# Activity Preserving Bijections between Spanning Trees and Orientations in Graphs

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RESUME Les principaux résultats de ce papier sont deux algorithmes duaux qui définissent une bijection entre l'ensemble des arbres couvrants d'activité interne 1 et d'activité externe 0 d'un graphe ordonné et l'ensemble de ses orientations acycliques avec un unique puits et une unique source adjacents. Plus généralement ces algorithmes s'étendent à une correspondance préservant les activités entre arbres couvrants et orientations. Pour certains ordres totaux de l'ensemble des arêtes, ils fournissent également une bijection entre arbres couvrants d'activité externe 0 et orientations acycliques ayant un unique puits donné.

ABSTRACT The main results of the paper are two dual algorithms which map in a 1-1 way the set of spanning trees with internal activity 1 and external activity 0 of an ordered graph onto the set of acyclic orientations with adjacent unique source and sink. More generally, these algorithms extend to an activity preserving correspondence between spanning trees and orientations. For certain linear orderings of the edges, they also provide a bijection between spanning trees with external activity 0 and acyclic orientations with a given unique sink.

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## 1. Introduction

The Tutte polynomial $t(G;x,y)$ of a graph $G$ is a two variable polynomial containing as specializations several fundamental numerical invariants of $G$ such as the numbers of spanning trees, $q$-colorings, acyclic orientations, etc. We refer the reader to [1] for a comprehensive survey of properties and applications of Tutte polynomials of graphs, and, more generally, matroids.

Suppose the edge-set of $G$ is linearly ordered. W.T. Tutte has shown that

$$t(G;x,y) = \sum_{i,j} t_{i,j} x^i y^j$$

where $t_{i,j}$ is the number of spanning trees of $G$ such that $i$ edges are smallest in their fundamental cocycle and $j$ edges are smallest in their fundamental cycle [14]. On the other hand, M. Las Vergnas has shown that

$$t(G;x,y) = \sum_{i,j} o_{i,j} 2^{-i-j} x^i y^j$$

where $o_{i,j}$ is the number of orientations of $G$ such that $i$ edges are smallest in some directed cocycle and $j$ edges are smallest in some directed cycle [11]. This last formula generalizes a well-known result of R. Stanley : the number of acyclic orientations of $G$ is equal to $t(G;2,0)$ [13].

Comparing these two expressions for $t(G;x,y)$ we get $o_{i,j} = 2^{i+j} t_{i,j}$ for all $i,j$. A natural question arises of a bijective proof for this formula [11]. The problem is to define a correspondence between spanning trees and orientations, preserving parameters $(i,j)$, called *activities* in the literature, and compatible with the above formula. More precisely, the desired correspondence should associate with a $(i,j)$-*active* spanning tree of $G$, a set of $2^{i+j}$ $(i,j)$-active orientations of $G$, in such a way that each orientation of $G$ is the image of a unique spanning tree. The main object of the present paper is to describe such a correspondence.

Spanning trees and orientations with $(1,0)$ activities (or, dually, $(0,1)$ activities) constitute the main case of our construction. Several papers of the literature deal with $(1,0)$-orientations of graphs, i.e. acyclic orientations with adjacent unique source and sink. Enumeration problems on $(1,0)$-orientations are considered by C. Greene and T. Zaslavsky in [10] in the contexts of graphs and hyperplane arrangements. In particular, they prove that the number of acyclic orientations of a graph with adjacent unique source and sink is $2\beta(G)$, or, equivalently, we have $o_{1,0} = 2t_{1,0}$ (implying that this number does not depend of the particular source and sink). In [6] bijective proofs are given to a result of [10] on acyclic orientations with unique sink (see below, and Section 6). Orientations with $(1,0)$ activities are studied in [5] for their relevance in several graph algorithms. On the other hand, the external activity of a spanning tree has recently retained some attention in relation with the chip-firing game and the sandpile model [3] (see also [2] for the particular case of $K_n$ and parking functions).

Section 3 contains the main results : two dual algorithms establish a bijection between spanning trees and orientations with $(1,0)$ activities. In Section 4, we obtain as a corollary, a bijection for $(0,1)$ activities. In Section 5, these bijections are extended to a correspondence between spanning trees and orientations consistent with the formula $o_{i,j} = 2^{i+j}t_{i,j}$, thus answering the above query. We point out that this correspondence not only preserves activities but also active elements. The construction uses reductions from general activities to the $(1,0)$ case. Finally, in Section 6, we show that the correspondence of Section 5 produces a bijection between internal spanning trees and acyclic orientations with a unique sink at a given vertex.

A bijection between acyclic orientations with a unique sink at a given vertex and internal spanning trees has recently appeared in the litterature [6]. The present one is different, as it preserves activies, whereas the bijection of [6] does not (see Section 6). The bijection of Section 3 constitutes an answer to a question of [6] (see (a) p.145). It should be mentioned that one of the present authors has already published an activity preserving correspondence between spanning trees and orientations in graphs [12], an extended abstract apparently overlooked in [6]. However, the correspondence in [12] is different from the present one (see Section 4). It has not been - and maybe cannot be - generalized beyond regular matroids, whereas the correspondence of this paper generalizes to oriented matroids [7][8][9], and thus, is probably more natural. We point out that the graphical case deserved the present specific treatment, as stronger orthogonality properties permit significative simplifications, and also for properties involving vertices (Section 6).

## 2. Notation and terminology

The present paper deals exclusively with graphs. We point out that all definitions and results of this section, except Minty's Lemma, have extensions to the wider context of matroid and oriented matroid theories. Throughout the paper, we will implicitly assume that graphs under consideration are connected, and that cycles and cocycles are elementary.

Let $G$ be a graph with edge-set $E$, and $T \subseteq E$ be a spanning tree of $G$. For $e \in E \setminus T$, we denote by $C(T;e)$ the *fundamental cycle* of $e$ with respect to $T$, i.e. the unique cycle contained in $T \cup \{e\}$, obtained from the unique path of $T$ joining the two vertices of $e$. For $e \in T$, we denote by $C^*(T;e)$ the *fundamental cocycle* of $e$ with respect to $T$, i.e. the unique cocycle contained in $(E \setminus T) \cup \{e\}$. The cocycle $C^*(T;e)$ is the set of edges of $G$ joining the two connected components of $T \setminus \{e\}$. For $e \in E \setminus T$ and $f \in T$, we have clearly $f \in C(T;e)$ if and only if $e \in C^*(T;f)$, and then $C(T;e) \cap C^*(T;f) = \{e, f\}$.

We say that a graph $G$ is *ordered* if its edge-set $E$ is linearly ordered. The notion of *activities* of a spanning trees $T$ in an ordered graph $G$ is due to W.T. Tutte [14]. The *internal activity* $\iota(T)$ is the number of edges $e \in T$ smallest in their fundamental cocycle $C^*(T;e)$, and the *external activity* $\varepsilon(T)$ is the number of edges $e \in E \setminus T$ smallest in their fundamental cycle $C(T;e)$. We denote by $t_{i,j}(G)$ the number of spanning trees of $G$ such that $\iota(T) = i$ and $\varepsilon(T) = j$.

Spanning tree activities have been introduced by Tutte to generalize, in a self-dual way, the chromatic polynomial of a graph. The *dichromate*, now called the *Tutte polynomial*, has been originally defined as

$$t(G;x,y) = \sum_{i,j} t_{i,j} x^i y^j$$

The main point in [14] is to prove that the coefficients $t_{i,j}$ are independent from the ordering of $E$. Nowadays, the simplest definition of the Tutte polynomial of a graph (or, more generally, a matroid) is by means of a closed formula in terms of subsets of edges, and the above formula is a theorem proved by deletion/contraction of the greatest edge (see [1]).

A cycle resp. cocycle in a directed graph is *directed* if all its edges are directed consistently. The *(primal) orientation activity* of an ordered directed graph $G$, or *$O$-activity*, denoted by $o(G)$, is the number of edges smallest in some directed cycle. The *dual orientation activity* of $G$, or *$O^*$-activity*, denoted by $o^*(G)$, is the number of edges smallest in some directed cocycle. We denote by $o_{i,j}(G)$ the number of orientations $\overrightarrow{G}$ of $G$ such that $o^*(\overrightarrow{G}) = i$ and $o(\overrightarrow{G}) = j$. The definitions of $O$- and $O^*$-activities have been introduced in [11] in view of the formula

$$t(G; x, y) = \sum_{i,j} o_{i,j} 2^{-i-j} x^i y^j$$

This formula implies that $o_{i,j}$ does not depend on the ordering, and that $o_{i,j} = 2^{i+j} t_{i,j}$. The proof in [11] is by deletion/contraction of the greatest edge.

Internal and external activities of spanning trees, and also the two types of orientation activities, are dual notions from the point of view of graph duality. If $G$ is a planar graph imbedded in the plane, and $G^*$ is a dual of $G$, we have $\varepsilon_{G^*}(T) = \iota_G(E \setminus T)$. If $G$ is directed, a *directed dual* of $G$ is a planar dual $G^*$ directed such that all directions of corresponding edges in $G$ and $G^*$ define rotations of the same type, clockwise or counterclockwise. Then, we have $o^*(G) = o(G^*)$. The graph $G$ is said *acyclic* if there is no directed cycle, i.e. if $o(G) = 0$, and, dually, is said *totally cyclic* (or *strongly connected*) if $o^*(G) = 0$.

In a directed graph, the two possible *cycle directions* along an elementary cycle $C$ can be distinguished by defining $C^+$ resp. $C^-$ as the set of edges of $C$ directed consistently resp. in the opposite direction. An elementary cocycle $D$ is the set of edges joining two subsets partitioning the vertex-set of $G$ into two connected subgraphs. There are two *cocycle directions* defined by an ordering of these two subsets. The two possible cocycle directions can be distinguished by defining $D^+$ resp. $D^-$ as the set of edges of $D$ directed from the first subset to the second resp. from the second to the first. In a directed graph, the notation $C(T; e)$ for $e \in E \setminus T$ resp. $C^*(T; e)$ for $e \in T$ can be precised by choosing the cycle direction resp. cocycle direction consistent with the direction of $e$, i.e. such that $e$ is in the positive part.

We make a crucial use in the proof of Theorem 3.4 (9) of the (directed) *graphical orthogonality property*

$$|C^+ \cap D^+| + |C^- \cap D^-| = |C^- \cap D^+| + |C^+ \cap D^-|$$

between a cycle $C$ and a cocycle $D$. In all other places, the weaker (directed) *orthogonality property* $C \cap D \neq \emptyset$ implies $(C^+ \cap D^+) \cup (C^- \cap D^-) \neq \emptyset$ and $(C^- \cap D^+) \cup (C^+ \cap D^-) \neq \emptyset$ suffices for our purpose. We mention that the graphical orthogonality property characterizes regular matroids (Minty 1975), whereas the orthogonality property characterizes oriented matroids (Bland-Las Vergnas 1978).

## 3. The bijection for $(1,0)$-activities

We recall that $t_{1,0}(G) \neq 0$ if and only if the graph $G$ is 2-connected and has no loop [1].

**Proposition 3.1**

*Let $G$ be an ordered directed graph, with smallest edge $e_1 = s's''$ directed from $s'$ to $s''$. Then $o^*(G) = 1$ and $o(G) = 0$ if and only if $G$ is acyclic, with unique source $s'$ and unique sink $s''$.* □

Proof

A directed graph has external activity 0 if and only if it is acyclic by definition. In an acyclic graph, $e_1$ belongs to a cocycle, so it is the smallest element of a cocycle. An acyclic graph has a source (otherwise one could construct easily a directed cycle). The set of edges having this source as an extremity is then a directed cocycle.

If the graph has dual activity 1 then this source must be an extremity of $e_1$ (because $e_1$ is the only possible minimal element of a cocycle). The same properties holding for the opposite orientation, the graph has a sink and any sink must be an extremity of $e_1$. This proves that the graph has unique source $s'$ and unique sink $s''$.

Conversely, suppose $G$ has a unique source $s'$ and a unique sink $s''$. The two connected subgraphs induced by the partition of $V$ defined by a cocycle are also acyclic. Hence, so they must have a source and a sink. If the cocycle is directed, there exist a source of $G$ in one component and a sink of $G$ in the other. Necessarily these two vertices are $s'$ and $s$", and so $e_1$ belong to the directed cocycle. □

**Proposition 3.2**

*Let $G$ be an ordered graph, and $T$ be a spanning tree of $G$. Set $T = \{b_1 < b_2 < \ldots < b_r\}$ and $E \setminus T = \{a_1 < a_2 < \ldots < a_{n-r}\}$.*

*(i) $\varepsilon(T) = 0$ if and only if $b_j = \mathrm{Min}\ (E \setminus \bigcup_{1\leq i<j} C^*(T;b_i))$ for $j = 1, 2, \ldots, r$.*

*(ii) $\iota(T) = 1$ if and only if $a_j = \mathrm{Min}\ ((E \setminus \{e_1\}) \setminus \bigcup_{1\leq i<j} C(T;a_i))$ for $j = 1, 2, \ldots, n-r$.*

Proof

(i) Let $e = \text{Min } (E \setminus \bigcup_{1\leq i<j} C^*(T;b_i))$, and suppose $e < b_j$. We have $e \notin T$, since $e \notin \{b_1, \ldots, b_{j-1}\}$ by definition. Set $C = C(T;e)$. If $b_i \in C$, we have $e \in C^*(T;b_i)$, therefore $C \cap \{b_1, \ldots, b_{j-1}\} = \emptyset$. It follows that $C \cap T \subseteq \{b_j, \ldots, b_r\}$, then $e = \text{Min } C$, hence $\varepsilon(T) > 0$.

Conversely, suppose $b_j = \text{Min } (E \setminus \bigcup_{1\leq i<j} C^*(T;b_i))$ for $j = 1, 2, \ldots, r$. Let $e \in E \setminus T$. Set $C = C(T;e)$, and let $b_j = \text{Min } C \cap T$. We have $e \notin \bigcup_{1\leq i<j} C^*(T;b_i))$, otherwise $b_i \in C$ for some $i < j$. Hence $b_j < e$, and $e$ is not externally active.

(ii) Let $e = \text{Min } ((E \setminus \{e_1\}) \setminus \bigcup_{1\leq i<j} C(T;a_i))$, and suppose $e < a_j$. We have $e \in T$, since $e \notin \{a_1, \ldots, a_{j-1}\}$ by definition. Set $D = C^*(T;e)$. If $a_i \in D$, we have $e \in C(T;a_i)$, therefore $D \cap \{a_1, \ldots, a_{j-1}\} = \emptyset$. It follows that $D \cap (E \setminus T) \subseteq \{a_j, \ldots, a_{n-r}\}$, then $e = \text{Min } D$, hence $\iota(T) > 1$.

Conversely, suppose $a_j = \text{Min } ((E \setminus \{e_1\}) \setminus \bigcup_{1\leq i<j} C(T;a_i))$ for $j = 1, 2, \ldots, n-r$. Let $e \in T \setminus \{e_1\}$. Set $D = C^*(T;e)$, and let $a_j = \text{Min } D \setminus T$. We have $e \notin \bigcup_{1\leq i<j} C(T;a_i))$, otherwise $a_i \in D$ for some $i < j$. Hence $a_j < e$, and $e$ is not internally active. $\square$

**Proposition-Definition 3.3**

*Let $G$ be an ordered graph, with edge-set $E = \{e_1 = s's'' < e_2 < \ldots < e_n\}$, and $T$ be a spanning tree of $G$ with internal activity 1 and external activity 0. The following two algorithms produce the same acyclic orientation of $G$, with unique source $s'$ and unique sink $s''$.*

*Step 0 (in both algorithms) : direct the smallest edge $e_1$ from $s'$ to $s''$.*

*(i) Algorithm 1*

*Let $E \setminus T = \{a_1 = e_2 < a_2 < \ldots < a_{n-r}\}$.*
*Step $i = 1, 2, \ldots, n-r$ : direct the undirected edges of $C(T;a_i)$ in the cycle direction opposite to the direction of its smallest edge.*

*(ii) Algorithm 2.*

*Let $T = \{b_1 = e_1 < b_2 < \ldots < b_r\}$.*
*Step 1 : direct all edges $\neq e_1$ of $C^*(T;b_1)$ in the cocycle direction defined by $e_1$*
*Step $i = 2, \ldots, r$ : direct the undirected edges of $C^*(T;b_i)$ in the cocycle direction opposite to the direction of its smallest edge.*

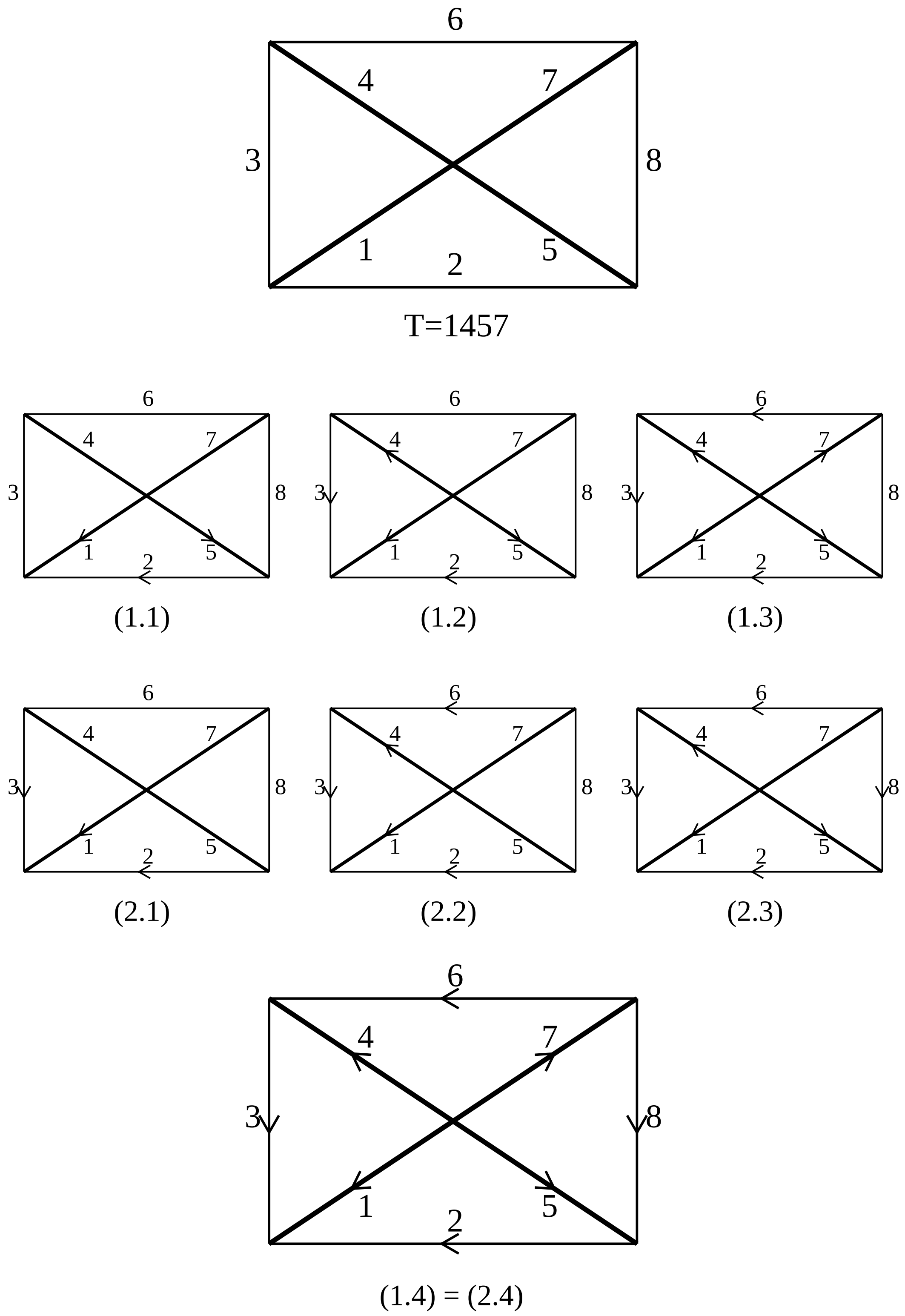


Figure 1

An example for Algorithms 1 and 2 applied to the 4-wheel $W_4$ is given by Figure 1.

Proof

Since $G$ has a spanning tree $T$ with $(1,0)$ activities, it has no isthmus or loop.

(1) *Algorithm 1 directs all edges of* $G$, *and* $(1')$ *Algorithm 2 directs all edges of* $G$

We show inductively that all edges $a_i$ $i = 1, 2, \ldots, n-r$ are directed by Algorithm 1. We have to check that after Step $i-1$ the edge $b = \mathrm{Min}\ C(T; a_i)$ is directed. This is clear for $i = 1$ since then $b = e_1$, so suppose $i \geq 2$. We have $b \in T$, otherwise $b = a_i$ would be externally active. If $b = e_1$, then $a_i$ is directed at Step $i$ of Algorithm 1. If $b \neq e_1$, then $b$ is not the smallest element of its fundamental cocycle since $\iota(T) = 1$. Set $a_j = \mathrm{Min}\ C^*(T; b)$. We have $a_j < b < a_i$, hence $a_j$ is directed after Step $i-1$ by induction. Since $b \in C(T; a_j)$, the edge $b$ has been directed by Algorithm 1 at a Step $\leq j < i$, hence $a_i$ is directed at Step $i$. On the other hand, since $G$ has no isthmus, we have $\bigcup_i (C(T; a_i) = E$, hence all edges of $T$ are directed by Algorithm 1.

The proof of $(1')$ is dual.

(2) *Algorithm 1 and Algorithm 2 produce the same orientation of* $G$

The proof is by induction on the rank in the ordering. Let $a \in E \setminus T$, and set $b = \mathrm{Min}\ C(T; a)$, $a' = \mathrm{Min}\ C^*(T; b)$. We have $b \in T$, otherwise $b = a$ is externally active, contradicting $\varepsilon(T) = 0$. If $a' \in T$, the edge $b = a'$ is internally active, hence $b = e_1$ since $\iota(T) = 1$. In this case $a$ and $e_1$ have opposite directions in $C(T; a)$ for Algorithm 1. We have $a \in C^*(T; e_1)$, hence $a$ and $e_1$ have the same direction in $C^*(T; e_1)$ by Step 1 of Algorithm 2. These directions agree by orthogonality. Otherwise, we have $b \neq e_1$ and $a' \in E \setminus T$. We have $a \in C^*(T; b)$ and $a' < b < a$. By Algorithm 1, the edges $b$ and $a$ have opposite directions in $C(T; a)$. We have $C(T; a_i) \cap C^*(T; b) = \{a, b\}$, hence by orthogonality $a$ and $b$ have the same direction in $C^*(T; b)$. As $b$ is the smallest edge in $T$ such that $a \in C^*(T; b)$, it follows that $a$ is undirected when $b$ is directed by Algorithm 2. Therefore $a$ and $b$ have same direction in $C^*(T; b)$ for Algorithm 2, opposite to the direction of $a'$. Since by induction, the directions of $b$ agree in Algorithms 1 and 2, the same conclusion holds for $a$.

The proof for $b \in T$ is similar and left to the reader.

Let $\overrightarrow{G}$ be the orientation of $G$ constructed by Algorithms 1 and 2.

(3) $o^*(\overrightarrow{G}) = 1$ *and* $(3')$ $o(\overrightarrow{G}) = 0$

Suppose there is a directed cocycle $D$ in $\overrightarrow{G}$ with $\mathrm{Min}\ D \neq e_1$, contradicting

$(3)$. Since $G$ has no isthmus, we have $\bigcup_i C(T;a_i) = E$. Let $i$ be the smallest integer such that $D \cap C(T;a_i) \neq \emptyset$. Let $b \in D \cap C(T;a_i) \setminus \{a_i\}$. Since $b \in C(T;a_i) \setminus \{a_i\}$, we have $b \in T$. By the choice of $i$, the edge $b$ is directed at step $i$ of Algorithm 1. Set $e = \text{Min } C(T;a_i)$. We have $e \neq a_i$ otherwise $a_i$ would be externally active, contradicting $\varepsilon(T) = 0$. If $i = 1$, we have $a_i = e_2$ and $e = b = e_1$, contradicting our assumption. Hence $i \geq 2$. By (1), the edge $e$ is directed after Step $i-1$ of Algorithm 1 and since $b$ is not we have $e \neq b$. Hence, by definition of Algorithm 1, both $b$ and $a_i$ are directed in the same direction of $C(T;a_i)$, opposite to the direction of $e$. It follows that all edges in $D \cap C(T;a_i)$ have the same direction in both $D$ and $C(T;a_i)$, contradicting orthogonality.

Suppose there is a directed cycle $C$ in $\vec{G}$, contradicting $(3')$. Since $G$ has no loop, we have $\bigcup_i C^*(T;b_i) = E$. Let $i$ be the smallest integer such that $C \cap C^*(T;b_i) \neq \emptyset$. Let $a \in C \cap C^*(T;b_i) \setminus \{b_i\}$. By the choice of $i$, the edge $a$ is directed at step $i$ of Algorithm 2. If $i = 1$, i.e. $b_1 = e_1$, then $a$ and $b_i$ have the same direction in $C^*(T;b_i)$ by definition of Step 1 of Algorithm 2. Suppose $i \leq 2$. Set $e = \text{Min } C^*(T;b_i)$. By $(1')$, the edge $e$ is directed after Step $i-1$ of Algorithm 2 and since $a$ is not, we have $e \neq a$. On the other hand, $e \neq b_i$ otherwise $b_i$ would be internally active, implying $i = 1$ since $\iota(T) = 1$. Hence, by definition of Step $i \geq 2$ in Algorithm 2, both $a$ and $b_i$ are directed in the same direction of $C^*(T;b_i)$, opposite to the direction of $e$. It follows that all edges in $C \cap C^*(T;b_i)$ have the same direction in both $C$ and $C^*(T;b_i)$, contradicting orthogonality. □

**Theorem 3.4**

*Let $G$ be an ordered graph. The application defined by Algorithms 1 and 2 is a bijection from the set of spanning trees of $G$ with $(1,0)$ activities onto the set of orientations of $G$ with $(1,0)$ activities such that the direction of the first edge is fixed.*

Proof

Since $2t_{1,0} = o_{1,0}$ by [10], it suffices to show that the application is injective. Suppose there exist two different spanning trees $T = \{b_1 < b_2 < ... < b_r\}$ and $T' = \{b'_1 < ... < b'r\}$ with $(1,0)$ activities such that Algorithms 1 and 2 produce the same directed graph.

(1) Let $k$ be the smallest integer such that $C^*(T;b_k) \neq C^*(T';b'_k)$. By Proposition 3.2, we have $b_i = b'_i$ for all $i \leq k$. Set $b = b_k = b'_k$, $D = C^*(T;b)$ and $D' = C^*(T';b)$. We have $b \in D^+ \cap D'^+$.

(2) $T \cap D' \subseteq \{b = b_k, \ldots, b_r\}$, *and* (1′) $T' \cap D \subseteq \{b = b'_k, \ldots, b'_r\}$

If $i < k$, by (1) we have $b_i = b'_i \notin C^*(T'; b'_k) = D'$.

(3) $T \cap D' \subseteq D'^+$, *and* (3′) $T' \cap D \subseteq D^+$

Let $b_i \in T \cap D'$. By (2), we have $i \geq k$. If $i = k$, then $b_i = b_k = b'_k = b \in D'^+$. Suppose $i > k$. Since $b_i \in D' = C^*(T'; b'_k)$, the edge $b_i$ is directed at a step $j \leq k$ of Algorithm 2 applied to $T'$. If $j < k$, we have $b'_j = b_j \in T$, hence $b_i \notin C^*(T; b_j) = C^*(T'; b'_j)$, so that $b_i$ cannot be directed at Step $j$.

Therefore $j = k$. If $k > 1$, the edges $b = b'_k$ and $b_i$ are directed by Algorithm 2 in the same cocycle direction of $D'$ (opposite to the direction of the smallest edge of $D'$), hence $b_i \in D'^+$. If $k = 1$, then, by definition of Step 1 in Algorithm 2, we have $D' = D'^+$.

(4) $|T \cap D'| \geq 2$ *and* (4′) $|T' \cap D| \geq 2$

Since $T$ is a spanning tree and $D'$ a cocycle, we have $|T \cap D'| \geq 1$. If $|T \cap D'| = 1$, then $D'$ is a fundamental cocycle of $T$, and necessarily, since $b = b_k \in T$, we have $D' = C^*(T; b) = D$, contradicting the definition of $k$. Therefore $|T \cap D'| \geq 2$.

(5) Let $a$ be the smallest element of the set

$$\bigcup_{e \in (T \cap D') \setminus \{b\}} C^*(T; e) \cup \bigcup_{e \in (T' \cap D) \setminus \{b\}} C^*(T'; e)$$

which is not empty by (4). By symmetry, we may suppose that $a = \text{Min } C^*(T; e)$ for some $e \in (T \cap D') \setminus \{b\}$. We have $e = b_\ell$ for some $\ell > k$ by (2). In particular $\ell > 1$.

(6) $a \notin T$

If $a \in T$, then $a = e$ and $a = \text{Min } C^*(T; a)$ is internally active. Hence $a = e_1 = b_1$, contradicting $\ell > 1$ (5).

Set $C = C(T; a)$.

(7) $a \notin T'$

Suppose $a \in T'$.

We have $a > b$ by (6). If $a \in D$, we have $a \in (T' \cap D) \setminus \{b\}$, hence $a \leq \text{Min } C^*(T'; a)$ by (5). Therefore $a$ is internally active, hence $a = e_1$, contradicting (6). So $a \notin D$. Since $a > b$, we have also $a \notin D'$.

Let $x \in C \cap D'$. We have $x \neq b$ since $a \notin D$, and $x \neq a$ since $a \notin D'$.

Therefore, $x \in (C \setminus \{a\}) \cap D') \setminus \{b\} \subseteq (T \cap D') \setminus \{b\}$. Hence $a \leq \text{Min } (C^*(T;x)$, and in fact $a = \text{Min } (C^*(T;x)$ since $x \in C = C(T;a)$ implies $a \in C^*(T;x)$. By Algorithm 2 applied to $T$, the edge $x$ is directed in the cocycle direction opposite to $a$ in the cocycle $C^*(T;x)$, hence by orthogonality $a$ and $x$ have the same cycle direction on $C$, i.e. $x \in C^+$. On the other hand, we have $x \in D' = C^*(T';b_k)$ and $x \notin C^*(T';b'_i) = C^*(T;b_i)$ for $i < k$, since $x$ in $T$. Hence, the edge $x$ is directed at Step $k$ of Algorithm 2 applied to $T'$. Since $x > b_k = b$, the edges $b$ and $x$ have the same cocycle direction in $D'$, i.e. $x \in D'^+$. It follows that $C \cap D' \subseteq C^+ \cap D'^+$.

By (5), we have $a \in C^*(T;e)$, hence $e \in C(T;a) = C$, and also $e \in D'$. We have $e \in C \cap D'$ and $C \cap D' \subseteq C^+ \cap D'^+$, contradicting the orthogonality property.

Set $C' = C(T';a)$. We have $a \in C^+ \cap C'^+$.

(8) $(C \cap D') \setminus \{a,b\} \subseteq C^+ \cap D'^+$ *and* (8′) $(C' \cap D) \setminus \{a,b\} \subseteq C'^+ \cap D^+$

We have $C \setminus \{a\} \subseteq T$, hence $(C \cap D') \setminus \{a,b\} \subseteq T \cap D' \subseteq D'^+$ by (3). Let $x \in (C \cap D') \setminus \{a,b\}$ We have $x \in (T \cap D') \setminus \{b\}$, hence $a \leq \text{Min } C^*(T;x)$ by (5). On the other hand $x \in C = C(T;a)$, hence $a \in C^*(T;x)$. It follows that $a = \text{Min } C^*(T;x)$. We have $x = b_i$ with $i > k$. By Algorithm 2 applied to $T$, at Step $i$ the edge $x = b_i$ is directed in the cocycle direction of $C^*(T;x)$ opposite to the direction of $a$. Now $C = C(T;a) \cap C^*(T;x) = \{x,a\}$, hence by orthogonality the edges $x$ and $a$ have the same cycle direction in the cycle $C$, i.e. $x \in C^+$.

(9) $C \cap D' \subseteq \{a,b\}$ *and* (9′) $C' \cap D \subseteq \{a,b\}$

Suppose $C \cap D' \subseteq \{a,b\} \neq \emptyset$. By (8) and graphical orthogonality, we have $a \in D'^-$ or $b \in C^-$, and both hold if $\{a,b\} \subseteq C \cap D'$.

Suppose $a \in D'^-$. Then $a \in C' \cap D' \subseteq \{a,b\}$, hence by orthogonality, we have $C' \cap D' = \{a,b\}$ and $b \in C'^+$. By (8') and graphical orthogonality applied to $C' \cap D$, we have $a \in D^-$. Then $a \in C \cap D \subseteq \{a,b\}$, hence by orthogonality, we have $C \cap D = \{a,b\}$ and $b \in C^+$. Therefore $\{a,b\} \subseteq C \cap D'$, both $a \in D'^-$ and $b \in C^-$ should hold : contradiction.

The case $b \in C^-$ is similar, and left to the reader.

(10) By (5), we have $a = \text{Min } C^*(T;e)$, with $e = b_\ell \in (T \cap D') \setminus \{b\}$ and $\ell > k$. We have $e \in C = C(T;a)$, hence $e \in C \cap ((T \cap D') \setminus \{b\}) \subseteq (C \cap D') \setminus \{b\} \subseteq \{a\}$ by (9). Therefore $a = e$. Hence $a = \text{Min } C^*(T;a)$, i.e. $a$ is internally active. Then, necessarily, $a = e_1 = b_1$, since $T$ and $T'$ have internal activity 1, contradicting $e = b_\ell$ with $\ell > 1$ (5). □

## 4. The bijection for $(0,1)$-activities

The case of $(0,1)$-activities can be reduced to $(1,0)$-activities by duality.

**Proposition 4.1**

*Let $G$ be an ordered graph with edge-set $\{e_1 < e_2 \ldots\}$.*

*(i) If $T$ is a spanning tree with $(1,0)$ activities, then $T \setminus \{e_1\} \cup \{e_2\}$ is a spanning tree with $(0,1)$ activities. The mapping defined by $T \mapsto T \setminus \{e_1\} \cup \{e_2\}$ is a bijection between the sets of spanning trees of $G$ with $(1,0)$ resp. $(0,1)$ activities.*

*(ii) If $\overrightarrow{G}$ is an orientation of $G$ with $(1,0)$ orientation activties, then $-_{e_1}\overrightarrow{G}$ has $(1,0)$ orientation activities. The mapping defined by $G \mapsto -_{e_1}\overrightarrow{G}$ is a bijection between the sets of orientations of $G$ with $(1,0)$ resp. $(0,1)$ activities.* $\square$

The proof of Proposition 4.1 is straightforward.

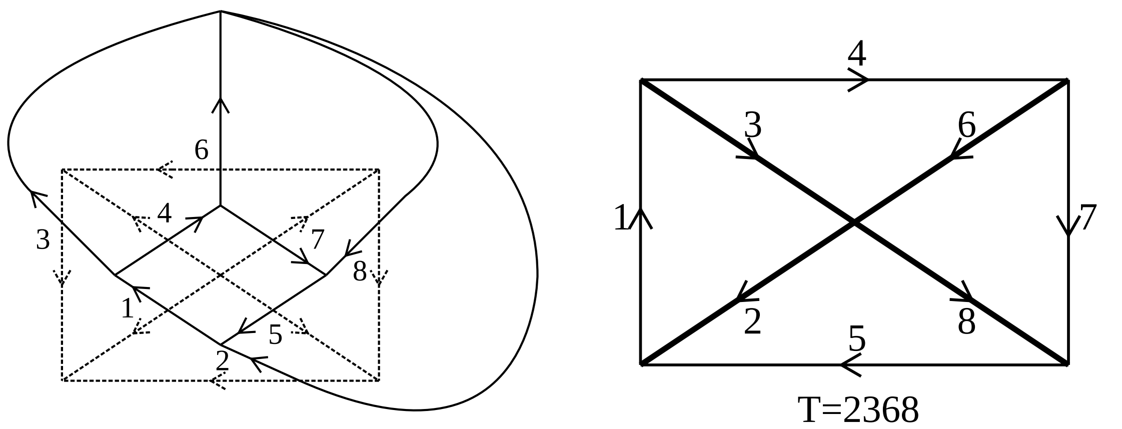


Figure 2

Figure 2 shows an application of Proposition 4.1 to the planar graph $W_4$ considered in Figure 1. We observe that the $(0,1)$-orientation associated with the spanning tree $T = 2368$ is different from the orientation associated with the same tree by the algorithm of [12] : the edge 8 of [12 Fig.4] is reversed in Figure 2.

## 5. The general correspondence

For short, we call *active correspondence*, the particular correspondence we construct in this section, associating with a general spanning tree of activities

$(i,j)$ a set of $2^{i+j}$ orientations with the same activities, in such a way that each orientation is the image of a unique spanning tree.

The main content of this section is that the construction of the active correspondence can be reduced to the $(1,0)$ case by means of *active partitions* of the edge-set. It turns out that, contrasting with Sections 3,4,6, where specific properties of graphs are used, Section 5 is a mere specialization to graphs of properties holding in matroids and oriented matroids. In consequence, we will only sketch the main results, and refer the reader to [7][8][9] for details and proofs.

Active partitions can be described either in terms of spanning trees in an ordered graph, or of orientations in an ordered directed graph. One main point is that if a spanning tree and an orientation are related by the active correspondence, then either definition produce the same active partition.

The definition of an active partition in terms of spanning trees is much more involved than its definition in terms of orientations. Let $G$ be an ordered graph with edge-set $E$, and $T$ be a spanning tree of $G$ with activities $(i,j)$. The first step is to construct a set $F$ such that $E \setminus F$ is the set of *internal* elements of $T$ and $F$ its set of *external* elements. For the convenience of the reader, we sketch the construction of $F$ (see [4] for more details and proofs).

Set $F_0 = \emptyset$, and define

$$F_{i+1} = F_i \cup \bigcup_{\substack{e \in E \setminus T \\ C_<(T;e) \subseteq F_i}} C(T;e)$$

where $C_<(T;e)$ is the set of elements of $C(T;e)$ strictly smaller than $e$. Clearly $F_1$ is the set of externally active elements of $T$, and we have $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_i \subseteq \ldots$. Set $F = \cup_i F_i$.

Then $F$ separates the internal and external activities : $T \setminus F$ is a spanning tree with $(i,0)$ activities of the contraction $G/F$ of $G$ by $F$, and $T \cap F$ is a spanning tree with $(0,j)$ activities of the subgraph $G(F)$ [4].

Let $\overrightarrow{G}$ be an orientation associated with $T$ by the active correspondence. By a result of G. Minty (1960), in a directed graph an edge belongs either to a directed cycle or to directed cocycle, but not to both. Then $F$ is the *totally cyclic part* of $\overrightarrow{G}$, i.e. the union of all directed cycles of $\overrightarrow{G}$, and $E \setminus F$ is the *acyclic part* of $G$, i.e. the union of all directed cocycles of $\overrightarrow{G}$.

It follows from this first reduction that without loss of generality, we may restrict the construction to $(i,0)$ or $(0,j)$ activities. Furthermore, internal and external elements, and also totally cyclic parts and acyclic parts, being related by duality, we may restrict ourselves to internal spanning trees and acyclic orientations.

The second step reduces the construction to $(1,0)$ activities. The reduction in terms of spanning trees is very similar to the above one, starting with active elements instead of the empty set (see [7][9]). For the sake of simplicity, we describe the reduction in terms of orientations. Of course, by so doing, we rather describe the reciprocal of the active correspondence. But this is clearly equivalent to the direct construction.

Let $\overrightarrow{G}$ be an acyclic orientations of the ordered graph $G$ with $\leq^* (\overrightarrow{G}) = i$, and let $a_1 < ... < a_i$ be its $O^*$-active edges. Then for $j = 1, 2, \ldots, i$ set

$$A_j = \bigcup_{\substack{D \text{ directed cocycle} \\ \text{Min } D = a_j}} D \setminus \bigcup_{\substack{D \text{ directed cocycle} \\ \text{Min } D > a_j}} D$$

Then the *active partition* of $\overrightarrow{G}$ for the orientation is the partition

$$E = A_1 + A_2 + \ldots + A_i$$

Set

$$\overrightarrow{G}_j = \overrightarrow{G}/(A_1 \cup A_2 \cup \ldots \cup A_{j-1}) \setminus (A_{j+1} \cup A_{j+2} \cup \ldots \cup A_i)$$

where, as usual $\setminus$ denotes the deletion, and $/$ denotes the contraction.

**Theorem 5.1**

*The graph $\overrightarrow{G}_j$ on the edge-set $A_j$ has $(1,0)$ orientation activities. By reversing the bijection of Section 3 on each graph $\overrightarrow{G}_j$ for $j = 1, 2, \ldots, i$, we associate with each $\overrightarrow{G}_j$ a spanning tree $T_j$ with $(1,0)$ activities. Then $T = T_1 + T_2 + \ldots + T_i$ is a spanning tree of $\overrightarrow{G}$ with $(i,0)$ activities. We define the active correspondence by associating the spanning tree $T$ with $\overrightarrow{G}$. This active correspondence has the desired properties, and moreover preserves active elements.*

The proof of Theorem 5.1, and the statement and proof of the mixed case, when both $F$ and $E \setminus F$ are not empty, can be found in [7][9] in the more general context of oriented matroids. We will illustrate its content in Section 6 on a example (Figures 3 and 4).

The fact that the bijection of Section 3 is actually (1-2) has an immediate consequence. By reversing in all possible ways, independently, all edge directions in each $A_j$, we get a set of $2^i$ orientations, called the *activity class* of $\overrightarrow{G}$. *All orientations in an activity class have the same active elements, the same active partition, and are associated with the same spanning tree.*

The activitiy classes constitute a partition of the set of orientations of a graph. The active correspondence induces an activity preserving bijection between spanning trees and activity classes of orientations.

## 6. Application to acyclic orientations with a unique sink

C. Greene and T. Zaslavsky have shown in [10] that the number of acyclic orientations of a graph $G$ with a unique sink at a given vertex is equal to $t(G;1,0)$. In [6], D.D. Gebhard and B.E. Sagan give three bijective proofs of this result. The third one [6 Th.4.1] is by means of an explicit bijection between acyclic orientations with a given unique sink and spanning trees with external activity 0, or *internal* spanning trees, as suggested by the relation $t(G;1,0) = \sum_i t_{i,0}$.

It turns out that the correspondence defined in Section 5 provides another bijection between internal spanning trees and acyclic orientations with a given unique sink, which moreover preserves active edges. The internally active edges of an internal tree becomes $O^*$-active edges of the orientation.

We say that a spanning tree $T$ in an ordered graph is *increasing with respect to a vertex $s$* if the edges increase for the ordering along any path of $T$ beginning at $s$.

**Proposition 6.1**

*Let $G$ be an ordered graph such that the lexicographically smallest spanning tree is increasing with respect to a vertex $s$.*

*Then there is exactly one acyclic orientation with a unique sink at $s$ in each activity class of acyclic orientations of $G$.*

Note that the hypothesis implies $s$ is an extremity of the smallest (non loop) edge of $G$.

**Lemma 6.1.1**

*In an ordered graph, the smallest edge of any cocycle belongs to the lexicographically smallest spanning tree.*

Proof

Let $G$ be an ordered graph, $T_0$ be its lexicographically smallest spanning tree, and $D$ be any cocycle. We have $D \cap T_0 \neq \emptyset$. Set $a = \text{Min } D$ and $b = \text{Min } D \cap T_0$. Then, $a \notin T_0$ implies $a < b$, hence $T_0 \cup \{a\} \setminus \{b\}$ is a spanning tree of $G$, lexicographically smaller than $T_0$, a contradiction. □

Proof of Proposition 6.1

Let $\overrightarrow{G} = (V, E)$ be an acyclic orientation of $G$ with a unique sink at $s$, and $D$ be a directed cocycle, with 2-partition $V = V_1 + V_2$ such that $s \in V_1$. We have $E = E_1 + D + E_2$, where $E_i$ is the set of edges of $\overrightarrow{G}$ having both extremities in $V_i$ for $i = 1, 2$. The directed graph $\overrightarrow{G}_2 = (V_2, E_2)$ is acyclic, hence has a sink $s_2$. If the edges of $D$ were directed from $V_1$ to $V_2$, then $s_2$ would be a sink of $\overrightarrow{G}$, which is impossible since $s_2 \neq s$. Hence the edges of $D$ must be directed from $V_2$ to $V_1$. With notation of Section 5, orientations in an activity class are obtained by reversing edge directions in all $A_j$ for $j \in J$ and all $J \subseteq \{1, 2, \ldots, i\}$. Set $A'_j = \bigcup_{j \leq k \leq i} A_j$ for $j = 1, 2, \ldots, i$. We have $A_j = A'_j \setminus A'_{j+1}$ for $j = 1, 2, \ldots, i-1$ and $A'_i = A_i$. It follows that activity classes can also be obtained by reversing edge directions in all $A'_j$ for $j \in J$ and $J \subseteq \{1, 2, \ldots, i\}$ (nb. some edges may be reversed several times in the second process). We observe that by construction the $A'_j$ are unions of directed cocycles. Hence by the above property, their state - reversed or not - is fixed if the orientation has a unique sink at a given vertex. Therefore $\overrightarrow{G}$ is the unique orientation of $G$ in its activity class having a unique sink at $s$.

Conversely, let $T_0$ be the lexicographically smallest spanning tree of $G$. By hypothesis $T_0$ is increasing. Let $\overrightarrow{G}$ be the graph in a given activity class of acyclic orientations of $G$ such that the $O^*$-active edges of this class are directed towards $s$ in $T_0$. The graph $\overrightarrow{G}$ is acyclic, hence has a sink $s'$. The edges with head-vertex $s'$ constitute a directed cocycle $D$ of $\overrightarrow{G}$, with minimal element $a \in T_0$. Then $a$ is directed towards the sink $s'$, and towards $s$ in $T_0$ by construction of $\overrightarrow{G}$. By hypothesis the paths of $T_0$ increase from the root $s$. If $s \neq s'$, there exists another edge $b < a$ on $T_0$ incident to $s'$, contradicting the minimality of $a$ in the directed cocycle $D$. Therefore $s = s'$, and $\overrightarrow{G}$ has a unique sink at $s$. □

**Theorem 6.2**

*Let $G$ be an ordered graph, such that the lexicographically smallest spanning tree is increasing with respect to a vertex $s$.*

*Then the mapping sending an internal spanning tree $T$ of $G$ to the unique acyclic orientation with a unique sink at $s$ belonging to the activity class of orientations associated with $T$ by the correspondence of Theorem 5.1, is an activity prerving bijection from the set of internal spanning trees of $G$ onto the set of acyclic orientations of $G$ with a unique sink at $s$.* $\square$

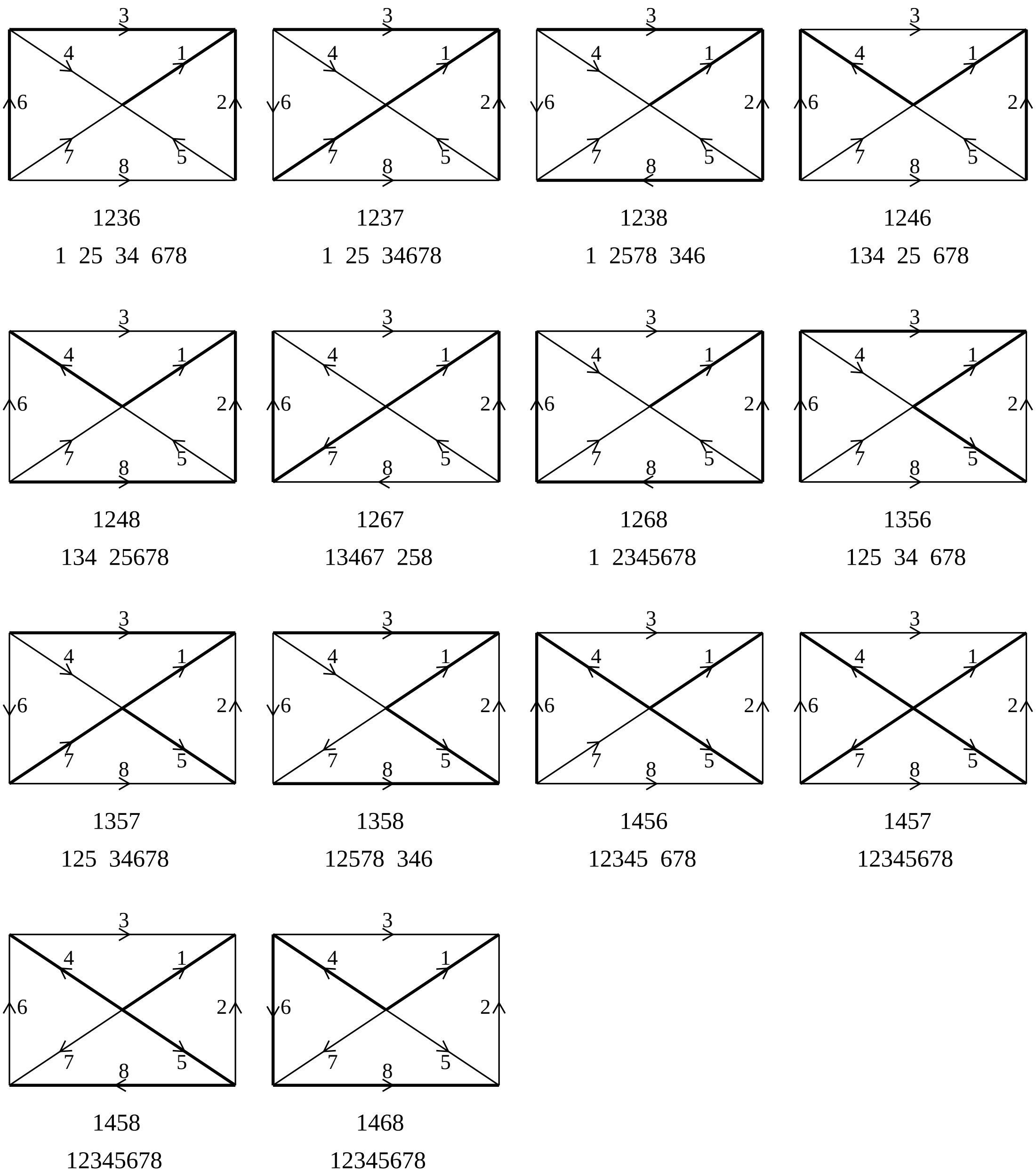


Figure 3

Theorem 6.2 is a straightforward corollary of Theorem 5.1 and Proposition 6.1. Note that given any spanning tree $T$ in a graph $G$, and a vertex $s$, it is always possible - and easy - to linearly order the edges of $G$ so that $T$ is the lexicographically smallest spanning tree and is increasing with respect to $s$. Label the edges of $T$ by consecutive integers $1, 2, \ldots$ in succesive layers defined by their distance to $s$. After $T$ has been labelled, label arbitrarily the edges not in $T$.

The bijections provided by Theorem 6.2 are different from the Gebhard-Sagan bijections. We observe that these bijections are activity preserving by construction, whereas Gebbard-Sagan bijections are not in general. The orientation in Figure 1 of [10 p.139] has $O^*$-activity 2, but the spanning tree constructed by the algorithm has internal activity 3.

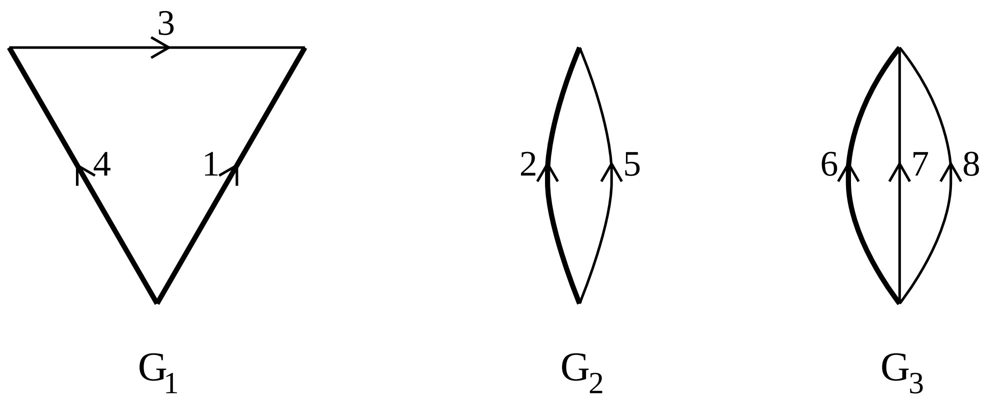


Figure 4

Figure 3 uses the graph $W_4$, also used in Figures 1 and 2, to give an example for Theorem 6.1, . The Tutte polynomial of $W_4$ is

$$t(W_4; x, y) = x^4 + y^4 + 4x^3 + 4x^2y + 4xy^2 + 4y^3 + 6x^2 + 9xy + 6y^2 + 3x + 3y$$

The graph $W_4$ has $t(W_4; 1, 0) = 14$ internal spanning trees.

The lexicographically smallest spanning tree 1236 is increasing with respect to the NE (North-East) vertex. For each acyclic orientation with unique sink at the NE vertex, we have indicated the internal spanning tree $T$ given by Theorem 6.1 (its edges are drawn in heavy lines). We have also indicated the active partition. The internal activity is the number of parts of the active partitions, and the active edges are the first element of each part. By reversing all edge directions in arbitrarily chosen parts of the active partition, we get the activity class associated $T$. By Theorem 6.1, in each activity class exactly one acyclic orientation has a unique sink at the NE vertex : this orientation is shown on Figure 3. Hence Figure 3 also illustrates the bijection from internal spanning trees to activity classes of acyclic orientation (or the active correspondence) defined in Section 5.

Figure 4 gives details of the construction of Section 5 for the spanning tree $T = 1246$. The active partition is 134+25+678. The graphs of Theorem 5.1 are $G_1 = G \setminus 25678$, $G_2 = G/134 \setminus 678$, $G_3 = G/12345$. The spanning trees with $(1,0)$ activities being unique in these very simple graphs one can check easily that we have $T_1 = 14$, $T_2 = 2$, $T_3 = 6$, and, of course, $1246 = 14 + 2 + 6$.

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