Applications of Nash Equilibria in Electricity Markets

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Outline of Presentation

- Review of Nash equilibria: the Variational Inequality approach
  - Facchinei and Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer-Verlag (New York 2003)
- Two recent application areas: basic models
  - electric power markets (with Ben Hobbs, mostly)
  - power control for digital subscriber lines (with Tom Luo)
- Extensions
  - price function conjectures
  - piecewise linear prices (such as price caps)
  - collusion, leading to a nonconvex optimization problem
  - generalized Nash problems, leading to quasi-variational inequalities
  - Stackelberg and multi-leader-follower games, leading to problems with equilibrium constraints
  - dynamics, requiring *differential variational inequalities*
  - stochastics, dealing with uncertainty

Selected References

Metzler, Hobbs, and Pang. Nash-Cournot equilibria in power markets on a linearized DC network with arbitrage: formulations and properties. *Networks and Spatial Economics* 3 (2003) 123–150.

Pang, Hobbs, and Day. Properties of oligopolistic market equilibria in linearized DC power networks with arbitrage and supply function conjectures. In E. Sachs, editor, *System Modeling and Optimization XX*, Kluwer Academic Publishers (2003) pp. 113–130.

Hobbs and Pang. Spatial oligopolistic equilibria with arbitrage, shared resources, and price function conjectures. *Mathematical Programming, Series B* 101 (2004) 57–94

Pang and Fukushima. Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games. *Computational Management Science* 1 (2005) 21–56.

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Selected References (continued)

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Review of Nash equilibria: the VI approach

- $\mathcal{F}$ a finite index set of players (firms)

- $X_f \subseteq \Re^{n_f}$ player f's strategy set; compact and convex

- $\pi_f : \Re^n \to \Re$ player f's profit function; $\pi_f(\cdot, q_{-f})$ is strictly concave for fixed but arbitrary

$$q_{-f} \equiv (\, q_t \,)_{f \neq t \in \mathcal{F}} \in X_{-f} \equiv \prod_{f \neq t \in \mathcal{F}} X_t,$$

where

$$n \equiv \sum_{f \in \mathcal{F}} n_f.$$

Player f's (parametric) optimization problem: solve for q_f in

$$\begin{array}{|ll|} \hline \text{maximize} & \pi_f(q_f, q_{-f}) \\ \text{subject to} & q_f \in X_f \\ \hline \end{array}$$

where q_{-f} is taken as exogenous by firm f.

A tuple $q^* \equiv (q_f^* : f \in \mathcal{F})$ is a *Nash equilibrium* if for all $f \in \mathcal{F}$,

$$\pi_f(q^*) \geq \pi_f(q_f, q_{-f}^*), \quad \forall\, q_f \in X_f;$$

i.e., no player can obtain a higher payoff by unilaterally deviating from such an equilibrium strategy.

Can players earn higher payoffs by **colluding**?

Existence, uniqueness, characterization, and computation of a Nash equilibrium

- (Existence) A Nash equilibrium $q^{\mathsf{N}} \equiv (q_f^{\mathsf{N}})_{f \in \mathcal{F}}$ exists.

- (Uniqueness) if the vector function

$$F(q) \equiv -(\nabla_{q_f} \pi_f(q) : f \in \mathcal{F}), \quad q \in X \equiv \prod_{f \in \mathcal{F}} X_f$$

is strictly monotone on X, then q^{N} is unique.

- (Variational description) q^{N} satisfies the variational inequality:

$$F(q^{\mathsf{N}})^T(q - q^{\mathsf{N}}) \geq 0, \quad \forall\, q \in X;$$

- (Computation) via variational inequality/complementarity methods.

Players in Cournot electric power models

- generator firms (producing and selling electricity)

- Independent System Operator
(operating and maintaining the power grid)

- resource allocator
(allocating resources, such as fuel and emissions allowances)

- market traders
(arbitragers for price differentials among regions)

- secondary markets
(such as emission and capacity markets)

Concatenating these players' profit maximization problems plus market clearing conditions yields the overall equilibrium problem.

The ISO's problem

Taking the transmision fee w^* as exogenous to his problem, the ISO solves a linear program to maximize revenue subject to technological constraints:

$$\begin{array}{ll} \text{maximize} & y^T w^* \\ \text{subject to} & Hy \leq h, \end{array}$$

where $y \in \Re^n$ is the vector of transmission flows, and H is, e.g., the matrix of *power distribution factors* (PDFs).

Optimality conditions:

$$\boxed{w^* = H^T z \text{ and } 0 \leq z \perp h - Hy \geq 0,}$$

The resource allocator's problem

Taking the resource price ρ^* as exogenous to his problem, the resource allocator solves a linear program to maximize revenue subject to technological constraints:

$$\begin{array}{ll} \text{maximize} & u^T \rho^* \\ \text{subject to} & Du \leq d, \end{array}$$

where $u \in \Re^m$ is the vector of resource allocations.

Optimality conditions:

$$\boxed{\rho^* = D^T v \text{ and } 0 \leq v \perp d - Du \geq 0,}$$

The arbitrager's problem

Taking $p^* \in \Re^n$, $w^* \in \Re^n$, and $\rho^* \in \Re^m$ as exogenous to his problem, the arbitrager solves a linear program to maximize revenue subject to resource constraints:

$$\begin{array}{ll} \text{maximize} & a^T(\,p^* - w^*\,) - (\,r^a\,)^T \rho^* \\ \text{subject to} & Ga = Ge^0 \text{ and } E^a\,a = r^a + \omega^a, \end{array}$$

where $a \in \Re^n$ is the vector of arbitrage amounts and $r^a \in \Re^m$ is arbitrager's resource usage, and $\omega^a \in \Re^m$ is the pre-allocated resources owned by the arbitrager.

Optimality conditions:

$$\boxed{\begin{bmatrix} 0 & -G^T \\ G & 0 \end{bmatrix} \begin{pmatrix} a \\ \lambda \end{pmatrix} = \begin{pmatrix} (\,E^a\,)^T \rho^* - p^* + w^* \\ Ge^0 \end{pmatrix}.}$$

Generator firms' problems

- objective is revenue less costs; in turn,
  - revenue = price x sales
  - costs include production, transmission, and resource usage
  - advanced models incorporate price conjectures

- models distinguish themselves in one of two types:

- endogenous arbitrage: firms anticipate arbitrage endogenously in their profit maximization (Stackelberg model)

- exogenous arbitrage: firms take arbitrage as an exogenous parameter in their profit maximization. (Nash model)

- forward positioning (not considered herein)

Firm f's problem: endogenous arbitrage

$$\text{maximize} \quad \sum_{i=1}^{n} [\, s_{fi}\, p_{fi} - c_{fi}(g_{fi}) - (\, s_{fi} - g_{fi}\,)\, w_i^* \,] - \sum_{j=1}^{m} \rho_j^*\, r_{fj}$$

subject to

$$\sum_{i=1}^{n} s_{fi} = \sum_{i=1}^{n} g_{fi}, \quad \text{(sales = generations)}$$

$$B^f s^f + E^f g^f = r^f + \omega^f, \quad \text{(resource constraint)}$$

$$0 \leq g^f \leq \text{CAP}^f, \quad \text{(generation capacity)}$$

$$0 \leq s^f,$$

$$\begin{bmatrix} 0 & -G^T \\ G & 0 \end{bmatrix} \begin{pmatrix} a^f \\ \lambda \end{pmatrix} = \begin{pmatrix} (\, E^a\,)^T \rho^* - p^f + w^* \\ G e^0 \end{pmatrix}$$

Price functions

- Perfect competition: $p_{fi} = p_i^*$, (price is exogenous);

- Basic Cournot model: $p_{fi} \equiv p_i(S_i + a_{fi})$, where

$$S_i \equiv \sum_{f \in \mathcal{F}} s_{fi}, \text{ total sales by all firms in region } i,$$

(price is a function of sales; players exhibiting market power);

- Extended Cournot – Supply function conjecture:

$$s_{-fi} = s^*_{-fi} + \beta_{fi}(p_i^*, s^*_{-fi})\,(p_{fi} - p_i^*),$$

(firms anticipate that a deviation of the power price from its equilibrium level will stimulate a deviation in supply from rival firms from their equilibrium supplies).

Forms of $\beta_{fi}(x,y)$

- a constant: $\beta_{fi}(x,y) = \beta_{fi}$; yielding a fixed-slope model:

$$p_{fi} = \left(p_i^* - \frac{s_{-fi}^*}{\beta_{fi}} \right) + \frac{1}{\beta_{fi}} s_{-fi};$$

$$\beta_{fi} = \infty \quad \Rightarrow \quad \text{perfect competition}$$

$$\beta_{fi} = 0 \quad \Rightarrow \quad \text{basic Cournot model;}$$

- a rational function $\beta_{fi}(x,y) = \dfrac{y}{x - \alpha_{fi}}$; yielding a variable-slope model:

$$p_{fi} = \alpha_{fi} + \frac{p_i^* - \alpha_{fi}}{s_{-fi}^*} s_{-fi}$$

and a **nonlinear** complementarity problem.

Market clearing conditions

- $p^f = p^*$ (firms' prices = market prices)

- $\sum_{h \neq f, h \in \mathcal{F}} s_{hi} = s_{-fi} = s^*_{-fi}$ for all $f \in \mathcal{F}$ (similarly for sales)

- $u = \sum_{f \in \mathcal{F}} r^f + r^a$ (balance of resource uses)

- $y = \sum_{f \in \mathcal{F}} (s^f - g^f) + a$ (flow balance).

Complementarity formulation

$$\begin{array}{ccccc} 0 & \leq & v & \perp \\ 0 & \leq & z & \perp \\ 0 & \leq & \boldsymbol{s} & \perp \\ 0 & \leq & \boldsymbol{g} & \perp \\ 0 & \leq & \boldsymbol{\eta} & \perp \\ \text{free} & & \varphi & \end{array} \begin{pmatrix} q_v \\ q_z \\ \boldsymbol{q}_s \\ \boldsymbol{q}_g \\ \text{CAP} \\ 0 \end{pmatrix} + \boldsymbol{M} \begin{pmatrix} v \\ z \\ \boldsymbol{s} \\ \boldsymbol{g} \\ \boldsymbol{\eta} \\ \varphi \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{F}_s(\boldsymbol{s}, v, z) \\ \text{vec}(c'_{fi}(g_{fi})) \\ 0 \\ 0 \end{pmatrix} \begin{array}{cc} \geq & 0 \\ \geq & 0 \\ \geq & 0 \\ \geq & 0 \\ \geq & 0 \\ = & 0. \end{array}$$

An exogenous arbitrage version leads to same formulation with $\boldsymbol{F}_s$ replaced by another function.

An extension: piecewise linear price functions

$$p_i(\tau) \equiv \begin{cases} P_{i0} - \beta_{i0}\,\tau & \text{if } -\infty < \tau \leq \alpha_{i1} \\ p_i(\alpha_{ij}) - \beta_{ij}\,(\,\tau - \alpha_{ij}\,) & \text{if } \left\{ \begin{array}{ccccc} \alpha_{ij} & \leq & \tau & \leq & \alpha_{ij+1} \\ 1 & \leq & j & \leq & m-1 \end{array} \right\} \\ p_i(\alpha_{im}) - \beta_{im}\,(\,\tau - \alpha_{im}\,) & \text{if } \alpha_{im} \leq \tau < \infty, \end{cases}$$

where

$$\alpha_{i1} < \alpha_{i2} < \ldots < \alpha_{im} \text{ and } \beta_{im} > \ldots > \beta_{i0} \geq 0$$

are, respectively, the breakpoints and the negatives of the slopes of $p_i(\tau)$ in the respective intervals of linearity. Let

$$\alpha'_{ij} \equiv \alpha_{ij} - \alpha_{ij-1}, \;\; \forall\, j = 1, \ldots, m+1,$$

$$\beta'_{ij} \equiv \beta_{ij} - \beta_{ij-1}, \;\; \forall\, j = 1, \ldots, m.$$

Recall firm's problem (without arbitrage):

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i \in \mathcal{N}} \left[\, s_{fi}\, p_i(S_i) - c_{fi}\, g_{fi} - (\, s_{fi} - g_{fi}\,)\, w_i^* \,\right] \\ \text{subject to} & \displaystyle\sum_{i \in \mathcal{N}} s_{fi} = \sum_{i \in \mathcal{N}} g_{fi}, \\ \text{and} & \left\{ \begin{array}{l} 0 \leq s_{fi};\;\; 0 \leq g_{fi} \leq \text{CAP}_{fi} \\ S_i \equiv \displaystyle\sum_{h \in \mathcal{F}} s_{hi} \leq \sigma_i \end{array} \right\} \; \forall\, i \in \mathcal{N}. \end{array}$$

- Need partial subgradients of the revenue function

$$r_{fi}(s_{fi}, s_{-fi}) \equiv s_{fi}\, p_i(s_{fi} + s_{-fi})$$

with respect to s_{fi}.

- The sales cap constraint involves all firms' variables. Resulting equilibrium problem is of the generalized Nash type.

Complementarity representation of subgradients

A scalar $a \in \partial_{s_{fi}} r_{fi}(s_{fi}, s_{-fi})$ if and only if scalars $\{\tau_{i1}, \cdots, \tau_{im}\}$ and $\{v_{fi1}, \cdots, v_{fim}\}$ exist such that

$$a = P_{i0} - \beta_{i0} \left(s_{fi} + S_i - \sum_{k=1}^{m} \tau_{ik} \right) - \sum_{k=1}^{m} \beta_{ik-1}\, \tau_{ik} - \sum_{k=1}^{m} \beta'_{ik}\, v_{fik}$$

and

$$\begin{aligned}
0 \leq v_{fi1} \quad & \perp \ \alpha'_{i1} - \left(S_i - \sum_{j=1}^{m} \tau_{ij} \right) \geq 0 \\
0 \leq v_{fij+1} \quad & \perp \ \alpha'_{ij+1} - \tau_{ij} \geq 0, \ \ \forall\, j = 1, \ldots, m-1 \\
0 \leq \tau_{ij} \quad & \perp \ s_{fi} - v_{fij} + \tau_{ij} \geq 0, \ \ \forall\, j = 1, \ldots, m.
\end{aligned}$$

$$
\begin{aligned}
0 \leq s_{fi} \quad &\perp \quad c_{f1} - P_{i0} + \beta_{i0} \left(s_{fi} + \sum_{h \in \mathcal{F}} s_{hi} - \sum_{j=1}^{m} \tau_{fij} \right) + \sum_{j=1}^{m} \beta_{ij-1}\, \tau_{fij} + \\
&\qquad \sum_{j=1}^{m} \beta'_{ij} v_{fij} + \sum_{k \in \mathcal{K}} \mathsf{PDF}_{ik} \lambda_k - \psi_f + \gamma_{f1} + \varphi_{fi} \geq 0, \ \forall (f,i) \in \mathcal{F} \times \mathcal{N} \\
0 \leq g_{fi} \quad &\perp \quad (c_{fi} - c_{f1}) - \sum_{k \in \mathcal{K}} \mathsf{PDF}_{ik} \lambda_k + \psi_f - \gamma_{f1} + \gamma_{fi} \geq 0, \ \forall (f,i) \in \mathcal{F} \times (\mathcal{N} \setminus \{1\}) \\
0 \leq \psi_f \quad &\perp \quad g_{f1} \equiv \sum_{i \in \mathcal{N}} s_{fi} - \sum_{1 \neq i \in \mathcal{N}} g_{fi} \geq 0, \ \ \forall f \in \mathcal{F} \\
0 \leq \lambda_k \quad &\perp \quad T_k - \sum_{h \in \mathcal{F}} \sum_{1 \neq i \in \mathcal{N}} \mathsf{PDF}_{ik} (s_{hi} - g_{hi}) \geq 0, \ \ \forall k \in \mathcal{K} \\
0 \leq \gamma_{f1} \quad &\perp \quad \mathsf{CAP}_{f1} - \sum_{i \in \mathcal{N}} s_{fi} + \sum_{1 \neq i \in \mathcal{N}} g_{fi} \geq 0, \ \ \forall f \in \mathcal{F} \\
0 \leq \gamma_{fi} \quad &\perp \quad \mathsf{CAP}_{fi} - g_{fi} \geq 0, \ \ \forall (f,i) \in \mathcal{F} \times (\mathcal{N} \setminus \{1\}) \\
0 \leq \varphi_{fi} \quad &\perp \quad \sigma_i - \sum_{h \in \mathcal{F}} s_{hi} \geq 0 \ \ \forall (f,i) \in \mathcal{F} \times \mathcal{N}
\end{aligned}
$$

plus complementarity conditions on (v_{fij}, τ_{fij}).

Failure of Lemke's algorithm

A 3-firm, 2-node problem with the nodal price functions being linear and given by:

$$p_1(S_1) = 1 - \frac{1}{2}S_1 \text{ and } p_2(S_2) = 2 - \frac{1}{4}S_2.$$

There are two common coupling constraints:

$$S_1 \leq 3/4 \text{ and } S_2 \leq 2.$$

The unit production costs are as follows:

$$(c_{11}, c_{12}) = (0.5, 1), \ (c_{21}, c_{22}) = (0.5, 1.5), \text{ and } (c_{31}, c_{32}) = (1.5, 0.5).$$

The generation capacities are all equal to one unit; $\text{CAP}_{fi} = 1$ for all $f = 1, 2, 3$ and $i = 1, 2$.

Resulting LCP is of size 21. Ray termination due to degeneracy.

A simpler example

$$\begin{array}{lll} \text{maximize}_{x_1} & x_1 \left(1 - \frac{1}{2}x_1 - \frac{1}{2}x_2\right) \\ \text{subject to} & x_1 + x_2 \le 1 \\ \text{and} & x_1 \ge 0 \end{array} \qquad \begin{array}{lll} \text{maximize}_{x_2} & x_2 \left(2 - \frac{1}{2}x_1 - \frac{1}{2}x_2\right) \\ \text{subject to} & x_1 + x_2 \le 1 \\ \text{and} & x_2 \ge 0. \end{array}$$

Note the common constraint $x_1 + x_2 \le 1$. The LCP formulation:

$$\begin{array}{rcllcl} 0 & \le & -1 + x_1 + \frac{1}{2}x_2 + \varphi_1 & \perp & x_1 & \ge & 0 \\ 0 & \le & -2 + \frac{1}{2}x_1 + x_2 + \varphi_2 & \perp & x_2 & \ge & 0 \\ 0 & \le & 1 - x_1 - x_2 & \perp & \varphi_1 & \ge & 0 \\ 0 & \le & 1 - x_1 - x_2 & \perp & \varphi_2 & \ge & 0. \end{array}$$

To initiate Lemke's algorithm, we augment the LCP by adding an artificial variable z_0 and an artificial column of all ones:

$$\begin{array}{lllll} 0 & \leq & y_1 = -1 + z_0 + x_1 + \frac{1}{2} x_2 + \varphi_1 & \perp & x_1 & \geq & 0 \\ 0 & \leq & y_2 = -2 + z_0 + \frac{1}{2} x_1 + x_2 + \varphi_2 & \perp & x_2 & \geq & 0 \\ 0 & \leq & \mu_1 = 1 + z_0 - x_1 - x_2 & \perp & \varphi_1 & \geq & 0 \\ 0 & \leq & \mu_2 = 1 + z_0 - x_1 - x_2 & \perp & \varphi_2 & \geq & 0. \end{array}$$

In the first pivot, z_0 enters the basis driving out y_2. The next entering variable is thus x_2. At this point, there is a tie in the ratio test, with both μ_1 and μ_2 being the candidate variables to become nonbasic. The system is as follows:

$$
\begin{array}{rclcll}
0 \le & y_1 & = & 1 + y_2 + \frac{1}{2}\,x_1 - \frac{1}{2}\,x_2^{\uparrow} + \varphi_1 - \varphi_2 & \perp & x_1 \ge 0 \\
 & z_0 & = & 2 + y_2 - \frac{1}{2}\,x_1 - x_2^{\uparrow} - \varphi_2 & & \\
0 \le & \mu_1 & = & 3 + y_2 - 1.5\,x_1 - 2\,x_2^{\uparrow} - \varphi_2 & \perp & \varphi_1 \ge 0 \\
0 \le & \mu_2 & = & 3 + y_2 - 1.5\,x_1 - 2\,x_2^{\uparrow} - \varphi_2 & \perp & \varphi_2 \ge 0.
\end{array}
$$

If μ_1 is chosen to be the leaving variable, then φ_1 becomes the next entering variable, and ray termination occurs.

Instead, if μ_2 is chosen to be the leaving variable, then the method successfully computes the equilibrium solution of $(x_1, x_2) = (1, 0)$ and $(\varphi_1, \varphi_2) = (0, 3/2)$. Note that the two multipliers φ_1 and φ_2 are not equal in this solution. The problem has an equilibrium solution $(x_1, x_2) = (0, 1)$ with equal multipliers $\varphi_1 = \varphi_2 = 1$. Finally, every solution to the problem must have $\varphi_2 > 0$.

A restricted-multiplier formulation

We restrict the multiplier associated with the joint constraint to depend only on the region i and applies to all firms; i.e.,

$$0 \leq \varphi_i \perp \sigma_i - \sum_{h \in \mathcal{F}} s_{hi} \geq 0 \;\; \forall\, i \in \mathcal{N}$$

This formulation ensures the successful termination of Lemke's algorithm.

Lesson learnt: Generalized Nash equilibrium problems are challenging!

The DSL game

Users maximize the transmission data rates subject to transmission line capacities. Employing Shannon's informational entropy function as the objective, player i solves the concave maximization problem in the variables p_k^i for fixed but arbitrary p_k^j:

$$\text{maximize} \quad \sum_{k=1}^{n} \log \left(1 + \frac{p_k^i}{\sigma_k^i + \sum_{j \neq i} \alpha_k^{ij} \, p_k^j} \right) \quad \text{(a nonlinear function)}$$

$$\text{subject to} \quad 0 \leq p_k^i \leq \text{CAP}_k^i, \quad \forall\, k = 1, \ldots, n$$

$$\text{and} \quad \sum_{k=1}^{n} p_k^i \leq P_{\text{max}}^i,$$

where p_k^i is user i's signal power of tone $k = 1, \ldots, n$.

Proposition. If $P^i_{\max} < \sum_{k=1}^{n} \text{CAP}^i_k$, then player i's problem is equivalent to a mixed linear complementarity problem (MLCP):

$$0 \leq p^i_k \perp \sigma^i_k + \sum_{j=1}^{m} \alpha^{ij}_k p^j_k + v_i + \varphi^i_k \geq 0, \quad \forall\, k = 1, \ldots, n$$

$$v_i \text{ free}, \qquad P^i_{\max} - \sum_{k=1}^{n} p^i_k = 0$$

$$0 \leq \varphi^i_k \perp \text{CAP}^i_k - p^i_k \geq 0, \quad \forall\, k = 1, \ldots, n.$$

Concatenating these MLCPs for all the players, and simplifying, get a standard LCP with a *copositive-plus* matrix $\boldsymbol{M}$:

$$0 \leq z \perp \boldsymbol{q} + \boldsymbol{M} z \geq 0,$$

ensuring the successful application of Lemke's algorithm for computing a DSL equilibrium solution. In practice, very slow convergence.

Goal of a **collusive game**: players want to earn more than the Nash profits by colluding.

- Can that be done?
- What is a collusive strategy?
- Which collusive strategy should the players choose?
- How does one compute such a strategy?

A collusive strategy should be one that allows the firms to obtain higher profits than the Nash profits and yet offers no incentive for the firms to deviate from.

Define, for a discount factor $\delta \in [0,1]$, the **collusive set**:

$$\Omega_\delta \equiv \Big\{ q \in \prod_{f \in \mathcal{F}} X_f \, : \, \pi_f(q) \geq (1-\delta)\,\pi_f^*(q_{-f}) + \delta\,\pi_f^{\mathrm{N}},\ \forall\, f \in \mathcal{F} \Big\},$$

where $\pi_f^*(q_{-f}) \equiv \max_{q_f \in X_f} \pi_f(q_f, q_{-f})$ is firm f's optimal value function parameterized by rival firms' strategy q_{-f}.

- $\Omega_0 = \{ q^{\mathrm{N}} \}$ and $\Omega_1 = \{ q \in X : \pi_f(q) \geq \pi_f^{\mathrm{N}},\ \forall\, f \in \mathcal{F} \}$;
- for all $0 \leq \delta_1 \leq \delta_2 \leq 1$, $\Omega_{\delta_1} \subseteq \Omega_{\delta_2}$.

The Collusive Optimization Problem

The Nash Bargaining Objective function (Nash 1950):

$$\boxed{\begin{array}{ll} \text{maximize} & \displaystyle\prod_{f \in \mathcal{F}} \left[\pi_f(q) - \pi_f^{\mathsf{N}} \right] \\ \text{subject to} & q \in \Omega_\delta \end{array}}.$$

Computational challenges

- An optimal solution exists, albeit problem is *nonconvex*.

- Constraint functions in Ω_δ are at best SC^1; more precisely, the gradient $\nabla \pi_f^*(q_{-f})$, while exists, is at best "semismooth" and *not* differentiable.

- $\pi_f^*(q_{-f})$ is only implicitly available.

Competitive Capacity Expansion under Uncertainty (a two-stage Nash game)

N firms compete in $\mathcal{N}$ regions.

Firm ν chooses a vector of regional production capacities $x^\nu \in X^\nu$ (a closed convex set of admissible capacities).

Capacity is installed prior to demand/price being realized.

A regional price $p_j(Q_j, \omega)$ that is a function of the random vector ω and $Q_j \equiv \sum_{\nu=1}^{N} q_j^\nu$ (the total production by all firms in region j).

For each realization of ω and capacity vector x^ν, firm ν determines its production quantities q^ν by maximizing its profit in anticipation of other firms' productions $q^{-\nu}$:

$$\begin{array}{ll} \text{maximize} & \pi_\nu(q,\omega) \equiv \sum_{j \in \mathcal{N}} \left[p_j(Q_j,\omega) q_j^\nu - c_j^\nu(q^\nu) \right] \\ \text{subject to} & q^\nu \in S^\nu(x^\nu,\omega) \end{array}$$

where $c_j^\nu(q^\nu)$ is firm ν's production cost and

$$S^\nu(x^\nu,\omega) \equiv \{ q^\nu \in \Re^{m_\nu} : W^\nu q^\nu \leq h^\nu(\omega) - B^\nu(\omega) x^\nu \}$$

is firm ν's set of admissible productions, with W^ν being a (constant) recourse matrix, $B^\nu(\omega)$ a random technology matrix, and $h^\nu(\omega)$ a random vector.

Let $q^*(x,\omega) \equiv (q^{\nu,*}(x,\omega))_{\nu=1}^N$ be the vector of the firms' equilibrium productions obtained by solving the Nash subgame defined by the maximization problems.

This is the **second-stage** game (in production).

Let $\vartheta_\nu(x,\omega) \equiv \pi_\nu(q^*(x,\omega),\omega)$ be firm ν's maximum profit as a function of all firms' capacity vector x and the random vector ω.

The **first-stage** game is to find a capacity tuple $x^* \equiv (x^{\nu,*})_{\nu=1}^N$ such that, for all $\nu = 1, \ldots N$,

$$\theta_\nu(x^*) \leq \theta_\nu(x^\nu, x^{-\nu,*}), \ \forall\, x^\nu \in X^\nu,$$

where

$$\theta_\nu(x) \equiv \eta_\nu(x^\nu) - \mathbb{E}_\omega \vartheta_\nu(x,\omega)$$

is firm ν's net cost of securing the capacity x^ν less the expected maximum profit.

Approximating $\mathbb{E}_\omega \vartheta_\nu(x,\omega)$ via Sampling:

Let $\{\omega^1, \cdots, \omega^\ell\}$ be ℓ independently and identically distributed samples of the random vector. Consider the random functions:

$$\begin{aligned} \theta_{\nu,\ell}(x) &\equiv \eta_\nu(x^\nu) - \frac{1}{\ell}\sum_{i=1}^{\ell} \vartheta_\nu(x,\omega^i) \\ &= \eta_\nu(x^\nu) - \frac{1}{\ell}\sum_{i=1}^{\ell} \pi_\nu(q^*(x,\omega^i),\omega^i) \end{aligned}$$

and the approximate Nash problem wherein player ν solves

$$\text{minimize } \theta_{\nu,\ell}(x) \text{ subject to } x^\nu \in X^\nu.$$

Let $\widehat{x}^\ell \equiv (\widehat{x}^{\nu,\ell})_{\nu=1}^N$ be an equilibrium solution to the approximated capacity game. (This solution is a random vector.)

Convergence via a new epi-convergence concept.

Concluding Remarks

We have discussed

- some realistic Nash equilibrium models in electricity markts;
- the failure of Lemke's algorithm on some of these models, one of which is known to be theoretically solvable by the algorithm;
- a capacity expansion model under uncertainty.

> Realistic Nash models are complex and involve many considerations, giving rise to practical challenges whose resolution remains elusive to date.