.

Let $i_1, i_2, \ldots, i_k$ be signed letters. We denote by $\mathcal{B}[i_1 i_2 \ldots i_k]$ respectively $\mathcal{HO}[i_1 i_2 \ldots i_k]$ the braid respectively hyperoctahedral arrangement defined by the variables $x_{i_j}$ $1 \leq j \leq k$. We note that the geometric (=unsigned) hyperplanes in $\mathcal{B}[i_1 i_2 \ldots i_k]$ depend on the signs of $i_1, i_2, \ldots, i_k$, but not those in $\mathcal{HO}[i_1 i_2 \ldots i_k]$.

For instance, we have $\mathcal{B}[\overline{3}41] = \{\overline{3}4, \overline{3}1, 41\}$ and $\mathcal{HO}[\overline{3}2] = \{\overline{3}3, \overline{3}2, \overline{3}\overline{2}, 2\overline{2}\}$.

**Lemma 6.2.1.** *Let $R$ be a region of $\mathcal{HO}_n$, and let $p = i_1 i_2 \ldots i_n$ be the signed permutation of $12 \ldots n$ associated with $R$. Let $F$ be a facet of $R$, supported by a hyperplane $i_k i_{k+1}$ for some $1 \leq k \leq n-1$ or by $i_n \overline{i}_n$ (in this latter case we consider that $k = n$). Let $v$ be the (unique) vertex of the simplex $R$ not contained in $F$. The cocircuit $C_v$, consisting of all hyperplanes not containing the vertex $v$, is given by*

$$C_v = \mathcal{HO}_n \setminus \big(\mathcal{B}[i_1 \ldots i_k] \cup \mathcal{HO}[i_{k+1} \ldots i_n]\big)$$

*The smallest hyperplane of $C_v$ is $a\overline{a}$, where $a$ is the first active letter starting from $i_k$ and going to the left.*

*Proof.* The facets of $R$ containing $v$ are supported by the hyperplanes $i_j i_{j+1}$ for $j = 1, 2, \ldots k-1, k+1, k+2, \ldots, n-1$ and $i_n \overline{i_n}$ if $k < n$. The transpositions associated with these hyperplanes generate the subgroups $A[i_1, i_2, \ldots, i_k]$ and $B[i_{k+1}, i_{k+2}, \ldots, i_n]$ of signed permutations. The hyperplanes defined by the transpositions of these subgroups, namely $\mathcal{B}[i_1 \ldots i_k]$ and $\mathcal{HO}[i_{k+1} \ldots i_n]$, constitute the set of hyperplanes of $\mathcal{HO}_n$ containing $v$. Hence, the cocircuit $C_v$ of $\mathcal{HO}_n$ associated with $v$, consisting of all hyperplanes not containing $v$, is equal to $\mathcal{HO} \setminus \mathcal{B}[i_1 \ldots i_k] \cup \mathcal{HO}[i_{k+1} \ldots i_n]$.

There is no letter $i$ in $p$ such that $|i| < |a|$ between $a$ and $i_k$, otherwise the smallest such letter would be active, contradicting the definition of $a$. Hence all letters $i$ of $p$ with $|i| < |a|$ are in the interval $[i_{k+1} \ldots i_n]$ of $p$. It follows that $\mathcal{HO}[i_{k+1} \ldots i_n]$ contains all hyperplanes of $\mathcal{HO}_n$ smaller than $a\overline{a}$ in the ordering. Since $a\overline{a}$ is in $C_v$, this hyperplane is the smallest hyperplane of $C_v$. □

**Lemma 6.2.2.** *With notation of Lemma 6.2.1, let $a_1, a_2, \ldots, a_k = i_1$ be the active letters of $p$, indexed such that $p = a_k \ldots a_{k-1} \ldots a_1 \ldots$. We have $|a_1| = 1 < |a_2| < \ldots |a_k|$ and $a_k = i_1$.*

*The active partition of $R$ is given by $A_1 = \mathcal{HO}[a_1 \ldots i_n]$, and by $A_j = \mathcal{HO}[a_j \ldots i_n] \setminus \mathcal{HO}[a_{j-1} \ldots i_n]$ for $2 \leq j \leq k$.*

*Proof.* For a vertex $v$, let $C_v$ denote the cocircuit of hyperplanes not containing $v$. Let $X_j$ be the union over all cocircuits $C_v$, where $v$ is vertex of the region $R$ whose smallest element is one of $a_j, a_{j+1}, \ldots, a_k$. By Lemma 6.2.1, we have

$$\begin{aligned} X_1 &= \bigcup_{i_\ell \in [i_1 \ldots i_n]} \big( (\mathcal{HO}[i_1 \ldots i_n] \setminus (\mathcal{B}[i_1 \ldots i_\ell] \cup \mathcal{HO}[i_{\ell+1} \ldots i_n]) \big) \\ &= \mathcal{HO}[i_1 \ldots i_n] \end{aligned}$$

and for $2 \leq j \leq k$

$$\begin{aligned} X_j &= \bigcup_{i_\ell \in [a_k = i_n \ldots a_{j-1}[} \big( (\mathcal{HO}[i_1 \ldots i_n] \setminus (\mathcal{B}[i_1 \ldots i_\ell] \cup \mathcal{HO}[i_{\ell+1} \ldots i_n]) \big) \\ &= \mathcal{HO}[i_1 \ldots i_n] \setminus \mathcal{HO}[a_{j-1} \ldots i_n] \end{aligned}$$

We have $A_k = X_k$, and $A_j = X_j \setminus X_{j+1}$ for $1 \leq j \leq k-1$. Lemma 6.2.2 follows. □

Let $i_1 i_2 \ldots i_{n-1}$ be a signed permutation of the letters $1, 2, \ldots, n-1$. Its fiber in $\mathcal{HO}_n$ is $p_1 = \overline{n} i_1 i_2 \ldots i_{n-1}$, $p_2 = i_1 \overline{n} i_2 \ldots i_{n-1}, \ldots$, $p_n = i_1 i_2 \ldots i_{n-1} \overline{n}$, $p_{n+1} = i_1 i_2 \ldots i_{n-1} n$, $p_{n+2} = i_1 i_2 \ldots n i_{n-1}, \ldots$, $p_{2n} = n i_1 i_2 \ldots i_{n-1}$.

Let $\mathcal{A}_i$ be the active partition of $p_i$ for $1 \leq i \leq 2n$. As easily seen, $p_2, p_3, \ldots, p_{2n-1}$ have the same active letters. This follows also from Proposition 3.3 and Lemma 6.2.1.

The colexicographic ordering of active partitions has been defined in Section 4.

**Lemma 6.2.3.** *We have $\mathcal{A}_i = \mathcal{A}_{2n-i+1}$ for $1 \leq i \leq n$, and*

$$\mathcal{A}_1 \geq \mathcal{A}_2 \geq \ldots \geq \mathcal{A}_n$$

*Furthermore, we have $\mathcal{A}_i > \mathcal{A}_{i+1}$ if and only if $n$ permutes with an active letter when going from $p_i$ to $p_{i+1}$.* □

We omit the proof, which is a straightforward consequence of Lemma 6.2.2.

*Proof of Theorem 6.2.* Let $k$ be the index such that $|i_k| = n$. The letter $i_k$ is active if and only if $k = 1$. In this case, by the Algorithm 3.3 and 4.2, we have $\alpha(t_n) = n\overline{n} = \alpha_1(t_n)$. Suppose $k \geq 2$, and let $a = i_j$ be the first active starting from $i_k$ and going to the left. By Lemma 6.2.3, the active interval $\lambda[R_1, R_2]$ is the interval $i_1 \dots i_{j-1}\overline{n}i_j \dots i_{k-1}i_{k+1} \dots i_n, \dots, i_1 \dots i_{j-1}ni_j \dots i_{k-1}i_{k+1} \dots i_n$ of the fiber path. The definition of an admissible linear ordering of $\mathcal{HO}_n$ immediately implies that the minimal edge of this symmetric interval is its middle edge $n\overline{n}$. Theorem 6.2 follows readily from Definition-Algorithm 4.2. □

The classical bijection $\underline{\alpha}$ between permutations of $12 \dots n$ and increasing trees on $n+1$ vertices labelled $01 \dots n$ [23] (see Section 5) readily provides a bijection between signed $n$-permutations and increasing trees with $n+1$ signed vertices. The increasing tree associated with a signed permutation $p$ is the image under $\underline{\alpha}$ of the permutation underlying $p$. Its vertices are signed in accordance with the signs of letters in $p$.

By removing the vertex labelled 0, we get an equivalent bijection between signed $n$-permutations and increasing forests on $n$ signed vertices.

We get from this bijection another equivalent one, between signed $n$-permutations and increasing forests on $n$ vertices with signed roots and signed edges, by keeping root signs and signing an edge of the forest plus if its two vertices have the same sign, and minus otherwise. Here, the definition of the edge-signature is chosen in accordance with the graphical representation of $\mathcal{HO}_n$ in Lemma 6.2.3.

Forgetting root signs, we obtain a mapping $\underline{\alpha}'$ from signed permutations to edge-signed increasing trees on $n$ vertices.

**Proposition 6.3.** *We have $\alpha = \alpha_1 = \underline{\alpha}'$.*

The graphical representation of $\mathcal{HO}_n$ is less classical than the graphical representation of $\mathcal{B}_n$. It can be done as follows. We represent a hyperplane $h_{i,j}$ of $\mathcal{HO}_n$ by the edge $ij$ of the complete graph $K_n$, a hyperplane $\overline{h}_{i,j}$ by the edge $ij$ endowed with a negative sign, and a hyperplane $h_i$ by the vertex $i$. This representation holds for geometric (unsigned) hyperplanes. To represent signed hyperplanes, we would have to consider directed edges and signed vertices.

We recall that a subset of the set of hyperplanes is internal if it contains no broken circuit, i.e. a circuit with its smallest element deleted.

**Lemma 6.3.1.** *A subset of $n$ hyperplanes of $\mathcal{HO}_n$ constitutes an internal simplex for an admissible linear ordering if and only if the associated edges constitute an increasing spanning forest of $K_n$, and the associated vertices are the smallest vertices of each tree of this forest.*

*Proof.* Let the $n$ hyperplanes be associated with a set $T$ of edges and a set $V$ of vertices of $K_n$. We have $|V| + |T| = n$. The hyperplanes represented by the edges of an elementary (i.e. without self- intersection) cycle of $T$ plus the hyperplane associated with one vertex $v$ of this cycle are linearly dependent, as can immediately be seen from their equations. Hence $v$ is not in $V$. If $v$ is chosen to be the smallest vertex of the cycle, we have a contradiction with the internal property, this vertex $v$ being minimal in its fundamental circuit for the admissible linear ordering. Therefore, the subgraph $T$ cannot contain any cycle, hence is a forest. A path with edges in $T$ joining two distinct vertices of $V$ produces a linear relation between the corresponding hyperplanes. Hence, a connected component of $T$ contains at most one vertex in $V$. Since $T$ is a forest, and $|V| + |T| = n$, it follows that $T$ is a spanning forest and each component contains exactly one vertex of $V$. It remains to show that the forest is increasing, and that a vertex of $V$ is the smallest vertex of its component of the forest. This follows easily as above from the internal property. □

The proof of Proposition 6.3 follows readily from Proposition 6.1, Theorem 6.2 and Lemma 6.3.1.

In Figure 10 the signed permutations with active letters positive appear at the top of each column, showing the active bijection (in blue).

Proofs of the following propositions are straightforward. We omit them.

**Lemma 6.4.1.** *Two signed permutations have the same image under the active mapping if and only if one can be obtained from the other by negating intervals of the form $p[a] \dots i_n$ for an active letter $a$ of $p$. Thus, an activity class contains exactly $2^k$ signed permutations, where $k$ is the activity of the class. It contains exactly one signed permutation with all active letters positive.* □

**Proposition 6.4.** *The active mapping is a bijection from the set of signed permutations with positive active letters onto the set of edge-signed increasing forests.* □

**Proposition 6.5.** *The number of edge-signed increasing forests on $n$ vertices is equal to $(2n-1)!! = (2n-1)(2n-3)\dots 3.1$.*

*Proof.* We extend $K_n$ to $K_{n+1}$ by adding an extra vertex labelled 0. By adding edges joining 0 to the roots of an increasing spanning forest of $K_n$, we get a bijection between increasing spanning forests of $K_n$ and increasing spanning trees of $K_{n+1}$. An increasing spanning tree of $K_{n+1}$ has its root 0 of degree $i$ if and only if its internal activity for the colexicographic ordering is $i$, since its active edges are exactly the edges incident to the root. Hence, by classical properties of the Tutte polynomial (see Section 2), the number of increasing spanning trees of $K_{n+1}$ with root of degree $i$ is the coefficient $t_{i,0}$ of the Tutte polynomial of $K_{n+1}$. There are $2^{n-i}$ ways of signing the $n-i$ edges in $K_n$ of an increasing spanning tree of $K_{n+1}$ having its root of degree $i$. Therefore the number

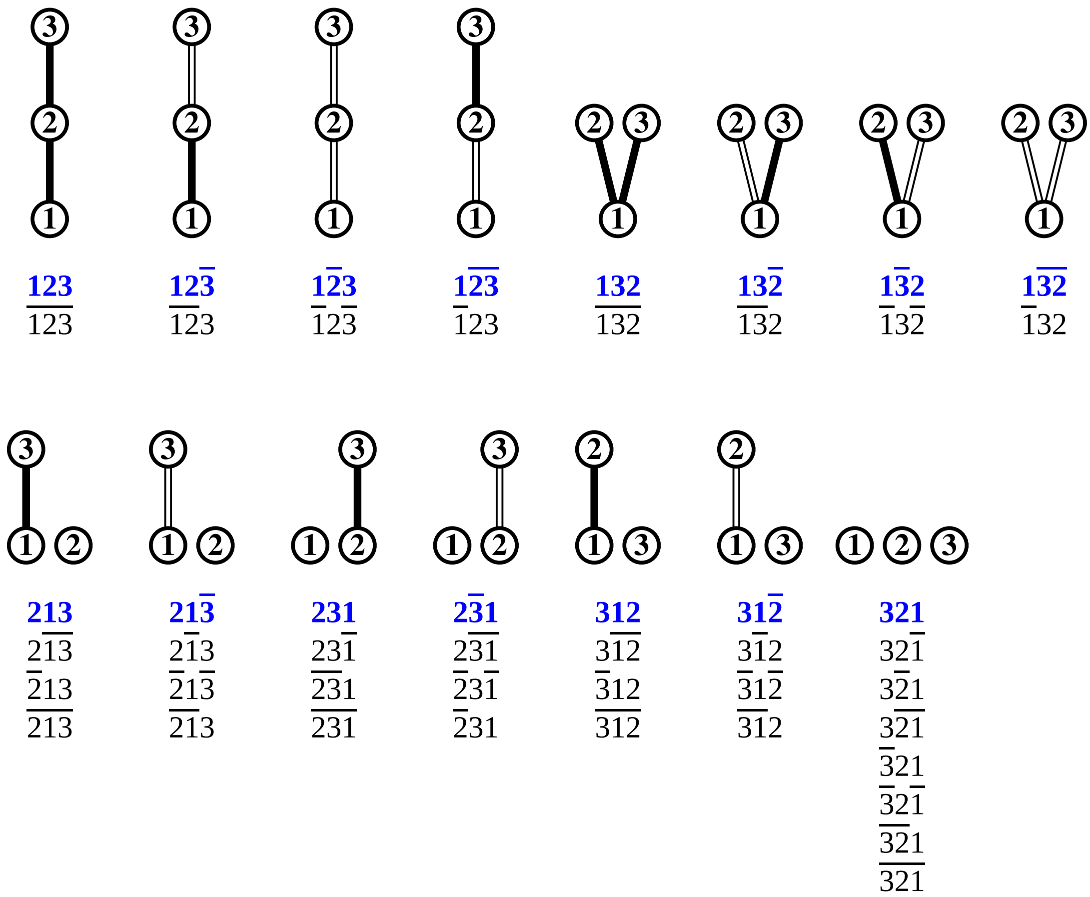


Figure 10. The active mapping $\alpha$ from signed 3-permutations
to edge-signed increasing forests on 3 vertices

of edge-signed increasing spanning forests of $K_n$ is $\sum_{i\geq 1} 2^{n-i} t_{i,0} = 2^n t(K_{n+1}; 1/2, 0)$. Since $|xt(K_{n+1}; x, 0)| = \chi(K_{n+1}; x) = x(x-1)\ldots(x-n)$, where $\chi(K_{n+1})$ denotes the chromatic polynomial of $K_{n+1}$, the number of edge-signed increasing spanning forests of $K_n$ is $2^n \left(\frac{1}{x}\chi(K_{n+1}; x)\right)_{x=\frac{1}{2}} = (2n-1)(2n-3)\ldots 3.1 = (2n-1)!!$. □

The same proof shows that the number of increasing spanning forests of $K_n$ with $q$-colored edges is equal to $q^n \left(\frac{1}{x}\chi(K_{n+1}; x)\right)_{x=1-\frac{1}{q}} = ((n-1)q+1)((n-2)q+1\ldots(q+1)$.

Properties (i) (ii) (iii) of Proposition 6.6 follow from the results of this section, (iv) follows from from [23] Prop. 1.3.16. For the definition of a descent in a permutation, see Section 5.

**Proposition 6.6.** *Let $p = i_1 i_2 \ldots i_n$ be a signed permutation of $12\ldots n$ with active letters $a_1, a_2, \ldots, a_k$.*

*(i) The number of components of the edge-signed increasing forest $T = \alpha(p)$ is equal to the activity $k$ of $p$.*

*(ii) The root of each component of* $T$ *is an active letter of* $p$

*(iii) The vertices of the component of* $T$ *with root* $a_j$, $1 \leq j \leq k$, *are the letters of the interval* $[a_j \ldots a_{j-1}[= p[a_j] \setminus p[a_{j-1}]$ *of* $p$.

*(iv) The leaves of* $T$ *are the descents of* $p$ *and the letter* $i_n$. □

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