THE TUTTE POLYNOMIAL OF A MORPHISM OF MATROIDS 5. COMPUTATIONAL COMPLEXITY

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Abstract. We determine the easy points of the 3-variable Tutte polynomial of a matroid perspective. It turns out that all but one of the sporadic easy points of the 3-variable Tutte polynomial proceed from the 8 sporadic easy points determined in the seminal paper of Jaeger-Vertigan-Welsh on the computational complexity of the Tutte polynomial of a matroid. The exceptional easy point, namely $(-1,-1,-1)$, can be evaluated with polynomial complexity for binary matroid perspectives by a previous result of the author.

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1. INTRODUCTION

The Tutte polynomial of a matroid - introduced by W.T. Tutte in 1954 for graphs - is a self-dual form of the generating function for cardinality and rank of subsets of elements. We have introduced in 1975 the Tutte polynomial of a matroid perspective, as a similar self-dual form of the generating function for cardinality and ranks in two matroids [6]. The Tutte polynomial of a matroid perspective is a 3-variable polynomial with non negative coefficients. For a general pair of matroids the 3-variable Tutte polynomial is a Laurent polynomial in $Z[x, y, z, z^{-1}]$, equivalent to the linking polynomial recently considered by Welsh and Kayibi [11]. Moreoften, the properties of the 3-variable Tutte polynomial of a matroid perspective generalize and unify properties of the usual 2-variable Tutte polynomial of a matroid. For instance, the evaluation $t(M, M'; 0, 0, 1)$ of an oriented matroid perspective $M \longrightarrow M'$ contains as particular cases both the counting of acyclic orientations of a graph and the counting of orientations with unique source and unique sink, and generalizations to hyperplane arrangements and oriented matroids [7].

The seminal paper of Jeager-Vertigan-Welsh on the computational complexity of the Tutte polynomial contains a wide range of results on the general intractability of the evaluation of the Tutte polynomial of a matroid except for a few listed special points and curves [5]. In the present note, we determine the easy points of the Tutte polynomial of a matroid perspective. It turns out that all but one of the sporadic easy points of the 3-variable Tutte polynomial proceed from the easy points of the 2-variable Tutte polynomial. The exceptional easy point, namely $(-1, -1, -1)$, has a polynomial evalution for represented binary matroid perspectives by a previous result of the author [3].

2. THE TUTTE POLYNOMIAL OF A MATROID PERSPECTIVE

Let M, M' be two matroids on a set E. The following properties (i)-(v) are equivalent (see [10] Section 7.3) :
(i) every flat of M' is a flat of M
(ii) every circuit of M is a union of circuits of M'
(iii) for every circuit C of M and cocircuit D' of M' we have $|C \cap D'| \neq 1$
(iv) $r_{M'}(X) - r_{M'}(Y) \leq r_M(X) - r_M(Y)$ for all $Y \subseteq X \subseteq E$
(v) there is a matroid N on a set $E \cup A$ such that $M = N \setminus A$ and $M' = N/A$

We write $M \longrightarrow M'$ when these equivalent properties hold, and say that $M \longrightarrow M'$ constitutes a *matroid perspective*. A matroid perspective is the particular case of a strong map of matroids when both matroids are on a same set. Note that no significant generality is lost, since it can easily be shown that any strong map is reducible to a perspective up to a bijection and adding loops and parallel elements. The matroid M' is often called a *quotient* of M in the literature [10]. Standard examples of matroid perspectives are obtained by identification of vertices in graphs or by embeddings of graphs in surfaces, and more generally from linear maps between vector spaces.

A matroid N as in (v) is called a *major* of $M \longrightarrow M'$. A matroid perspective is said to be*graphic* resp. *binary* if it has a graphic resp. binary major. Let M be a matroid on a set E. We denote by $\mathbf{0}$ the rank zero matroid on E, and by $\mathbf{1}$ the free matroid of rank $\mid E \mid$ on E. Then $M \longrightarrow M$, $M \longrightarrow \mathbf{0}$ and $\mathbf{1} \longrightarrow M$ are matroid perspectives on E. As is easily seen, if M is graphic, these matroid perspectives are also graphic. We will use them in Section 3.

The Tutte polynomial of a matroid perspective defined in [6][8][9] is a variant of the rank generating function of two matroids. We have

$$t(M, M'; x, y, z) = \sum_{A \subseteq E} (x-1)^{r(M') - r_{M'}(A)} (y-1)^{|A| - r_M(A)} z^{r(M) - r(M') - (r_M(A) - r_{M'}(A))}$$

In general, the function $t(M,M')$ is a Laurent polynomial in $Z[x,y,z,z^{-1}]$, and its coefficients may be positive or negative integers. In the case of a matroid perspective $M \longrightarrow M'$, the function $t(M,M')$ is a polynomial in x,y,z with non negative integer coefficients (proof by deletion/contraction), and many fundamental properties of the usual Tutte polynomial generalize [6][8][9].

Another variant of the rank generating function of two matroids, a 4-variable polynomial called the *linking polynomial*, has been recently considered by D.J.A. Welsh and K.K. Kayibi [11]. The linking polynomial is equivalent to the Tutte polynomial of two matroids [11] (see Property (14) p. 394) [12]. By this equivalence, properties can be stated in terms of either polynomials. On an expression of the Tutte polynomial of a strong map in terms of activities, Th. 6.1 in [8] and Th. 8.1 in [9], see the acknowledgement [12].

When $M \longrightarrow M'$, the polynomial $t(M,M')$ can also be considered as the Tutte polynomial of a matroid pointed by a subset of elements [6][8][9]. With notation of (v) in the equivalences defining a matroid perspective, we have $t(M,M';x,y,z) = t(N;A;x,y,z)$ (see details in [9] section 3). When $r(M) = r(M')+1$, or, equivalently, when A is reduced to one element, say $A = \{e\}$, the polynomial $t(M,M') = t(N;\{e\})$ is equivalent to the 4-variable polynomial introduced by T. Brylawski in [1]. Generalizations of certain results of Brylawski to Tutte polynomials of set-pointed matroids are studied in [2].

3. EASY POINTS

Two results of the literature will be used in the proof of Theorem 1.

Theorem A (F. Jaeger, D. Vertigan, D.J.A. Welsh [5])

The problem of evaluating the Tutte polynomial of a graph at a point in the (x,y)-plane is #P-hard except when $(x-1)(y-1) = 1$ or when (x,y) equals $(1,1)$ $(-1,-1)$ $(0,-1)$ $(-1,0)$ $(i,-i)$ $(-i,i)$ (j,j^2) (j^2,j) *where* $j = e^{2\pi i}/3$.

We refer the reader to [5] for the interpretation of the special points in Theorem A (see also [4] for (j,j^2) and (j^2,j)).

Theorem B (G. Etienne, M. Las Vergnas [3] Th. 6.2)

Let $M \longrightarrow M'$ be a binary matroid perspective, i.e. such that $M = S(V)$ and $M' = S(V')$ for binary subspaces $V \subseteq V' \subseteq GF(2)^E$, where $S(V)$ denotes the matroid on E whose circuits are the inclusion-minimal supports of non zero vectors of V. We have

$$t(M,M';-1,-1,-1) = \begin{cases} 0 & \text{if } 1_E \notin V + V'^{\perp} \\ (-1)^{|E|-\dim(V\cap V^{\perp})}2^{\dim(V\cap V'^{\perp})} & \text{if } 1_E \in V + V'^{\perp} \end{cases}$$

Extending definitions of Jaeger-Vertigan-Welsh, we say that a point (a,b,c) of the complex 3-space is an *easy point* of the 3-variable Tutte polynomial of a matroid perspective if there is a polynomial algorithm to evaluate $t(M,M';a,b,c)$ on graphic matroid perspectives $M \longrightarrow M'$.

Theorem 1

The easy points of the 3-variable Tutte polynomial of a matroid perspective are

(i) *all points of the curve* $(t+1, 1/t+1, t)$

(ii) *15 points obtained from the 8 sporadic easy points of the 2-variable Tutte polynomial of a matroid, namely for each (a,b) in the list of Jaeger-Vertigan-Welsh the points$(a,b,a-1)$ and $(a,b,1/(b-1))$ if $b \neq 1$*

(iii) $(-1,-1,-1)$.

Proof

Let M be a graphic matroid. Then $M \longrightarrow \mathbf{0}$ and $\mathbf{1} \longrightarrow M$ are graphic matroid perspectives. By straightforward substitutions we have $t(M,\mathbf{0};x,y,z) = t(M;z+1,y)$ and $t(\mathbf{1},M;x,y,z) = z^{|M|-r(M)}t(M;x,1/z+1)$

([9] (5.4) (5.5)). It follows that if (a, b, c) is an easy point of the 3-variable Tutte polynomial then $(c+1, b)$ and $(a, 1/c+1)$ are easy points of the 2-variable Tutte polynomial. By Theorem A, we have $c(b-1) = 1$ or $(c+1, b) \in \mathcal{L} = \{(1,1), (-1,-1), (0,-1), (-1,0), (i,-i), (-i,i), (j,j^2), (j^2,j)\}$, and $(a-1)/c = 1$ or $(a, 1/c+1) \in \mathcal{L}$. Therefore either $c(b-1) = 1$ and $(a-1)/c = 1$ - case (i),or $c(b-1) = 1$ and $(a, 1/c+1) \in \mathcal{L}$, equivalently $(a,b) \in \mathcal{L}$ and $c = 1/(b-1)$,or $(a-1)/c = 1$ and $(c+1, b) \in \mathcal{L}$, equivalently $(a,b) \in \mathcal{L}$ and $c = a-1$ - case (ii),or $(a, 1/c+1) \in \mathcal{L}$ and $(c+1, b) \in \mathcal{L}$ - case (iii). If $(a, 1/c+1) \in \mathcal{L}$ then $c \in \{-1/2, -1, -1/2+i/2, -1/2-i/2, -1/3+j/3, -1/3+j^2/3\}$and if $(c+1, b) \in \mathcal{L}$ then $c \in \{0, -2, -1, -1+i, -1-i, -1+j, -1+j^2\}$. The intersection of the two lists is $c = -1$, and then $a = b = -1$.

We prove that conversely each point in (i)-(ii)-(iii) is easy. Let $M \to M'$ be a matroid perspective on a set E.

(i) We have $t(M, M'; t+1, 1/t+1, t) = t^{r(M)-r(M')} \sum_{A \subseteq E} t^{-|A|} = t^{r(M)-r(M')} t^{-|E|}$, hence $(t+1, 1/t+1, t)$ is easy for any matroid perspective.

(ii) By straightforward substitutions in the formula defining $t(M, M')$ we have $t(M, M'; x, y, x-1) = t(M; x, y)$ and $t(M, M'; x, y, 1/(y-1) = (y-1)^{-(r(M)-r(M'))} t(M'; x, y)$. It follows that the 15 points of case (ii) amount to easy points of 2-variable Tutte polynomials, hence are easy by Theorem A.

(iii) With notation of Theorem B, if V and V' are defined by bases, all necessary computations to evaluate $t(M, M'; -1, -1, -1)$ can be made by means of polynomial algorithms. It follows that $(-1, -1, -1)$ can be polynomially evaluated for binary matroid perspectives with a succint presentation in the sense of [5], hence is an easy point for the 3-variable Tutte polynomial of a matroid perspective.□

An alternate proof of Theorem 1 is obtained by using the perspective $M \longrightarrow M$ in place of $M \longrightarrow \mathbf{0}$ resp. $\mathbf{1} \longrightarrow M$.

When $(a,b) = (1,1)$, the point $(a, b, 1/(b-1)$ is not in the complex 3-space. However since $(y-1)^{r(M)-r(M')} t(M, M'; x, y, 1/(y-1)) = t(M'; x, y)$, we may consider that the limit evaluationat $(1, 1, \infty)$ is also an easy point, dual to the evaluation at $(1, 1, 0)$. This limit is equal to the evaluation at $(1,1)$ of the 2-variable polynomial coefficient of $z^{r(M)-r(M')}$ in $t(M, M'; x, y, z)$. With this convention there are 16 easy points in case (ii).

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