Convex Integer Optimization
and
Universality of Multiway Polytopes

Shmuel Onn

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Based on several papers joint with various subsets of
{De Loera, Hemmecke, Rothblum, Weismantel}

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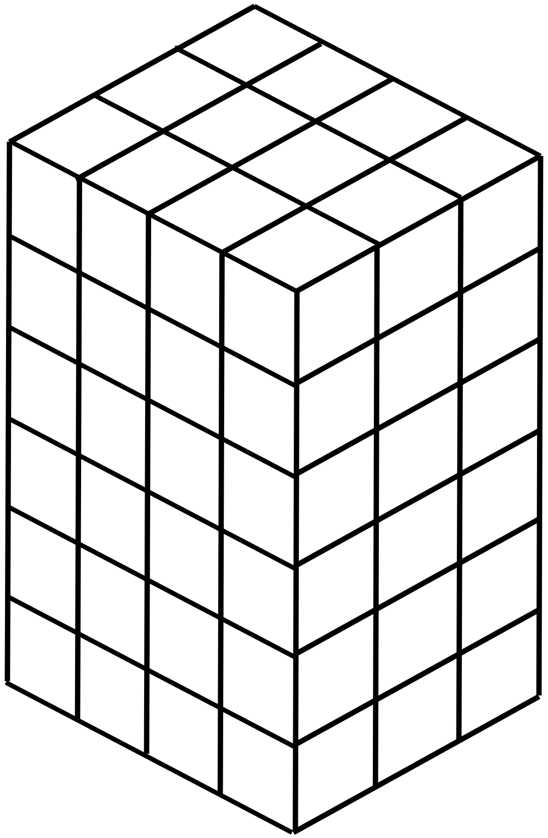

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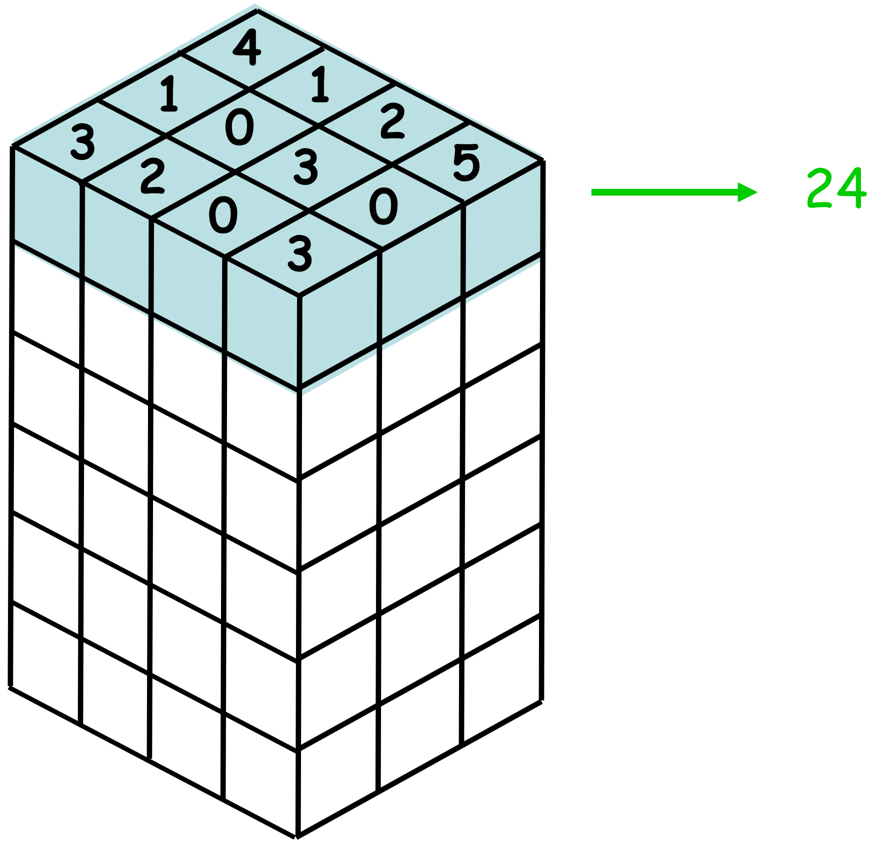

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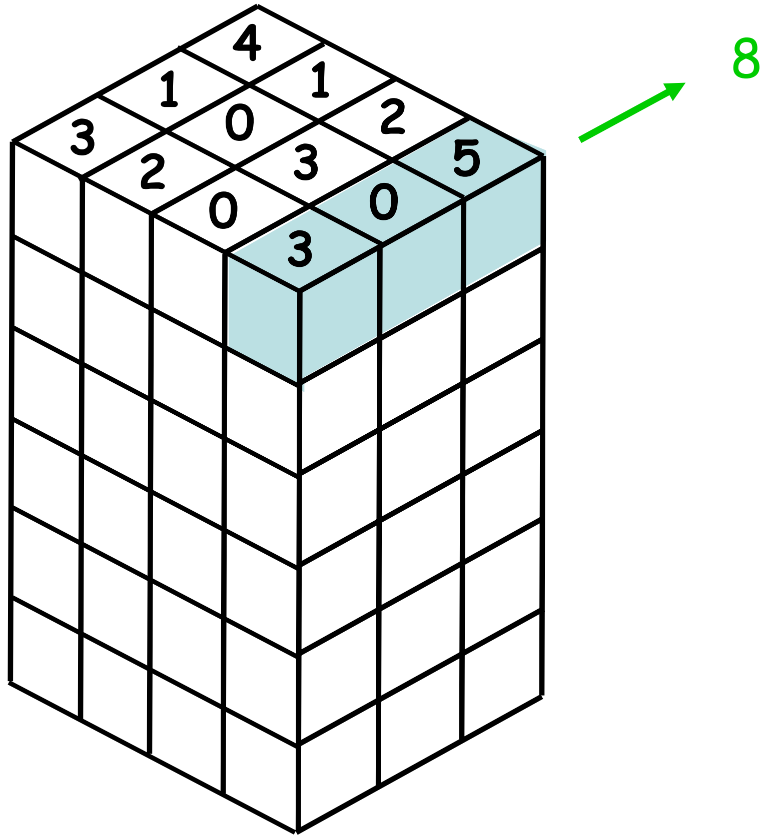

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Two contrasting results:

Universality Theorem: Any rational polytope is an $r \times c \times 3$ line-sum polytope.

Optimization Theorem: (Convex) Integer Programming over $m_1 \times \cdots \times m_k \times n$ polytopes is solvable in polynomial time.

Universality and its Consequences

Universality Theorem for Short 3-Way Polytopes

Theorem: Any rational polytope $P = \{y \in \mathbb{R}^m_+ : Ay = b\}$ is polytime representable as an $r \times c \times 3$ line-sum polytope

$$T = \left\{ x \in \mathbb{R}^{r \times c \times 3}_+ : \sum_i x_{i,j,k} = w_{j,k} \,,\, \sum_j x_{i,j,k} = v_{i,k} \,,\, \sum_k x_{i,j,k} = u_{i,j} \right\}$$

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→ Implications on the rational version of Hilbert's 10th problem on the decidability of the realization problem for polytopes ?

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Question: how does the set of values that can occur in a specific entry in all tables with the released margins look like ?

Fact: for k-way tables with fixed hyperplane-sums, the set of values in an entry is always an interval.

Example: the values 0, 2 occur in an entry:

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Therefore, also the value 1 occurs in that entry:

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In contrast we have the following universality:

Theorem: For every finite set S of nonnegative integers, there are r, c and line-sums for $r \times c \times 3$ tables such that the set of values occurring in a fixed entry in all possible tables with these line-sums is precisely S.

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Proof: Given $S = \{s_1, \ldots, s_m\}$, let

$$P := \{y \in \mathbb{R}_+^{m+1} \ : \ y_0 - \sum_{i=1}^{m} s_i y_i = 0\,,\ \sum_{i=1}^{m} y_i = 1\}.$$

Lift P using the universality theorem to $r \times c \times 3$ line-sum polytope T.

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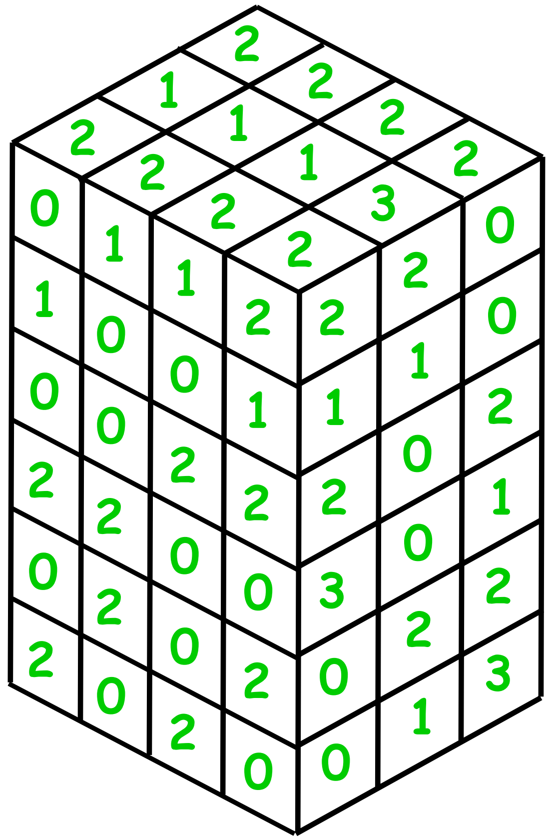
Consider the following line-sums for **6** x **4** x **3** tables:



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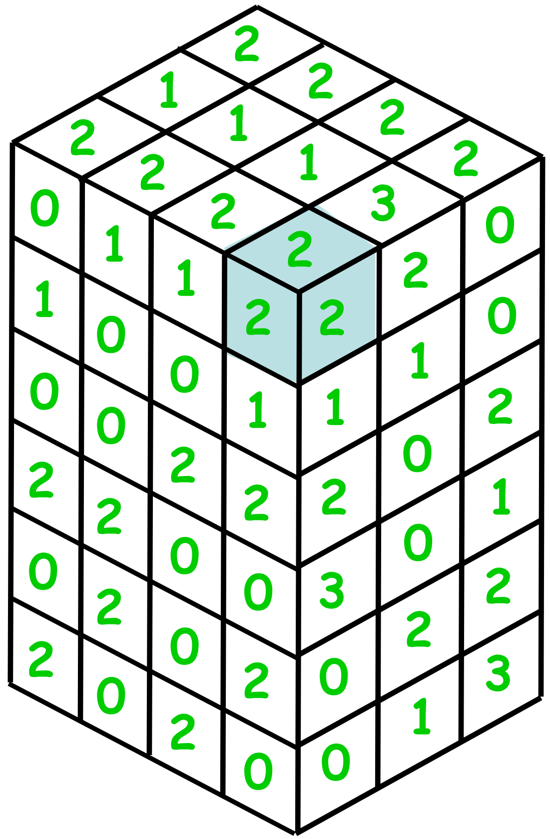

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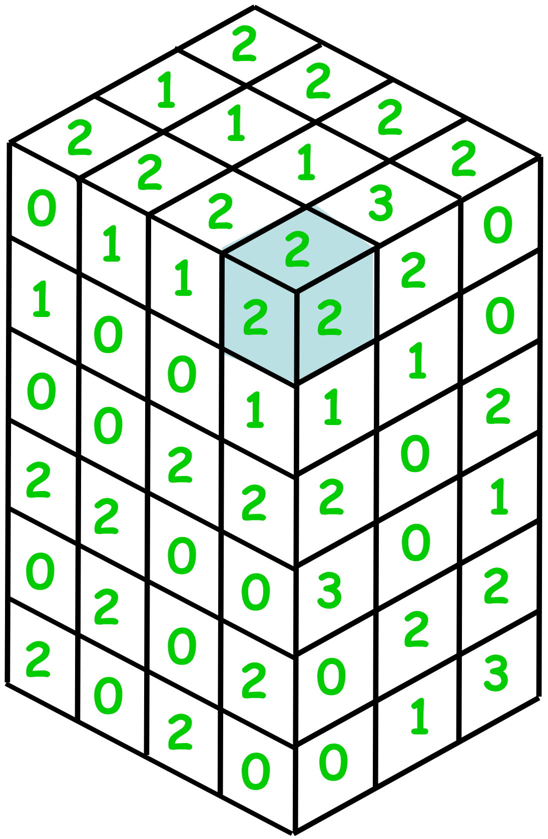

The **only values** occurring in that entry in all possible tables with these line-sums are **0**, **2**

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More Universality Consequences

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Universality Theorem for Bitransportation Polytopes:

Theorem: Any rational polytope $P = \{y \in \mathbb{R}^m_+ : Ay = b\}$ is polytime representable as an $n \times n$ bitransportation polytope

$$B = \left\{ (x^1, x^2) \in \oplus_2 \mathbb{R}^{n \times n}_+ \ : \ \sum_j x^k_{i,j} = r^k_i \,,\ \sum_i x^k_{i,j} = c^k_j \,,\ x^1_{i,j} + x^2_{i,j} \le u_{i,j} \right\}$$

Convex

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Optimization

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We consider the following convex integer programming problem:

$$\max \{c(w_1 x, \ldots, w_d x) : x \geq 0,\ Ax = b,\ x \text{ integer}\}$$

where $w_1, \ldots, w_d$ are linear forms and c is a convex functional on R^d.

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Nonetheless, as a consequence of our more general theorem below, we obtain the following Optimization Theorem for long multiway polytopes:

Theorem: Fix $d, m_1, \ldots, m_k$. Then convex integer programming over any $m_1 \times \cdots \times m_k \times n$ multiway polytope is solvable in polynomial oracle-time for any margins, $w_1, \ldots, w_d$, and convex c presented by comparison oracle.

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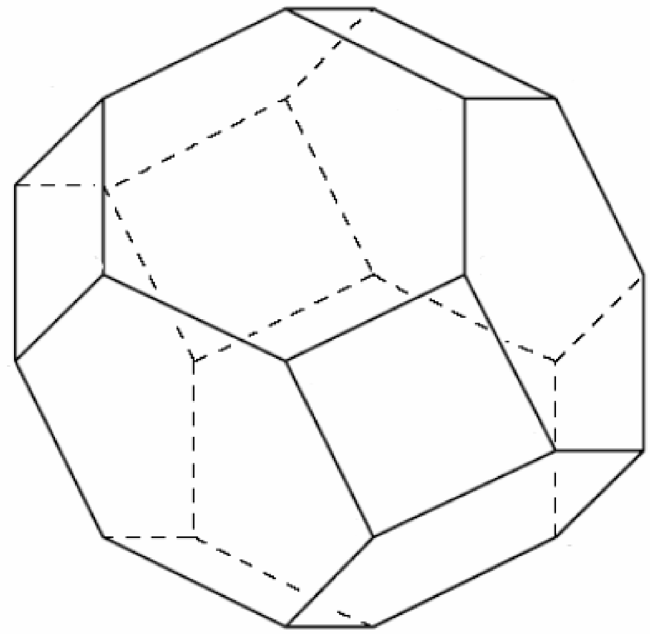

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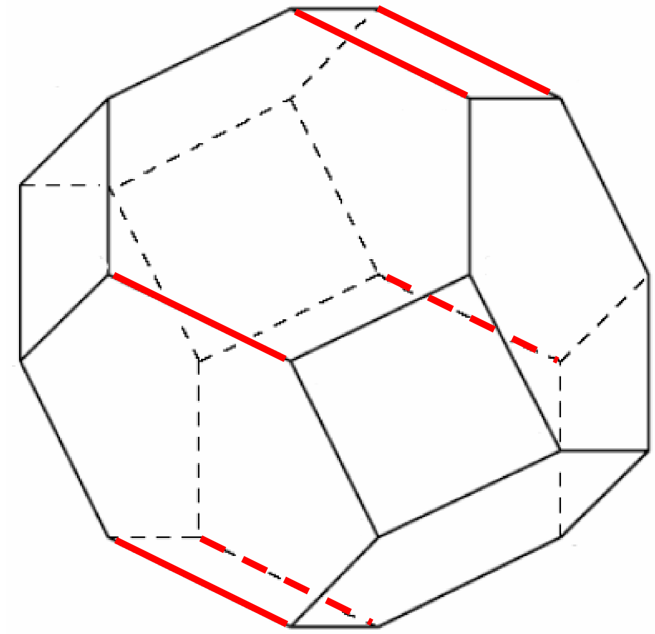

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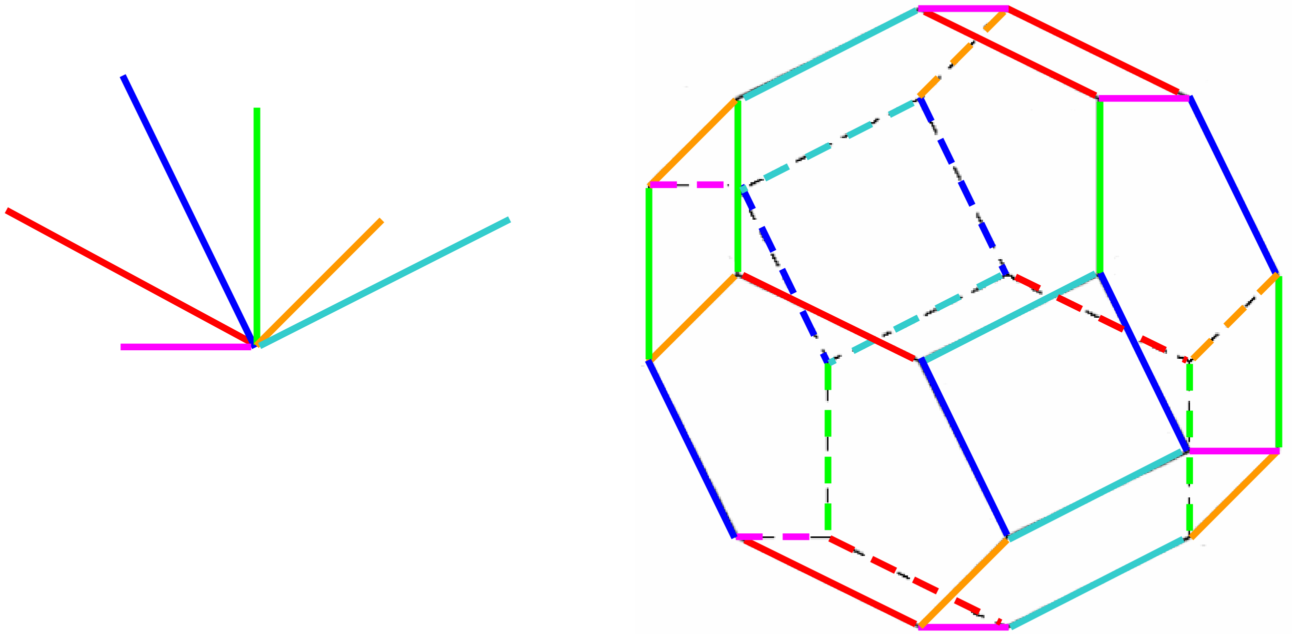

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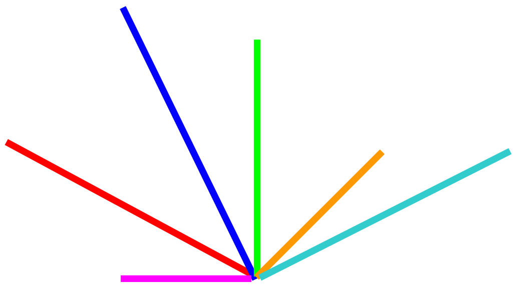

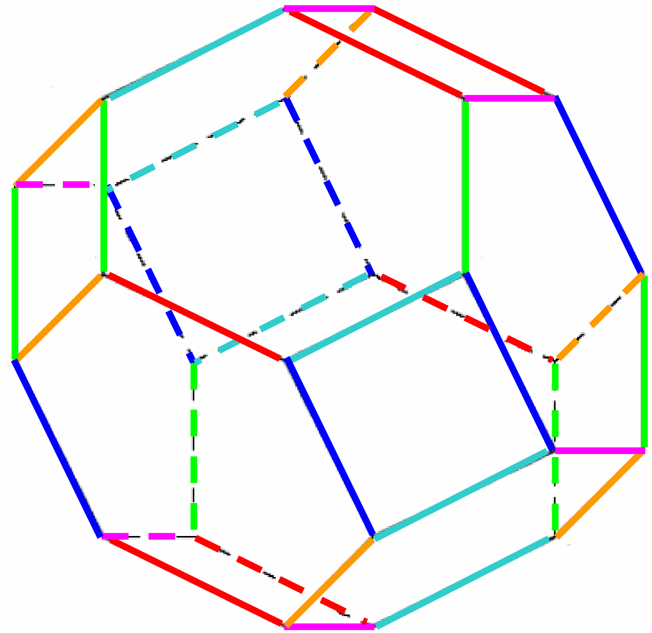

Lemma 1: Fix d. Then, given a set E covering all edge-directions of P, the convex integer programming problem over P is reducible to solving polynomially many linear integer programming counterparts over P.

Zonotope Refinement and Construction

Prop. 1: If $E = \{e^1, \ldots, e^m\}$ covers all edge-directions of a polytope P then the zonotope $Z = [-1, 1]\, e^1 + \ldots + [-1, 1]\, e^m$ is a refinement of P.

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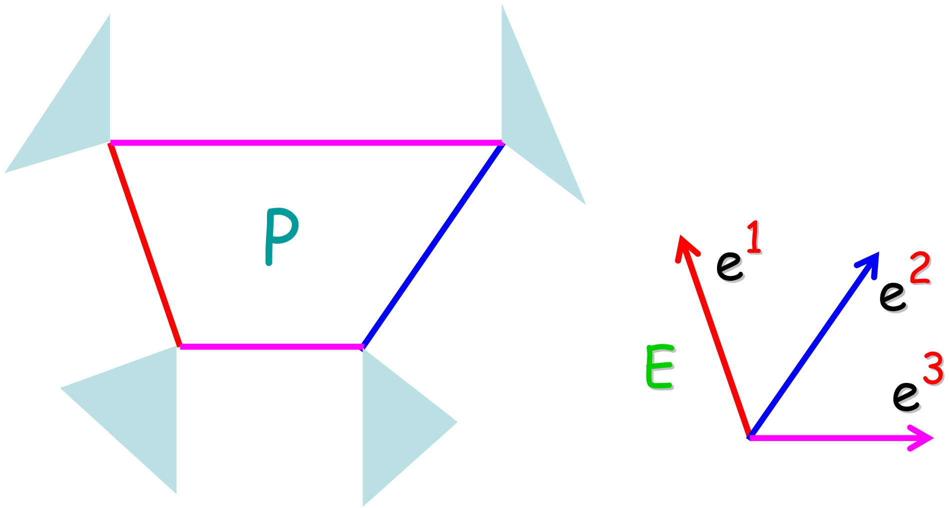

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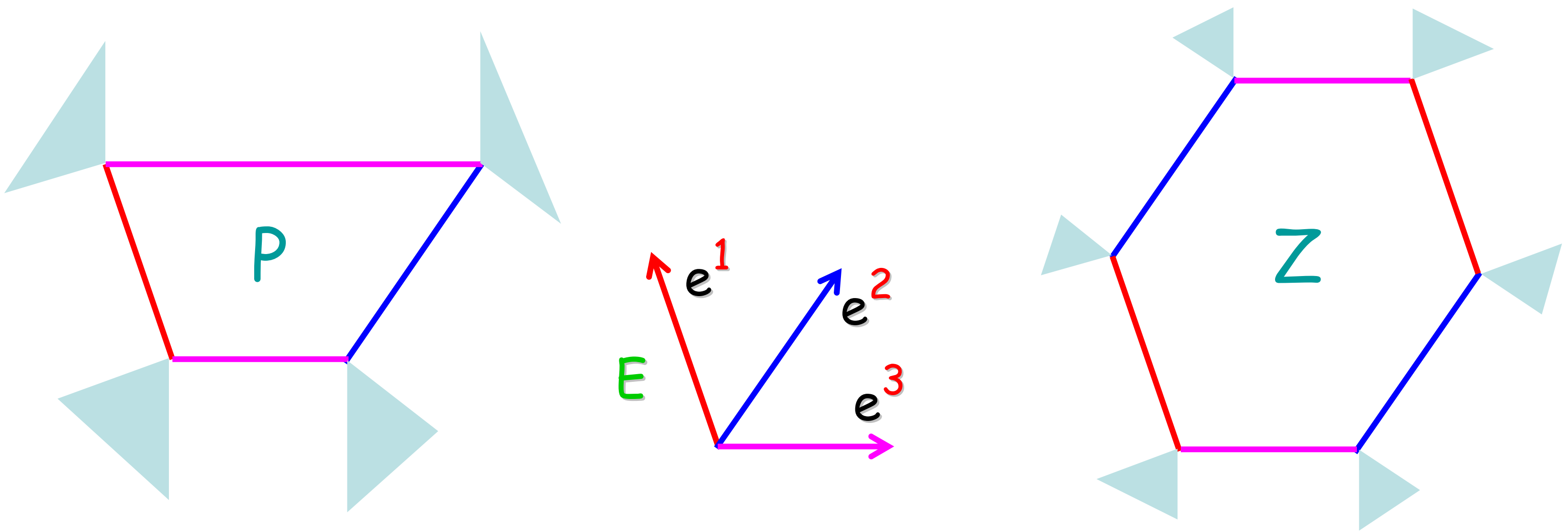

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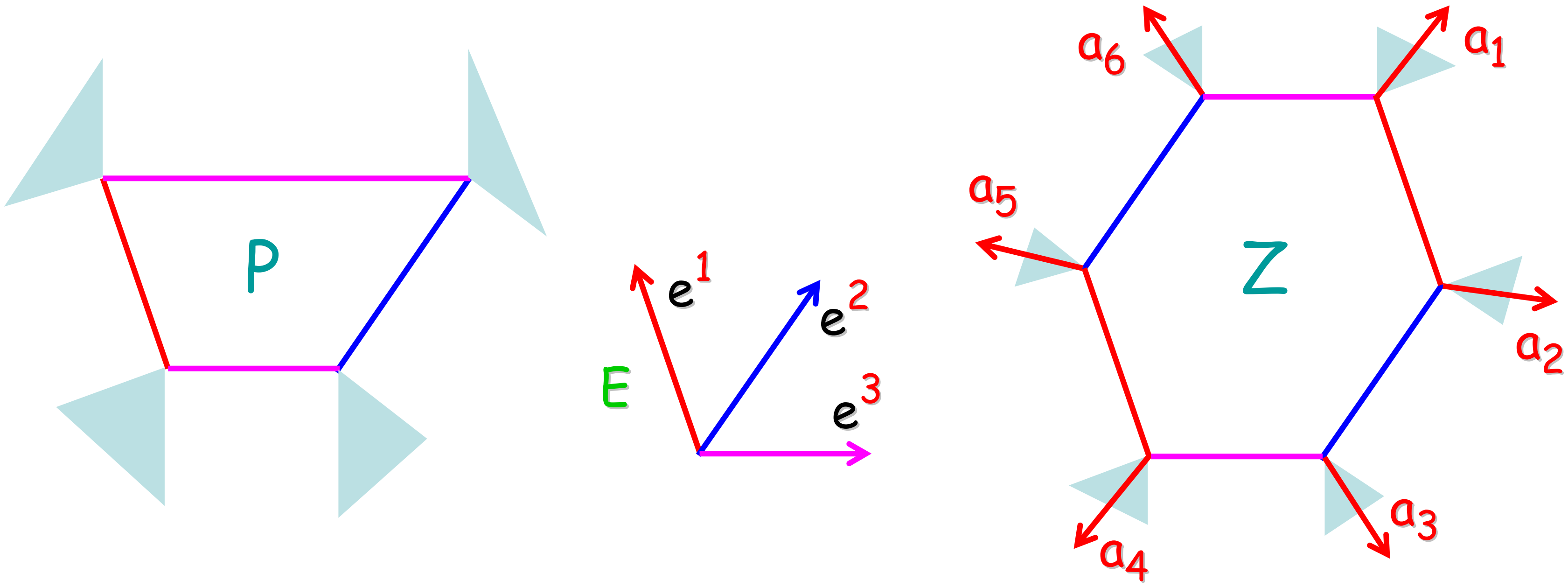

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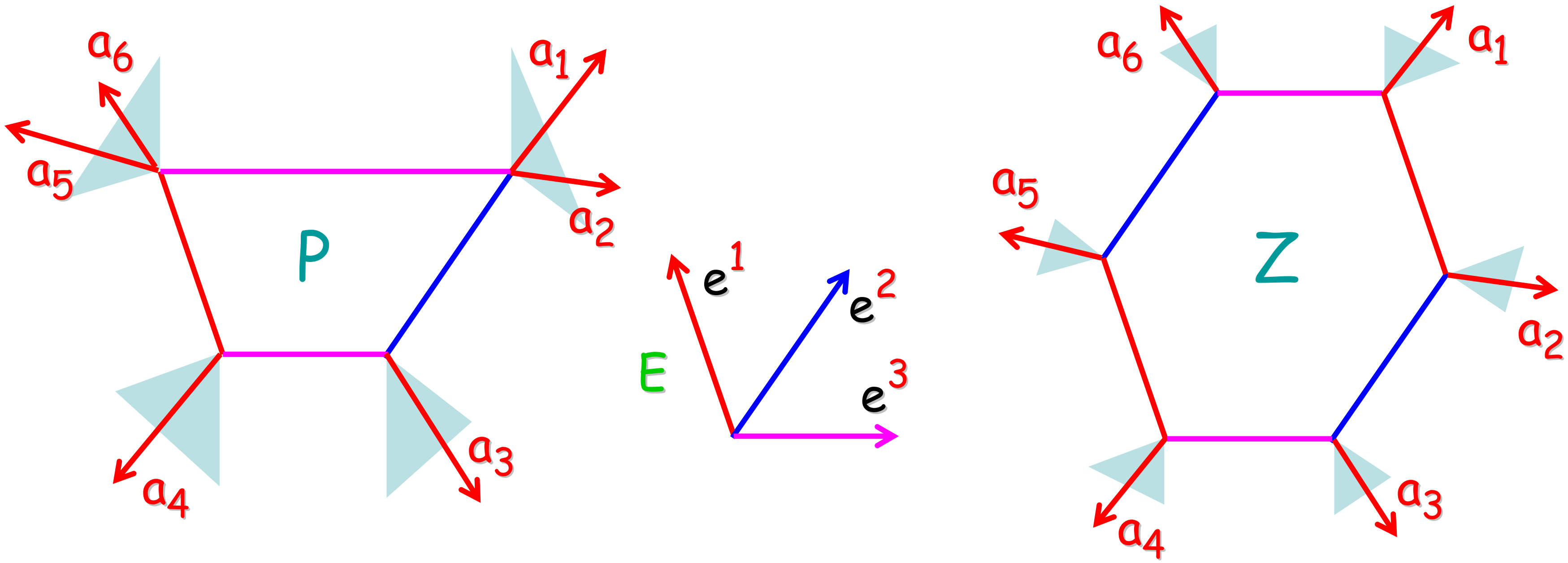

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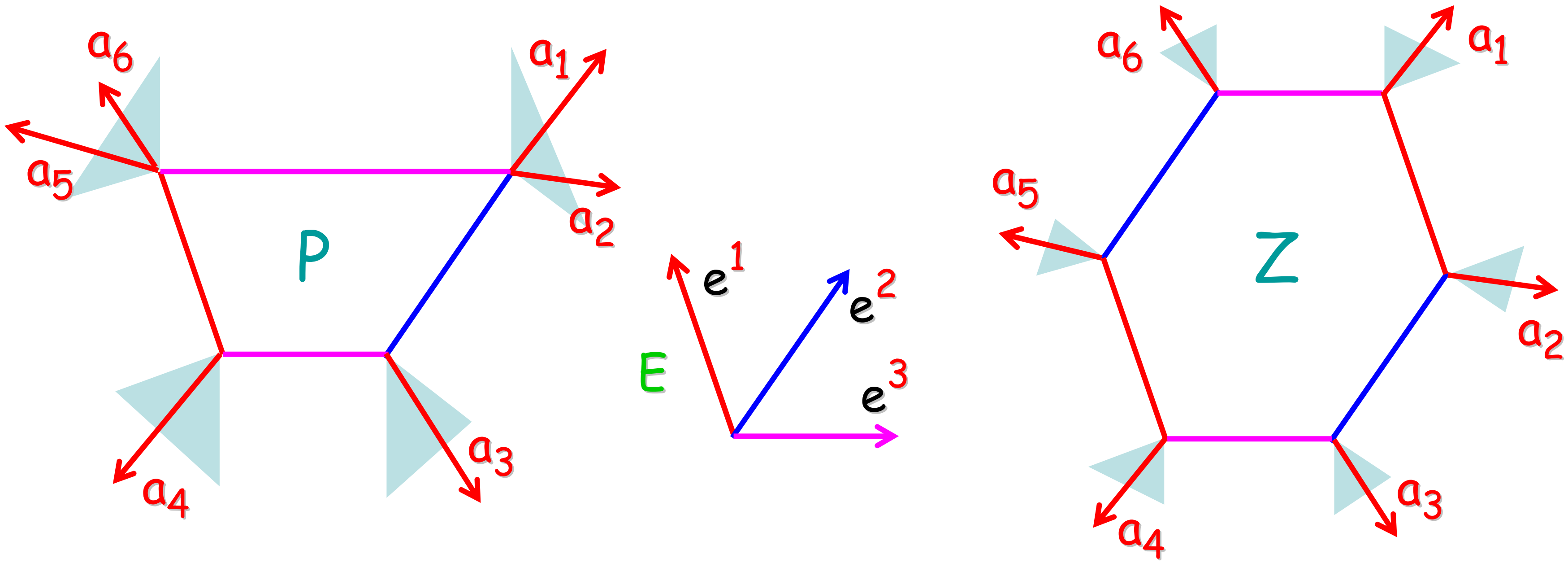

Prop. 2: In R^d, the zonotope Z can be constructed from $E = \{e^1, \ldots, e^m\}$ along with a vector a_i in the cone of every vertex in $O(m^{d-1})$ operations.

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The Algorithm Establishing Lemma 1

Input: Polytope P in R^n given via A,b, set E covering its edge-directions, d x n matrix w, and convex functional c on R^d given by comparison oracle.

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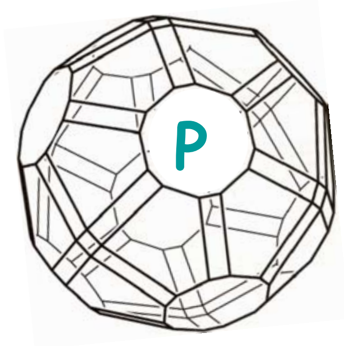

R^n

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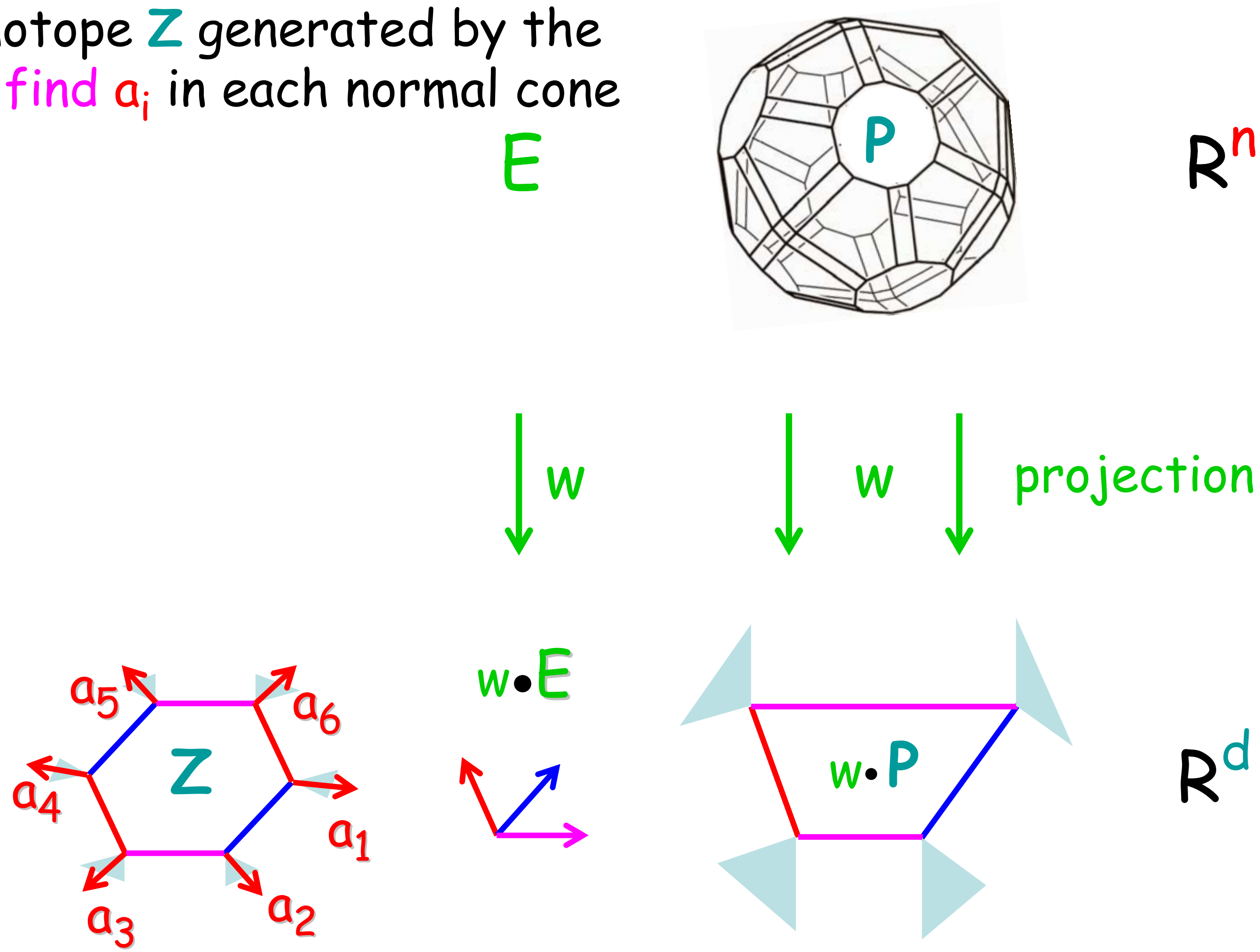

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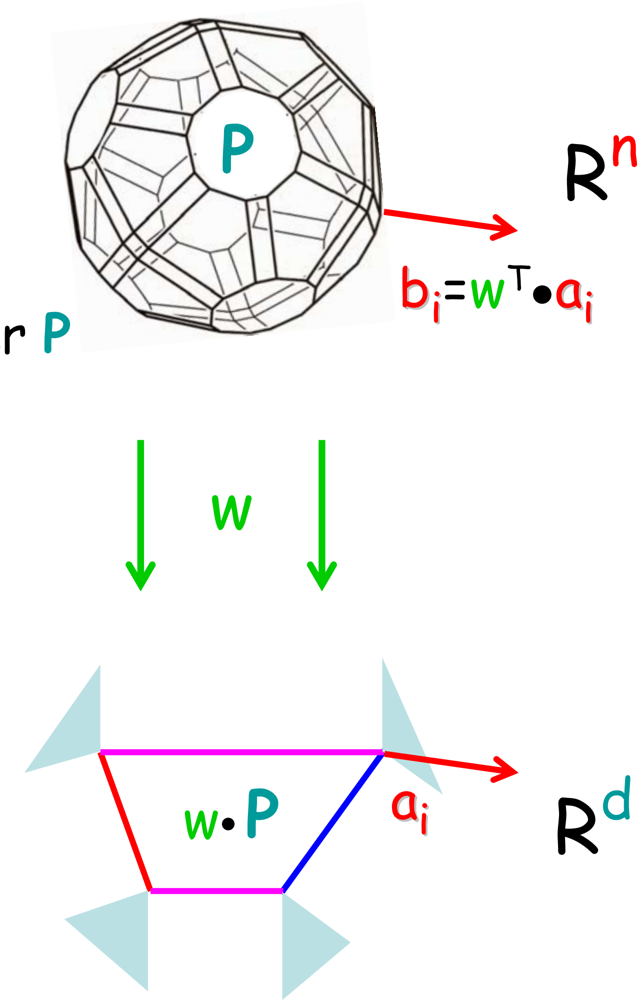

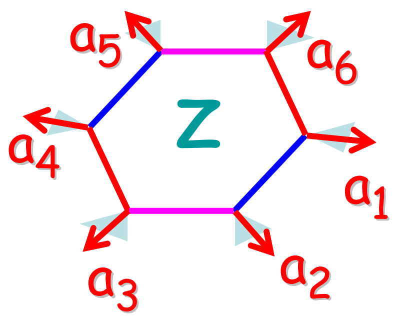

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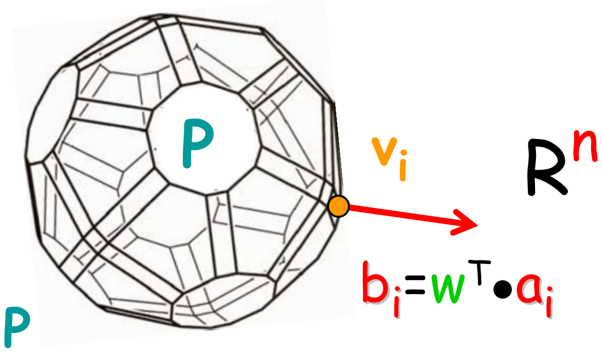

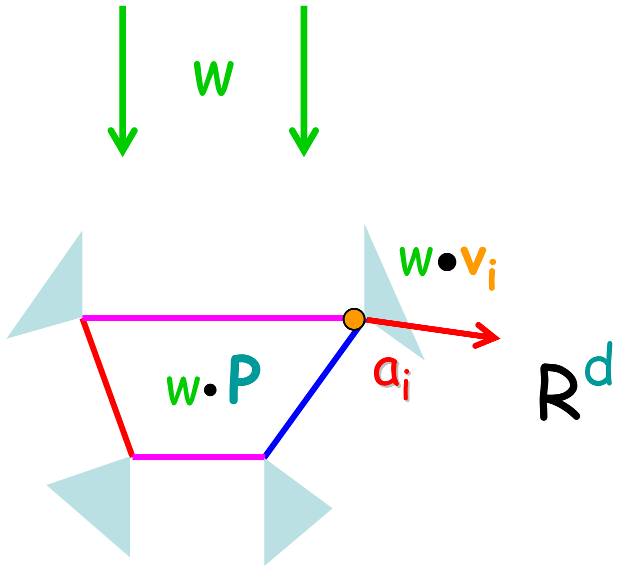

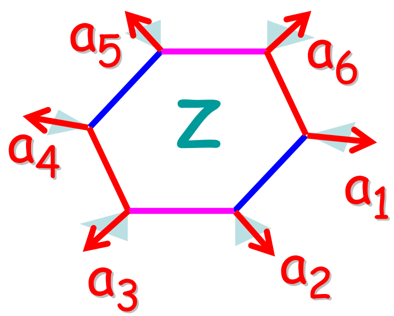

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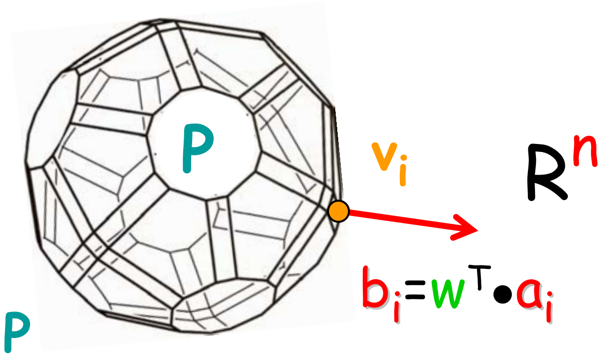

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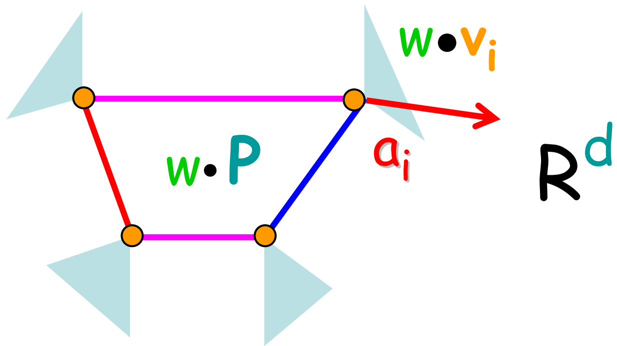
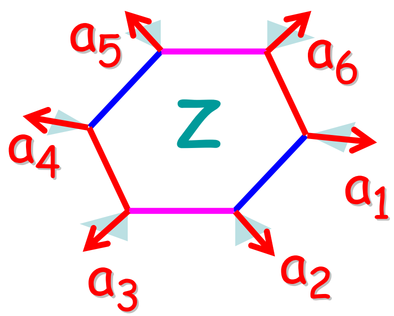


4. Output any v_i attaining maximum value $c(w \bullet v_i)$ using comparison oracle



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Proof Ingredient 2: Graver Bases

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The Graver basis of an integer matrix A is the set of conformal-minimal nonzero integer dependencies on A, i.e. vectors with $Av = 0$. For instance, the Graver basis of $A = [1\ 2\ 1]$ is $\pm\{\ [2\ {-1}\ 0], [0\ {-1}\ 2], [1\ 0\ {-1}], [1\ {-1}\ 1]\ \}$.

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Theorem: For any fixed d and $(r+s) \times t$ matrix A, there is a polynomial oracle-time algorithm that, given n, b, $w_1, \ldots, w_d$, and convex c presented by comparison oracle, solves the convex integer programming problem

$$\max \{ c(w_1 x, \ldots, w_d x) \; : \; A^{(n)} x = b, \; x \text{ in } N^{nt} \}$$

Application 1: Multiway Tables

The margin equations for any $m_1 \times \cdots \times m_k \times n$ polytope form an n-fold system defined by a suitable matrix A, where A_1 controls the equations of margins involving summation over layers, whereas A_2 controls the equations of margins involving summation within a single layer at a time.

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We deduce the optimization theorem for long k-way polytopes:

Theorem: Fix $d, m_1, \ldots, m_k$. Then convex integer programming over any $m_1 \times \cdots \times m_k \times n$ multiway polytope is solvable in polynomial oracle-time for any margins, $w_1, \ldots, w_d$, and convex c presented by comparison oracle.

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Recall that in contrast, short 3-way polytopes are universal:

Theorem: Any rational polytope is an $r \times c \times 3$ line-sum 3-way polytope.

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Partition n items evaluated by k criteria to p players, to maximize social utility which is convex on the sums of values of items each player gets.

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The social utility of π is $c(A^\pi) = 244432$

All 90 partitions π of items {1, …,6} To 3 players where each player gets 2 items

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[13 24 56]	[13 25 46]	[13 26 45]	[13 45 26]	[13 46 25]	[13 56 24]
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[15 23 46]	[15 24 36]	[15 26 34]	[15 34 26]	[15 36 24]	[15 46 23]
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[23 14 56]	[23 15 46]	[23 16 45]	[23 45 16]	[23 46 15]	[23 56 14]
[24 13 56]	[24 15 36]	[24 16 35]	[24 35 16]	[24 36 15]	[24 56 13]
[25 13 46]	[25 14 36]	[25 16 34]	[25 34 16]	[25 36 14]	[25 46 13]
[26 13 45]	[26 14 35]	[26 15 34]	[26 34 15]	[26 35 14]	[26 45 13]
[34 12 56]	[34 15 26]	[34 16 25]	[34 25 16]	[34 26 15]	[34 56 12]
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[23 14 56]	[23 15 46]	[23 16 45]	[23 45 16]	[23 46 15]	[23 56 14]
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Theorem: Partitioning problems with fixed p and k are polytime solvable.

Bibliography: most papers are available at

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Also related:

- Markov bases of three-way tables are arbitrarily complicated (J. Symb. Comp. 2006)
- Convex combinatorial optimization (Disc. Comp. Geom. 2004)
- The Hilbert zonotope and a polynomial time algorithm for universal Gröbner bases (Adv. App. Math. 2003)