# Introduction to Optimization

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# Overview

An optimization problem is to find a maximizer or minimizer of a given function subject to a given set of constraints that must be satisfied by any solution. Mathematically, it can be written in the form:

$$\begin{array}{ll} \text{maximize (or minimize)} & f(x) \\ \text{subject to} & x \in \Omega, \end{array}$$

where $f$ is a given function from a general multidimensional space $\mathbb{R}^d$ to the set of reals $R$, and $\Omega$ is a subset of $\mathbb{R}^d$ defined by various conditions. For example, the following is an instance of the optimization problem:

$$\begin{array}{lrcl} \text{maximize} & f((x_1, x_2)) := 3x_1^2 & + & x_2^2 \\ \text{subject to} & x_1 \geq 0 & & \\ & & & x_2 \geq 0 \\ & x_1 & + & x_2 \leq \frac{7}{3} \\ & x_1 \text{ is integer.} & & \end{array}$$

In this example, the dimension $d$ of the underlying space is 2, and the region of all "feasible" solutions is

$$\Omega = \{x \in R^2 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq \frac{7}{3}, x_1 \text{ is integer}\}.$$

One of the main goals is to find algorithms to find an optimal solution, that is, a vector $x \in \Omega$ maximizing (or minimizing) the objective function $f$. Furthermore, whenever possible, we look for an "efficient" algorithm. Efficiency can be defined in many different ways but for the users of optimization techniques, the most important one is (loosely defined) practical efficiency that allows a computer implementation to return a (correct) optimal solution in a practically acceptable time. We will study a theoretical efficiency in this lecture that provides excellent guidelines for practical efficiency.

The optimization problem itself is a very general problem which cannot be treated uniformly. We must consider various classes of special subproblems, defined by function types that can appear in the formulation, or restricted by whether some variables take only integer values, etc. The optimization problem contains many "easy" classes of problems that admit efficient algorithms. Among them are the linear programming problem, network flow problems and convex programming problems. On the other hand there are many "hard" classes of problems, such as the integer programming problem and non-convex optimization problems, that demand much more sophisticated techniques and require much more time than the easy problems of the same size.

One important emphasis is to understand **certificates** for optimality. When an algorithm correctly solves an optimization problem, it finds not only an optimal solution but a certificate that guarantees the optimality. In general, easy optimization problems admit a simple ("succinct") certificate so that the verification of optimality is easy. We shall study various types of certificates for efficiently solvable optimization problems. On the other hand, for hard problems that do not seem to admit a succinct certificate, we shall study algorithms that search for optimal or approximative solutions using exhaustive search, heuristic search and other techniques.

## Three Main Themes of Optimization

1. **Linear Programming or Linear Optimization (LP)**

$$\begin{array}{lrcl} \text{maximize} & c^T x & & \qquad (x \in R^n) \\ \text{subject to} & Ax & \leq & b \\ & x & \geq & \mathbf{0}. \end{array}$$

   Solvable by highly efficient algorithms. Practically no size limit. The duality theorem plays a central role.

2. **Combinatorial Optimization**

$$\begin{array}{lrcl} \text{maximize} & c^T x & & \\ \text{subject to} & x & \in & \Omega. \end{array}$$

   Here $\Omega$ is a "discrete" set, e.g.

$$\Omega = \{x \in R^n : Ax \leq b, x_j = 0 \text{ or } 1 \text{ for all } j\}.$$

   Includes both easy and hard problems, i.e. P (polynomially solvable) and NP-Complete. Must learn how to recognize the hardness of a given problem, and how to select appropriate techniques.

3. **Nonlinear Programming or Nonlinear Optimization (NLP)**

$$\begin{array}{llll} \text{maximize} & f(x) & & \\ \text{subject to} & g_i(x) \leq & 0 & \qquad \text{for } i = 1, \ldots, m, \end{array}$$

   where $f(x)$ and $g_i(x)$ are real-valued functions: $R^n \rightarrow R$.

   Convexity plays an important role. Interior-point algorithms solve "convex" NLP efficiently, including LP.

# Chapter 1

# Introduction to Linear Programming

## 1.1 Importance of Linear Programming

- Many applications

  Optimum allocation of resources
  – optimum production/allocation of resources, production scheduling, diet planning

  Transportation problems
  – Transsphipment problems, minimum cost flows, maximum flows, shortest path problems

  Work force planning
  – Optimal assignment of jobs, scheduling of classes

- Large-scale problems solvable

  **Solution methods**

  | | | |
  |---|---|---|
  | – Simplex method | Dantzig | 1947 |
  | – Interior-point methods | Karmarkar et al. | 1984 – |
  | – Combinatorial methods | Bland et al. | 1977 – |

  One can solve LP's with a large number (up to millions) of variables and constraints, and there are many reliable LP codes:

  – CPLEX, IMSL, LINDO, MINOS, MPSX, XPRESS-MP, etc.

- LP techniques can be used to solve much harder problems:

  – combinatorial optimization, integer programming problems, etc.

- Beautiful theory behind it!

## 1.2 Examples

**Example 1.1** *Optimum allocation of resources*

*Chateau ETH produces three different types of wines, Red, Rose and White, using three different types of grapes planted in its own vineyard. The amount of each grape necessary to produce a unit amount of each wine, the daily production of each grape, and the profit of selling a unit of each wine is given below. How much of each wines should one produce to maximize the profit? We assume that all wines produced can be sold.*

| | *wines* | | | |
|---|---|---|---|---|
| | *Red* | *White* | *Rose* | |
| *grapes* | | | | *supply* |
| *Pinot Noir* | *2* | *0* | *0* | *4* |
| *Gamay* | *1* | *0* | *2* | *8* |
| *Chasselas* | *0* | *3* | *1* | *6* |
| | | *(ton/unit)* | | *(ton/day)* |
| | *3* | *4* | *2* | |
| | *profit (K sf/unit)* | | | |

- Trying to produce the most profitable wine as much as possible.
  Limit of 2 units of white.
- The remaining resources allows 2 units of red. So,
  2 units of red, 2 units of white.
- By reducing 1 unit of white, one can produce
  2 units of red, 1 unit of white, 3 units of rose.

| | |
|---|---|
| Question 1 | Is this the best production? A proof? |
| Question 2 | Maybe we should sell the resources to wine producers? |
| Question 3 | How does the profitability affect the decision? |
| | the production quantities $\cdots$ ? |

**Vineyard's Primal LP (Optimize Production)**

$x_1$: Red, $x_2$: White, $x_3$: Rose (units).

$$\boxed{\begin{array}{lrcrcrcrl} \text{max} & 3x_1 & + & 4x_2 & + & 2x_3 & & & \Leftarrow \text{Profit} \\ \text{subject to} & 2x_1 & & & & & \leq & 4 & \Leftarrow \text{Pinot} \\ & x_1 & & & + & 2x_3 & \leq & 8 & \Leftarrow \text{Gamay} \\ & & & 3x_2 & + & x_3 & \leq & 6 & \Leftarrow \text{Chasselas} \\ & x_1 \geq 0, & & x_2 \geq 0, & & x_3 & \geq & 0 & \end{array}}$$

**Remark 1.1** *When any of the variable(s) above is restricted to take only integer values* 0, 1, 2, ..., *the resulting problem is called an integer linear program (IP or ILP) and much harder to solve in general because it belongs to the class NP-complete, the notion discussed in Chapter 5. There are some exceptions, such as the assignment problem and the maximum flow problem, that can be solved very efficiently.*

**Remark 1.2** *It is uncommon to mix red and white grapes to produce a rose wine. A typical way is by using only red grapes and removing skins at an early fermentation stage. Thus, our wine production problem (Example 1.1) does not reflect the usual practice. Nevertheless, mixing different types of grapes, in particular for red wines, is extremely common in France, Italy and Switzerland.*

**Example 1.2** *Optimum allocation of jobs*

*ETH Watch Co. has $P$ workers who are assigned to carry out $Q$ tasks. Suppose the worker $i$ can accomplish $m_{ij}$ times the work load of task $j$ in one hour ($m_{ij} > 0$). Also it is required that the total time for the worker $i$ cannot exceed $C_i$ hours. How can one allocate the tasks to the workers in order to minimize the total amount of working time?*

Mathematical Modeling

Let $x_{ij}$ be the time assigned to worker $i$ for task $j$.

$$\begin{array}{ll} \text{min} & \displaystyle\sum_{i,j} x_{ij} \\ \text{subject to} & \\ & \displaystyle\sum_{j=1}^{Q} x_{ij} \leq C_i \quad (i = 1, \ldots P), \\ & \displaystyle\sum_{i=1}^{P} m_{ij} x_{ij} = 1 \quad (j = 1, \ldots, Q), \\ & x_{ij} \geq 0 \quad (i = 1, \ldots P; j = 1, \ldots, Q). \end{array}$$

## 1.3 Linear Programming Problems

linear function

$$f(x_1, x_2, \cdots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

where $c_1, c_2, \cdots, c_n$ are given real numbers and $x_1, x_2, \cdots, x_n$ are variables.

linear equality

$$f(x_1, x_2, \cdots, x_n) = b$$

where $f$ is a linear function and $b$ is a given real number.

linear inequality

$$\begin{aligned} f(x_1, x_2, \cdots, x_n) &\geq b \\ f(x_1, x_2, \cdots, x_n) &\leq b \end{aligned}$$

A linear constraint means either a linear equality or inequality.

Linear Programming Problem or LP

It is a problem to maximize or minimize a linear function over a finite set of linear constraints:

$$\begin{array}{lll} \max & c_1x_1 + c_2x_2 + \cdots + c_nx_n & \\ \text{subject to} & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i & (i = 1, \cdots, k) \\ & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i & (i = k+1, \cdots, k') \\ & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i & (i = k'+1, \cdots, m) \end{array}$$

Here, $c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is called the objective function.

Quiz Decide for each of the following problems whether it is an LP or not.

1.

$$\begin{array}{llcll} \text{max} & 2x & + & 4y & \\ \text{subject to} & x & - & 3y & = 5 \\ & & & y & \leq 0 \end{array}$$

2.

$$\begin{array}{llclll} \text{max} & 2x & + & & 4y & \\ \text{subject to} & x & - & & 3y & = 5 \\ & x & \geq 0 & \text{or} & y & \leq 0 \end{array}$$

3.

$$\begin{array}{llclcll} \text{max} & x & + & y & + & z & \\ \text{subject to} & x & + & 3y & - & 3z & < 5 \\ & x & - & 5y & & & \geq 3 \end{array}$$

4.

$$\begin{array}{llclcll} \text{min} & x^2 & + & 4y^2 & + & 4xy & \\ \text{subject to} & x & + & 2y & & & \leq 4 \\ & x & - & 5y & & & \geq 3 \\ & \multicolumn{5}{l}{x \geq 0, \quad y \geq 0} \end{array}$$

5.

$$\begin{array}{llclcll} \text{min} & x_1 & + & 2x_2 & - & x_3 & \\ \text{s. t.} & x_1 & \geq 0 & x_2 & \geq 0 & & \\ & x_1 & + & 4x_2 & & & \leq 4 \\ & & & x_2 & + & x_3 & \leq 4 \\ & \multicolumn{6}{l}{x_1, x_2, x_3 \text{ are integers.}} \end{array}$$

6.

$$\begin{array}{llclcll} \text{min} & 2x_1 & - & x_2 & - & 3x_3 & \\ \text{s. t.} & x_1 & + & 4x_2 & & & \leq 4 \\ & & & x_2 & + & x_3 & \leq 4 \\ & x_1 & \geq 0 & x_2 & \geq 0 & & \\ & \multicolumn{6}{l}{x_1 \text{ is integer.}} \end{array}$$

7.

$$\begin{array}{llclclll} \text{min} & x_1 & + & 2x_2 & - & x_3 & & \\ \text{s. t.} & x_1 & + & 4x_2 & + & x_3 & \leq & 4 \\ & 3x_1 & & x_2 & + & x_3 & \leq & 4 \\ & \multicolumn{7}{l}{x_1, x_2, x_3 \text{ are either 0 or 1.}} \end{array}$$

## 1.4 Solving an LP: What does it mean?

Key words
optimal, unbounded, infeasible

Optimal Production Problem of Chateau ETH

$x_1$: Red, $x_2$: White, $x_3$: Rose (units).

$$
\begin{array}{llcccccl}
\text{max} & 3x_1 & + & 4x_2 & + & 2x_3 & & \Leftarrow \text{Profit} \\
\text{subject to} & 2x_1 & & & & & \leq 4 & \Leftarrow \text{Pinot} \\
& x_1 & & & + & 2x_3 & \leq 8 & \Leftarrow \text{Gamay} \\
& & & 3x_2 & + & x_3 & \leq 6 & \Leftarrow \text{Chasselas} \\
& x_1 \geq 0, & & x_2 \geq 0, & & x_3 & \geq 0 &
\end{array}
$$

- Feasible solution
  a vector that satisfies all constraints:

$$
\begin{array}{ll}
(x_1, x_2, x_3) = (0, 0, 0) & \textit{yes} \\
(x_1, x_2, x_3) = (1, 1, 1) & \textit{yes} \\
(x_1, x_2, x_3) = (2, 1, 3) & \textit{yes} \\
& \\
(x_1, x_2, x_3) = (3, 0, 0) & \textit{no} \\
(x_1, x_2, x_3) = (2, -1, 0) & \textit{no}
\end{array}
$$

- Feasible region
  the set $\Omega$ of all feasible solutions $x = (x_1, x_2, x_3)^T$. Figure 1.1 shows this region. Geometrically the feasible region is a *convex polyhedron*.

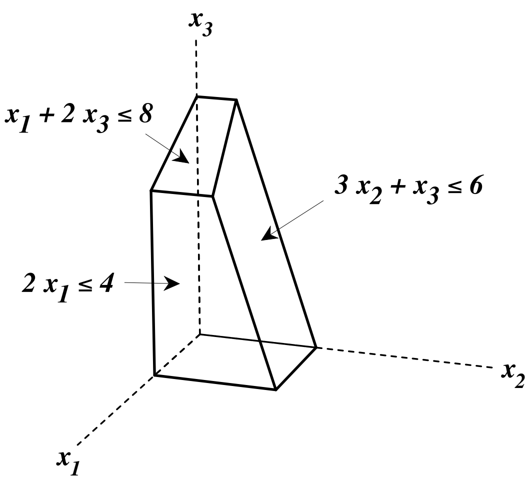


Figure 1.1: Feasible Region $\Omega$

- Optimal solution
  a feasible solution that optimizes (maximizes or minimizes) the objective function among all the feasible solutions.

- LP may not admit an optimal solution. There are two such cases:

  (1) Infeasible case

$$
\begin{array}{lrcrcrl}
\max & x_1 & + & 5x_2 & & & \\
\text{subject to} & x_1 & + & x_2 & \geq & 6 & \leftarrow \text{conflicting} \\
& -x_1 & - & x_2 & \geq & -4 & \leftarrow \text{constraints}
\end{array}
$$

  This LP has no feasible solution. $\Longrightarrow$ It is said to be infeasible.

  (2) Unbounded case

$$
\begin{array}{lrcrcr}
\max & 2x_1 & - & x_2 & & \\
\text{subject to} & -x_1 & + & x_2 & \leq & 6 \\
& -x_1 & - & 3x_2 & \leq & -4
\end{array}
$$

  The objective function is not bounded (above for maximization, below for minimization) in the feasible region. More formally it means that for any real number $k$ there exists a feasible solution whose objective value is better (larger for maximization, smaller for minimization) than $k$.

  $\Longrightarrow$ An LP is said to be unbounded

- Fundamental Theorem

**Theorem 1.3** *Every LP satisfies exactly one of the three conditions:*

(1) *it is infeasible;*

(2) *it is unbounded;*

(3) *it has an optimal solution.*

- Solving an LP means

| |
|---|
| Derive the conclusion 1, 2 or 3, and exhibit its certificate. |

For example, the simplex method is a method solving an LP. A certificate is an extra information with which one can prove the correctness of the conclusion easily. We shall see certificates for 1, 2 and 3 in Chapter 2.

## 1.5 History of Linear Programming

| Military | Economy/Industry | Linear Programming | Mathematics |
|---|---|---|---|
| | | | Inequality Theory |
| Military<br>20th Century | Input-Output Model<br>Leontief (1936) | | Fourier (1923)<br>Gordan (1873)<br>Farkas (1902)<br>Motzkin (1936)<br>Game Theory<br>von Neumann &<br>Morgenstern (1944) |
| Linear Programming<br>(1947) | | Simplex Method<br>Danzig (1947) | |
| | Economic Model<br>Koopmans (1948) | Duality Theory<br>von Neumann (1947) | |
| | (Nobel Prize<br>Koopmans<br>Kantorovich<br>(1975)<br>Opt. resourse alloc.) | Combinatorial Algo.<br>Bland etc. (1977 )<br>Polynomial Algo.<br>Khachian (1979) | |
| | | New Polynomial Algo.<br>Karmarkar (1984) | |

Note: A polynomial or polynomial-time algorithm means a theoretically efficient algorithm. Roughly speaking, it is defined as an algorithm which terminates in time polynomial in the binary size of input. This measure is justified by the fact that any polynomial time algorithm runs faster than any exponential algorithm for problems of sufficiently large input size. Yet, the polynomiality merely guarantees that such an algorithm runs not too badly for the worst case. The simplex method is not a polynomial algorithm but it is known to be very efficient method in practice.

# Chapter 2

# LP Basics I

## 2.1 Recognition of Optimality

$$
\begin{array}{lrcrcrcr}
\text{max} & 3x_1 & + & 4x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
\text{E1:} & 2x_1 & & & & & \leq & 4 \\
\text{E2:} & x_1 & & & + & 2x_3 & \leq & 8 \\
\text{E3:} & & & 3x_2 & + & x_3 & \leq & 6 \\
\text{E4:} & x_1 & \geq 0, & x_2 & \geq 0, & x_3 & \geq & 0
\end{array}
$$

How can one convince someone (yourself, for example) that the production (Red 2, White 1 and Rose 3 units) is optimal?

$$
\begin{aligned}
&(x_1, x_2, x_3) = (2, 1, 3)\\
&\text{profit} = 3 \times 2 + 4 \times 1 + 2 \times 3 = 16
\end{aligned}
$$

- Because we have checked many (say 100,000) feasible solutions and the production above is the best among them...
- Because CPLEX returns this solution and CPLEX is a famous (and very expensive) software, it cannot be wrong.
- We exhausted all the resources and thus we cannot do better.

Are these reasonings correct?

- An inequality that is satisfied by any feasible solution.

  Every feasible solution $(x_1, x_2, x_3)$ satisfies E1 $\sim$ E4, and thus in particular it must satisfy any positive combinations of E1 and E3:

$$
\begin{array}{lrcrcrcr}
2 \times \text{E1:} & 4x_1 & & & & & \leq & 8 \\
2 \times \text{E3:} & & & 6x_2 & + & 2x_3 & \leq & 12
\end{array}
$$

whose sum gives:

$$(2.1) \qquad 2 \times \text{E1} + 2 \times \text{E3}: \quad 4x_1 + 6x_2 + 2x_3 \leq 20$$

The LHS of this inequality can be related to the objective function.

$$(2.2) \qquad \text{profit} = 3x_1 + 4x_2 + 2x_3$$

In fact, it **OVERESTIMATE**s the objective value for any feasible solution, since the coefficients of $x_1, x_2, x_3$ in (2.1) are greater than or equal to the corresponding terms in the objective function, and all variables are restricted to be nonnegative.

Therefore, we know

$$\text{profit} = 3x_1 + 4x_2 + 2x_3 \leq 4x_1 + 6x_2 + 2x_3 \leq 20$$

is valid for any feasible solution $(x_1, x_2, x_3)$. More precisely,

> By taking a linear combination of the constraints, we concluded that the objective value cannot exceed 20.

Can we do better than this to lower the upper bound to 16? This would prove the optimality of $(x_1, x_2, x_3) = (2, 1, 3)$. In fact this is possible. Add the inequalities E1, E2, E3 with coefficients $4/3, 1/3, 4/3$:

$$\boxed{\frac{4}{3} \times \text{E1} + \frac{1}{3} \times \text{E2} + \frac{4}{3} \times \text{E3}: \quad 3x_1 + 4x_2 + 2x_3 \leq 16.}$$

Finding such coefficients is a mystery (for the moment). Nevertheless, we could prove the optimality of the production $(x_1, x_2, x_3) = (2, 1, 3)$.

By solving an LP by the simplex algorithm or by any reasonable algorithm, we obtain a vector of these mysterious coefficients as well as an optimal solution. This vector is called the dual price.

## 2.2 Dual Problem

In the previous section, we showed how one can prove the optimality of our small LP by taking a proper linear combination of the constraints.

The **DUAL** problem of an LP is in fact an LP of finding mysterious coefficients of the original constraints to get the best upper bound of the objective function.

We use nonnegative variables $y_1$, $y_2$ and $y_3$ as the (unknown) coefficients of E1, E2 and E3, respectively, to obtain a general inequality.

$$\boxed{\begin{array}{l} y_1 \times \mathrm{E1} + y_2 \times \mathrm{E2} + y_3 \times \mathrm{E3}: \\ \qquad (2y_1 + y_2)x_1 + (3y_3)x_2 + (2y_2 + y_3)x_3 \le 4y_1 + 8y_2 + 6y_3 \end{array}}$$

where $y_1 \ge 0$, $y_2 \ge 0$, $y_3 \ge 0$.

For the RHS of the inequality to be an upper bound of the objective value (and thus for the LHS to become an overestimate of the objective function), the following conditions are sufficient:

$$\boxed{\begin{array}{rcrcrcl} 2y_1 & + & y_2 & & & \ge & 3 \\ & & & & 3y_3 & \ge & 4 \\ & & 2y_2 & + & y_3 & \ge & 2 \end{array}}\quad.$$

Therefore, the problem of finding the best (smallest) upper bound is again an LP:

**Example 2.1 (The Dual of Chateau ETH Problem:)**

$$\boxed{\begin{array}{lrcrcrcl} \min & 4y_1 & + & 8y_2 & + & 6y_3 & & \\ \text{subject to} & 2y_1 & + & y_2 & & & \ge & 3 \\ & & & & & 3y_3 & \ge & 4 \\ & & & 2y_2 & + & y_3 & \ge & 2 \\ & y_1 & \ge 0, & y_2 & \ge 0, & y_3 & \ge & 0 \end{array}}$$

This problem is defined as the *dual problem* of the LP.

For any LP in canonical form:

$$\begin{array}{lrcrcccrcl} \max & c_1x_1 & + & c_2x_2 & + & \cdots & + & c_nx_n & & \\ \text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \le & b_1 \\ & \vdots & & \vdots & & & & \vdots & & \vdots \\ & a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \le & b_m \\ & x_1 & \ge 0, & x_2 & \ge 0, & & \ldots & x_n & \ge 0, & \end{array} \tag{2.3}$$

we define the dual problem as the LP:

$$\begin{array}{lrcrcccrcl} \min & b_1y_1 & + & b_2y_2 & + & \cdots & + & b_my_m & & \\ \text{subject to} & a_{11}y_1 & + & a_{21}y_2 & + & \cdots & + & a_{m1}y_m & \ge & c_1 \\ & \vdots & & \vdots & & & & \vdots & & \vdots \\ & a_{1n}y_1 & + & a_{2n}y_2 & + & \cdots & + & a_{mn}y_m & \ge & c_n \\ & y_1 & \ge 0, & y_2 & \ge 0, & & \ldots & y_m & \ge 0. & \end{array} \tag{2.4}$$

The original LP is sometimes called the *primal* problem to distinguish it from the dual problem. Using matrices, one can write these LPs as:

$$\begin{array}{lll} \max & c^T x & \\ \text{subject to} & Ax & \leq b \\ & x & \geq \mathbf{0}, \end{array} \tag{2.5}$$

$$\begin{array}{lll} \min & b^T y & \\ \text{subject to} & A^T y & \geq c \\ & y & \geq \mathbf{0}. \end{array} \tag{2.6}$$

Here $b$ and $c$ are the column vectors $(b_1, b_2, \ldots, b_m)^T$ and $(c_1, c_2, \ldots, c_n)^T$, and $A$ is the $m \times n$ matrix having $a_{ij}$ in the $(i, j)$ position. $\mathbf{0}$ denotes a column vector of all 0's of appropriate size. A vector inequality (or equality) means the component-wise simultaneous inequalities (equalities), for example, $u \geq v$ means $u_j \geq v_j$ for all $j$.

The canonical form of an LP (2.3) is a special form of the general LP problem: it is a maximization problem with no equality constraints, all variables restricted to be nonnegative and all other inequalities in one form LHS $\leq$ RHS.

The dual LP is not in canonical form as it is. However, there is a trivial transformation to a canonical form LP. Simply replace the objective function with its negative, minimization with maximization, and replace the reversely oriented inequalities with their $-1$ multiplications. One can transform any LP problem to an equivalent LP in canonical form.

Quiz Show that the dual problem of the dual LP is equivalent to the primal problem.

The following theorem is quite easy to prove. In fact, we proved it for our small LP and the same argument works for the general case.

**Theorem 2.1 (Weak Duality)** *For any pair of primal and dual feasible solutions $x = (x_1, x_2, \cdots, x_n)^T$ and $y = (y_1, y_2, \cdots, y_m)^T$*

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i \qquad (c^T x \leq b^T y). \tag{2.7}$$

One important consequence of the weak duality is:

> If the equality is satisfied in (2.7) by some primal and dual feasible solutions $x$ and $y$, then they are both optimal.

Prove it by using the definition of optimality.

The following theorem shows that the equality is always satisfied by some pair of feasible solutions if they exist. This means that the optimality of a solution to an LP can be ALWAYS verified by exhibiting a dual optimal solution (certificate for optimality).

**Theorem 2.2 (Strong Duality)** *If an LP has an optimal solution $\overline{x} = (\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_n)^T$ then the dual problem has an optimal solution $\overline{y} = (\overline{y_1}, \overline{y}_2, \cdots, \overline{y}_m)^T$ and their optimal values are equal:*

$$\sum_{j=1}^{n} c_j \overline{x}_j = \sum_{i=1}^{m} b_i \overline{y}_i \qquad (c^T \overline{x} = b^T \overline{y}). \tag{2.8}$$

This is considered as the most important theorem in LP theory. Unlike the weak duality theorem, the strong duality is not easy to prove. We leave the proof to the Advanced Part (Chapter 4) of the lecture note.

There is an alternative way to write the optimality criterion for a dual pair of feasible solutions: $c^T \overline{x} = b^T \overline{y}$, which is sometimes more useful.

**Theorem 2.3 (Complementary Slackness Conditions)** *For a dual pair of feasible solutions $\overline{x}$ and $\overline{y}$ the following conditions are equivalent:*

(a) *both $\overline{x}$ and $\overline{y}$ are optimal solutions;*

(b) $c^T \overline{x} = b^T \overline{y}$;

(c) $\overline{y}^T (b - A\overline{x}) = 0$ *and* $\overline{x}^T (A^T \overline{y} - c) = 0$.

(c') $\overline{y}_i (b - A\overline{x})_i = 0$ *for all* $i$ *and* $\overline{x}_j (A^T \overline{y} - c)_j = 0$ *for all* $j$.

(c") $\overline{y}_i > 0$ *implies* $(b - A\overline{x})_i = 0$, *for all* $i$ *and*
$\overline{x}_j > 0$ *implies* $(A^T \overline{y} - c)_j = 0$, *for all* $j$.

## 2.3 Recognition of Infeasibility

Consider the following LP:

**Example 2.2**

$$\begin{array}{lrcrcrcr}
\text{max} & 3x_1 & + & 4x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
\text{E1:} & 2x_1 & & & & & \leq & 4 \\
\text{E2:} & -x_1 & & & - & 2x_3 & \leq & -15 \\
\text{E3:} & & & 3x_2 & + & x_3 & \leq & 6 \\
\text{E4:} & x_1 & \geq 0, & x_2 & \geq 0 & x_3 & \geq & 0
\end{array}$$

How can one prove the infeasibility of this LP? Clearly

- I have checked 100,000,000 candidates for feasibility and none is feasible. Thus the LP is infeasible.

has no sense!

Actually we can use essentially the same technique as for the optimality. That is to use linear combinations.

Taking the combination of E1, E2, E3 with coefficient 1/2, 1 and 2, we obtain an inequality $6x_2 \leq -1$ which must be satisfied by any feasible solution. Now this inequality contradicts with $x_2 \geq 0$. Therefore there is no feasible solution. This kind of proof is in fact always possible by the following theorem:

**Theorem 2.4 (Farkas' Lemma)** *A system of linear inequalities* $\{Ax \leq b$ *and* $x \geq \mathbf{0}\}$ *has no solution if and only if the system* $\{y \geq \mathbf{0},\ A^T y \geq \mathbf{0}$ *and* $b^T y < 0\}$ *has a solution.*

One can easily verify: if there exists $y \in R^m$ such that $y \geq \mathbf{0}$, $A^T y \geq \mathbf{0}$ and $b^T y < 0$, then there is no solution to the system $Ax \leq b$ and $x \geq \mathbf{0}$. The hard part of the proof is the converse.

## 2.4 Recognition of Unboundedness

Consider the LPs:

**Example 2.3**

$$
\begin{array}{lrcrcrcl}
\text{max} & 3x_1 & + & 4x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
\text{E1:} & 2x_1 & & & & & \leq & 4 \\
\text{E2:} & x_1 & & & + & 2x_3 & \leq & 8 \\
\text{E3:} & & & -3x_2 & + & x_3 & \leq & 6 \\
\text{E4:} & x_1 & \geq 0, & x_2 & \geq 0 & x_3 & \geq & 0
\end{array}
$$

**Example 2.4**

$$
\begin{array}{lrcrcrcl}
\text{max} & 3x_1 & - & 4x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
\text{E1:} & -2x_1 & & & & & \leq & 4 \\
\text{E2:} & x_1 & & & - & 2x_3 & \leq & 8 \\
\text{E3:} & & & -3x_2 & + & x_3 & \leq & 6 \\
\text{E4:} & x_1 & \geq 0, & x_2 & \geq 0 & x_3 & \geq & 0
\end{array}
$$

It is easy to see that these two problems are feasible. For example, the origin $x = (0,0,0)^T$ is feasible for both. What about unboundedness?

By a little observation, one can see the objective function is not bounded above for the first problem. One can increase the value of $x_2$ by any positive $\alpha$ at any feasible solution, e.g. $(0, \alpha, 0)^T$, we obtain a feasible solution whose objective value is increased by $4\alpha$. Since $\alpha$ can take any positive value, the objective function is unbounded above in the feasible region.

Thus we have a certificate of unboundedness, namely, one feasible solution together with a "direction" $(0,1,0)^T$ which can be added to the feasible solution with any positive multiple to stay feasible and to increase the objective value.

For the second problem, one has to be a little bit more careful to find such an unbounded direction. Consider the direction $(1,1,1)^T$. If any positive ($\alpha$) multiple of this direction is added to any feasible solution, the objective value increases by $\alpha$ ($= (3-4+2)\alpha$). On the other hand the feasibility will be preserved as well (why?).

It turns out that for any unbounded LP, the same certificate exists and thus one can easily verify the unboundedness.

**Theorem 2.5 (Unboundedness Certificate)** *An LP*

$$\max\ c^T x \textit{ subject to } Ax \le b \textit{ and } x \ge \mathbf{0}$$

*is unbounded if and only if it has a feasible solution $x$ and there exists (a direction) $z$ such that $z \ge \mathbf{0}$, $Az \le \mathbf{0}$ and $c^T z > 0$.*

Quiz Solve the above LPs, Example 2.3 and Example 2.4 by an LP code and study the results. Does it give a certificate for infeasibility/unboundedness?

## 2.5 Dual LP in Various Forms

In Section 2.2, we defined the dual problem of an LP in canonical form. In this section, we present the dual problems of LPs in other forms, that can be obtained by first converting them to canonical form, applying the definition of the dual problem, and then doing some simple equivalence transformations. These allow a direct application of the duality theorems to LPs in different forms.

First of all, we remark two equivalences of linear constraints:

(2.9) (Equality) $a^T x = b \Longleftrightarrow a^T x \le b$ and $-a^T x \le -b$

(2.10) (Free variable) $x_j$ free $\Longleftrightarrow x_j = x'_j - x''_j, x'_j \ge 0$ and $x''_j \ge 0$.

**Proposition 2.6** *For each (P*) of the LPs in LHS, its dual LP is given by the corresponding LP (D*) in RHS below:*

| | |
|---|---|
| (P1) $\begin{array}{llrl} \max & & c^T x & \\ \text{s.t.} & & Ax & = b \\ & & x & \ge \mathbf{0} \end{array}$ | (D1) $\begin{array}{llrl} \min & & b^T y & \\ \text{s.t.} & & y & \text{free} \\ & & A^T y & \ge c \end{array}$ |
| (P2) $\begin{array}{llrl} \max & & c^T x & \\ \text{s.t.} & & Ax & \le b \\ & & x & \text{free} \end{array}$ | (D2) $\begin{array}{llrl} \min & & b^T y & \\ \text{s.t.} & & y & \ge \mathbf{0} \\ & & A^T y & = c \end{array}$ |
| (P3) $\begin{array}{llcrl} \max & (c^1)^T x^1 & + & (c^2)^T x^2 & \\ \text{s.t.} & A^{11} x^1 & + & A^{12} x^2 & = b^1 \\ & A^{21} x^1 & + & A^{22} x^2 & \le b^2 \\ & x^1 & \text{free} & & \\ & & & x^2 & \ge \mathbf{0} \end{array}$ | (D3) $\begin{array}{llcrl} \min & (b^1)^T y^1 & + & (b^2)^T y^2 & \\ \text{s.t.} & & & y^1 & \text{free} \\ & & & y^2 & \ge \mathbf{0} \\ & (A^{11})^T y^1 & + & (A^{21})^T y^2 & = c^1 \\ & (A^{12})^T y^1 & + & (A^{22})^T y^2 & \ge c^2 \end{array}$ |

**Proof.** Consider the form (P1). By (2.9), it is equivalent to an LP in canonical form:

$$\text{(P1')} \quad \begin{array}{lrl} \max & c^T x & \\ \text{s.t.} & Ax & \le b \\ & -Ax & \le -b \\ & x & \ge \mathbf{0}. \end{array}$$

By the definition (2.4) of the dual, we obtain

$$\text{(D1')} \quad \begin{array}{lrl} \max & b^T y' - b^T y'' & \\ \text{s.t.} & A^T y' + (-A^T) y'' & \ge c \\ & y', y'' & \ge \mathbf{0}. \end{array} \quad \begin{array}{lllrll} [ & \Longleftrightarrow & \max & b^T (y' - y'') & & ] \\ [ & \Longleftrightarrow & & A^T (y' - y'') & \ge c & ] \\ [ & \Longleftrightarrow & & y', y'' & \ge \mathbf{0}. & ] \end{array}$$

By (2.10), the problem (D1') is equivalent to (D1). Proofs for the rest are left for exercise. ∎

# Chapter 3

# LP Basics II

## 3.1 Interpretation of Dual LP

Chateau EPFL is interested in purchasing high quality grapes produced at Chateau ETH.

> In order for Chateau EPFL to buy the grapes from Chateau ETH, how should they decide the prices?

| | Red | wines White | Rose | |
|---|---|---|---|---|
| grapes | | | | supply |
| Pinot Noir | 2 | 0 | 0 | 4 |
| Gamay | 1 | 0 | 2 | 8 |
| Chasselas | 0 | 3 | 1 | 6 |
| | | (ton/unit) | | (ton/day) |
| | 3 | 4 | 2 | |
| | | profit (K sf/unit) | | |

First of all, we set the prices as variables:

| | | |
|---|---|---|
| Pinot | $y_1$ | (K sf/ton) |
| Gamay | $y_2$ | (K sf/ton) |
| Chasselas | $y_3$ | (K sf/ton). |

Chateau ETH can generate 3K francs profit by production of one unit of red wine, the total sale price of the grapes for the production (Pinot 2 tons and Gamay 1 ton) should not be lower than that.

$$2y_1 + 1y_2 \geq 3 \qquad \text{Red wine constraint} \tag{3.1}$$

Similarly, we must have

$$3y_3 \geq 4 \qquad \text{White wine constraint} \tag{3.2}$$
$$2y_2 + 1y_3 \geq 2 \qquad \text{Rose wine constraint.} \tag{3.3}$$

Clearly Chateau EPFL's main interest is to minimize the purchase cost of the grapes, and so the pricing problem is the LP:

$$\boxed{\begin{array}{lrcrcrcl} \text{min} & 4y_1 & + & 8y_2 & + & 6y_3 & & \text{(Minimizing the total cost)} \\ \text{subject to} & 2y_1 & + & y_2 & & & \geq & 3 \\ & & & & & 3y_3 & \geq & 4 \\ & & & 2y_2 & + & y_3 & \geq & 2 \\ & y_1 & \geq 0, & y_2 & \geq 0, & y_3 & \geq & 0. \end{array}}$$

This problem is somewhat familiar, isn't it? In fact it is precisely the dual problem we defined in the previous section.

Chateau ETH's problem:

$$\begin{array}{lrcrcrcl} \text{max} & 3x_1 & + & 4x_2 & + & 2x_3 & & \\ \text{subject to} & 2x_1 & & & & & \leq & 4 \\ & x_1 & & & + & 2x_3 & \leq & 8 \\ & & & 3x_2 & + & x_3 & \leq & 6 \\ & x_1 & \geq 0, & x_2 & \geq 0, & x_3 & \geq & 0 \end{array} \qquad \text{opt. sol. } \overline{x} = (2,1,3)^T$$

Its dual = Chateau EPFL's problem:

$$\begin{array}{lrcrcrcl} \text{min} & 4y_1 & + & 8y_2 & + & 6y_3 & & \\ \text{subject to} & 2y_1 & + & y_2 & & & \geq & 3 \\ & & & & & 3y_3 & \geq & 4 \\ & & & 2y_2 & + & y_3 & \geq & 2 \\ & y_1 & \geq 0, & y_2 & \geq 0, & y_3 & \geq & 0 \end{array} \qquad \text{opt. sol. } \overline{y} = (4/3, 1/3, 4/3)^T$$

The weak duality theorem says:

- The total purchase cost of Chateau EPFL cannot be less than the total profit of production at Chateau ETH.

The strong duality theorem says:

- If both parties behave optimally, the total purchase cost for Chateau EPFL is equal to the total profit of wine production at Chateau ETH.
- Thus, for Chateau ETH it does not make any difference in profit by producing wines or selling the grapes.

## 3.2 Exercise(Pre-sensitivity Analysis)

Since Chateau ETH has a long relation with their neighbor Chateau EPFL, they have decided to sell Gamay grape to Ch. EPFL who has a very little grape harvest this year. The selling price is fixed to theoretically sound 1/3 (K sf/ton), but Ch. ETH wants to maintain the same total profit. Can they sell any amount of Gamay with this price?

Change the amount of Gamay sold to EPFL gradually, solve the resulting LP's with an LP code and graph in Firgure 3.1 the total profit (sum of wine production profit and grape selling profit) to check the critical point(s).

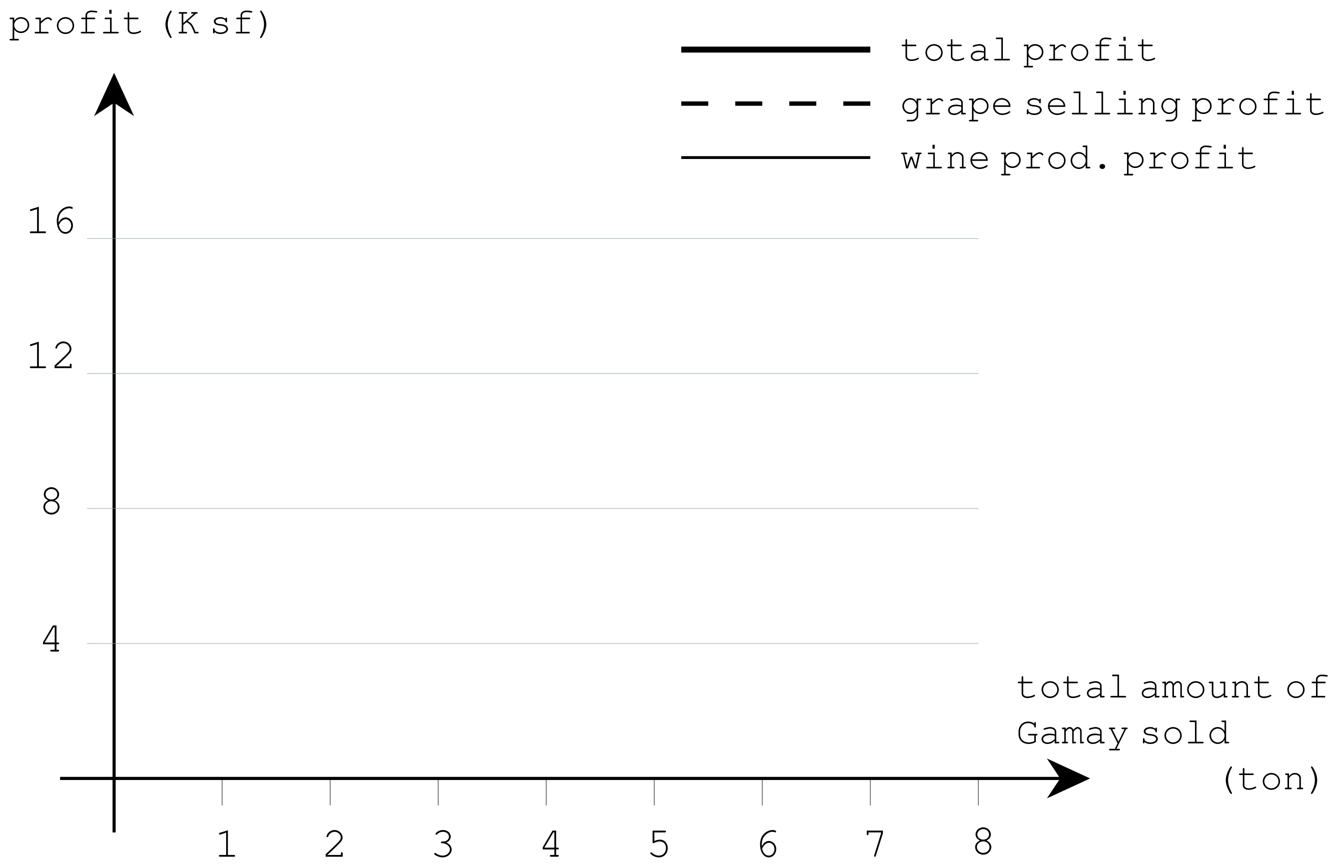


Figure 3.1: Profit Analysis on Gamay Selling with the Fixed Price 1/3