Recap:

1.) RDFTs, Yevgen's question

2.) Infinite space models

## Infinite space model (continued)

Four cases ($C = T, U, V, W$) with monomial left signal extension.

Consider $C = T$ ($T$-transform)

$T$: $1, x, 2x^2-1, 4x^3-3x, \ldots\ldots$, $T_{-n} = T_n$
$\quad T_0 \quad T_1 \quad T_2 \quad T_3$

parameterization 1 (power form): $x = \frac{u+u^{-1}}{2}$, $T_k(x) = \frac{u^k+u^{-k}}{2}$

" 2 (trigon. form): $x = \cos\theta$, $T_k(x) = \cos k\theta$

Filtering:

$$\left(\sum_{k\geq 0} h_k T_k\right)\left(\sum_{n\geq 0} s_n T_n\right) = \sum_{n\geq 0}\sum_{k+\ell=n} h_k s_\ell (T_k T_\ell) = \ldots = \sum_{n\geq 0} t_n T_n$$

or $$\left(\sum h_k \frac{u^k+u^{-k}}{2}\right)\left(\sum s_n \frac{u^n+u^{-n}}{2}\right) = \ldots\ldots = \sum t_n \frac{u^n+u^{-n}}{2}$$

shows that ($T$-transform) space convolution

$$\hat{h} * \hat{s} = \hat{t}, \quad \hat{h}, \hat{s} \text{ right-sided}$$

works as:

a.) extend $\hat{h}, \hat{s}$ symmetrically

b.) perform time convolution ($\rightarrow$ symmetric result)

c.) extract right half

# Spectrum / FT:

**time** (use $u$ as variable) $\qquad$ **space** (T-trafo)

signals: $s = S(u) = \sum_{n \in \mathbb{Z}} s_n u^n \xrightarrow{x = \frac{u+u^{-1}}{2}} s = S(x) = \sum_{n \geq 0} s_n T_n(x)$

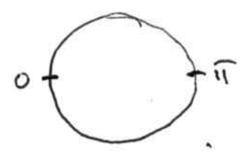
pure frequency: $E_\omega(u) = \sum_{n \in \mathbb{Z}} e^{j\omega n} u^n$, $\omega \in [-\pi, \pi)$ $\qquad$ $E_\omega(x) = \frac{1}{2} + \sum_{n \geq 1} \cos n\omega \, T_n(x)$, $\omega \in [0, \pi)$

FT (coord. free): $S(u) \mapsto \left(S(e^{j\omega}) E_\omega(u)\right)_\omega$ $\qquad$ $S(x) \mapsto \left(S(\cos\omega) E_\omega(x)\right)_\omega$

FT (coordinates): $\hat{s} \mapsto \left(S(e^{j\omega})\right)_{\omega \in [-\pi, \pi)}$ $\qquad$ $\hat{s} \mapsto \left(S(\cos\omega)\right)_{\omega \in [0, \pi]}$

$= \omega \mapsto S(e^{j\omega})$ $\qquad$ $= \omega \mapsto S(\cos\omega)$

function on: $u = e^{j\omega}$: $\qquad$ function on: $x = \frac{u+u^{-1}}{2} = \cos\omega$ :





$S(e^{j\omega}) = \sum_{n \in \mathbb{Z}} s_n e^{jn\omega}$ $\qquad$ $S(\cos\omega) = \sum_{n \geq 0} s_n \cos n\omega$

Frequ. response of $h$: $h = H(u)$, $\left(H(e^{j\omega})\right)_{\omega \in [-\pi, \pi)}$ $\qquad$ $h = H(x)$, $\left(H(\cos\omega)\right)_{\omega \in [0, \pi]}$

$= \omega \mapsto H(e^{j\omega})$ $\qquad$ $= \omega \mapsto H(\cos\omega)$

namely: $H(u) \cdot E_\omega(u) = H(e^{j\omega}) \cdot E_\omega(u)$ $\qquad$ $H(x) \cdot E_\omega(x) = H(\cos\omega) E_\omega(x)$

# Finite space models

**shift / linear extension / realization:**

yields signals $\sum s_e C_e$ , $C \in \{T, U, V, W\}$, but

$$C_0 - C_1 \cdots C_{n-2} - C_{n-1} \quad ?$$

Need for b.c.:

$$C_n = \sum_{k=0}^{n-1} \beta_k C_k$$

$$\Rightarrow \mathcal{A} = \mathcal{M} = \mathbb{C}[x] / C_n - \sum_{k=0}^{n-1} \beta_k C_k$$

**Lemma:** The above b.c. yields a monomial right signal extension if and only if $C_n = C_{n-2}$ or $C_n = 0$ or $C_n = \pm C_{n-1}$ ($C \in \{T, U, V, W\}$).

proof (sketch): ....

Notes:
- the right b.c.'s are precisely the mirrored versions of the left b.c.'s
- total of 16 cases

**Resulting signal models:**

We define a total of 16 finite space models for
- $C \in \{T, U, V, W\}$
- $p \in \{C_n - C_{n-2}, C_n, C_n - C_{n-1}, C_n + C_{n-1}\}$

namely:

$$\mathcal{A} = \mathbb{C}[x]/p(x) = \Big\{ \sum_{k=0}^{n-1} h_k T_k(x) \;\Big|\; \hat{h} \in \mathbb{C}^n \Big\}$$

$$\mathcal{M} = \quad " \quad = \Big\{ \sum_{e=0}^{n-1} s_e C_e(x) \;\Big|\; \hat{s} \in \mathbb{C}^n \Big\}$$

$$\Phi : \mathbb{C}^n \to \mathcal{M}$$

$$\hat{s} \mapsto \sum_{e=0}^{n-1} s_e C_e$$

# Overview table

| left b.c. \ right b.c. | | $s_n = s_{n-2}$ | $s_n = 0$ | $s_n = s_{n-1}$ | $s_n = -s_{n-1}$ |
|---|---|---|---|---|---|
| T | $s_{-1} = s_1$ | DCT-1 | DCT-3 | types 5-8 | |
| U | $s_{-1} = 0$ | | | | |
| V | $s_{-1} = s_0$ | types 5-8 | | DCT-2 | DCT-4 |
| W | $s_{-1} = -s_0$ | | | | |

yields DCTs/DSTs type 1-8 as FTs.

## Example 1: DCT-3

signal model: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/T_n(x)$, T-basis $b = \{T_0, \dots, T_{n-1}\}$

shift matrix:

$$\varphi(x) = \begin{pmatrix} 0 & 1/2 & & & \\ 1 & 0 & \ddots & & \\ & 1/2 & \ddots & 1/2 & \\ & & \ddots & 0 & 1/2 \\ & & & 1/2 & 0 \end{pmatrix}$$

visualization:  (scaled by 1/2)

signal extension:

left: $T_{-\ell} = T_\ell$ (WS)

right: $T_{n+\ell} = -T_{n-\ell}$ (WA)

spectrum and FT: zeros of $T_n$?

$T_n(x) = \cos(n \operatorname{acos} x) \Rightarrow$ zeros: $\cos \frac{k+1/2}{n}\pi$, $0 \le k < n$

$$\mathcal{F}: \mathbb{C}[x]/T_n(x) \longrightarrow \bigoplus_{k=0}^{n-1} \mathbb{C}[x]/x - \cos\frac{k+1/2}{n}\pi$$

as matrix

$$\mathcal{P}_{b,\alpha} = [T_\ell(\alpha_k)] = \left[\cos\frac{(k+\frac{1}{2})\ell}{n}\pi\right] = \mathrm{DCT\text{-}3}_n$$