Recap:

$$\begin{tikzcd}
 & & \text{regular signal model} \arrow[dl, "\varphi(x)"'] \\
\text{weighted graph} \arrow[r, shift left, "\text{adjac. matrix}"] & \text{matrix } A \arrow[l, shift left] \arrow[ur, dashed, bend right, "m_A(x)=p_A(x)"'] \arrow[dr, dashed, "A\geq 0"] & \\
 & & \text{Markov chain} \arrow[ul, bend left, "\text{prob. transition matrix}"]
\end{tikzcd}$$

Next topic: linear statistic models $\overset{?}{\longleftrightarrow}$ algebraic models (also linear)

KLT — FT

covariance matrix — filter matrix

Reminder: random variables and vectors

$\boxed{(a+bi)^* = a-bi}$

random vars

random variable x (function $S \to \mathbb{C}$)

expected value $\mu_x = E(x)$

variance $var(x) = \sigma_x^2 = E(|x-\mu_x|^2)$

random vectors

random vector: $\hat{x} = (x_0, \dots, x_{n-1})^T$, x_i random vars

expected value: $E(\hat{x}) = \mu_{\hat{x}} = (\mu_{x_0}, \dots, \mu_{x_{n-1}})^T$

(auto)covariance: $\Sigma_{\hat{x}} = E((\hat{x}-\mu_{\hat{x}})(\hat{x}-\mu_{\hat{x}})^H) = [\gamma_{i,j}]_{0\leq i,j<n}$

- $\gamma_{i,j} = E((x_i-\mu_{x_i})(x_j-\mu_{x_j})^*)$ "covariance between x_i and x_j"

- $\gamma_{i,i} = \sigma_{x_i}^2$

$| \cdot — = \square$

- $\rho_{i,j} = \dfrac{\gamma_{i,j}}{\sigma_i \sigma_j}$ "correlation coefficient"

(always ≤ 1, $\rho_{i,i} = 1$)

- $\Sigma_{\hat{x}}$ diagonal: the random vars $x_0, \dots, x_{n-1}$ are decorrelated (white)
- $\Sigma_{\hat{x}}$ is hermitian ($A = A^H$) and even positive semidefinite (hermitian + no negative Eigenvalues)

$\Rightarrow$ diagonalized by a unitary matrix + Eigenvalues are real

- conversely: every positive semi-definite matrix is a covariance matrix

cross-covariance:

$$\Sigma_{\hat{x},\hat{y}} = E\left((\hat{x}-\mu_{\hat{x}})(\hat{y}-\mu_{\hat{y}})^H\right)$$

<u>linear transformations</u>

$\hat{y} = A\hat{x}, \quad A \in \mathbb{C}^{n\times n}$

$$\Rightarrow \mu_{\hat{y}} = E(A\hat{x}) = A\cdot E(\hat{x}) = A\mu_{\hat{x}} \qquad \text{(linearity of } E)$$

$$\Sigma_{\hat{x},\hat{y}} = E\left((\hat{x}-\mu_{\hat{x}})(\hat{y}-\mu_{\hat{y}})^H\right) = E\left((\hat{x}-\mu_{\hat{x}})(A\hat{x}-A\mu_{\hat{x}})^H\right)$$

$$= \Sigma_{\hat{x}}\cdot A^H$$

$$\Sigma_{\hat{y},\hat{x}} = A\cdot\Sigma_{\hat{x}}$$

$$\Sigma_{\hat{y}} = A\cdot\Sigma_{\hat{x}}\cdot A^H$$

Choosing A as the unitary matrix that diagonalizes $\Sigma_{\hat{x}}$, $\hat{x}$ can be <u>decorrelated</u>. This A is called the Karhunen-Loève transform (KLT) for $\hat{x}$.
(KLT ~ mid 1940s)

- The same idea underlies principal component analysis (PCA)

<u>Random fields</u>

Assume n random variables $\hat{s} = (s_0,\dots,s_{n-1})^T$ (the signal)
n random variables $\hat{\nu} = (\nu_0,\dots,\nu_{n-1})^T$ (the noise or error)

which are related as

$$s_k = a_{k,k-m}s_{k-m} + \dots + a_{k,k}s_k + \dots + a_{k,k+m}s_{k+m} + \nu_k, \quad 0 \le k < n$$

In words: s_k is a linear combination of its nearest neighbours up to an error ν_k.

If s_i exceeds the boundary, i.e., $i<0$ or $i \geq n$, b.c.'s are assumed and reflected in above equations.

In vector form:

$$\hat{s} = A\hat{s} + \hat{\nu} \iff (I - A)\hat{s} = \hat{\nu}, \qquad A = [a_{k,\ell}]_{0 \leq k, \ell < n}$$

We assume $\hat{\nu}$ (and hence $\hat{s}$) is zero-mean Gaussian.

Under these assumptions, we call $\hat{s}$ an m-th order <u>Gauss-Markov random field</u> (GMRF).

In the following we assume A is real.

Terminology: $a_{k,k+i} = 0 \Rightarrow$ causal, $a_{k,\ell}$ independent of $k \Rightarrow$ homogeneous/stationary

<u>Minimum-mean square error (MMSE) estimate</u>

Clearly: $\Sigma_{\hat{\nu}} = (I_n - A)\Sigma_{\hat{s}}(I_n - A)^T$

<u>Case 1:</u> $I_n - A$ positive semi-definite (includes symmetric)

$\Rightarrow (I_n - A)\hat{s} = \hat{\nu}$ is in its MMSE representation (since $I_n - A$ is a valid covariance matrix)

solution: $\Sigma_{\hat{\nu}} = \sigma^2(I_n - A)$ (noise is correlated)

$\Sigma_{\hat{s}} = \sigma^2(I_n - A)^{-1}$

$\Sigma_{\hat{s},\hat{\nu}} = \sigma^2 I_n$ in this case

<u>Case 2:</u> Else.

$$\Rightarrow \underbrace{(I_n - A)^T(I_n - A)}_{\text{valid covariance matrix}}\hat{s} = (I_n - A)^T\hat{\nu} \quad \text{is the MMSE representation}$$

solution: $\Sigma_{\hat{\nu}} = \sigma^2 I_n$

$\Sigma_{\hat{s}} = \sigma^2(I_n - A)^{-1}(I_n - A^T)^{-1}$

<u>Recap:</u> - $\hat{x} = (x_0, \ldots, x_{n-1})^T$ random vector

- $\mu_{\hat{x}}$ mean
- $\Sigma_{\hat{x}} = E((\hat{x}-\mu_{\hat{x}})(\hat{x}-\mu_{\hat{x}})^H)$ covariance matrix (pos. semidef)
- KLT: unitary matrix that diagonalizes $\Sigma_{\hat{x}}$

GMRF: $\hat{s} = (s_0, \ldots, s_{n-1})^T$ random vector, assume zero-mean, real

$$s_k = a_{k,k-m} s_{k-m} + \ldots + a_{k,k-1} s_{k-1} + a_{k,k+1} s_{k+1} + \ldots + a_{k,k+m} s_{k+m} + \nu_k$$

- mth order
- vector form: $\hat{s} = A\hat{s} + \hat{\nu} \iff (I - A)\hat{s} = \hat{\nu}$

implies $\Sigma_{\hat{\nu}} = (I_n - A)\Sigma_{\hat{s}}(I_n - A)^T$

<u>Usual setups:</u>

Forward: GMRF is given through A and $\Sigma_{\hat{\nu}}$

Inverse: - given $\Sigma_{\hat{s}}$, m

- find A such that the minimum mean square error (MMSE)

$$E(\nu_i^2) = tr(\Sigma_{\hat{\nu}}) \text{ is minimized}$$

this is equivalent to making $\hat{s}, \hat{\nu}$ orthogonal

$$(I_n - A)\Sigma_{\hat{s}} = \Sigma_{\hat{\nu},\hat{s}} \approx \sigma^2 I_n$$

Case 1: solved with pos. sdef $I_n - A$ (implies symmetric)

$$\Rightarrow \Sigma_{\hat{s}} = \sigma^2 (I_n - A)^{-1}$$

$$\Sigma_{\hat{\nu}} = \sigma^2 (I_n - A)$$

Case 2: solved with other $I_n - A$

- transform: $\underbrace{(I_n - A)^T (I_n - A)}_{\text{pos. semidef}} \hat{s} = \underbrace{(I_n - A)^T \hat{\nu}}_{\hat{\mu}}$

- solve with previous case:

$$\Sigma_{\hat{s}} = \sigma^2 (I_n - A)^{-1} (I_n - A)^{-T}$$

$$(I_n - A)^T \Sigma_{\hat{\nu}} (I_n - A) = \Sigma_{\hat{\mu}} = \sigma^2 (I_n - A)^T (I_n - A) \Rightarrow \Sigma_{\hat{\nu}} = \sigma^2 I_n$$

Summary: For a GMRF (its MMSE estimate), the KLT diagonalizes

case 1: $(I_n - A)^{-1}$ i.e. $I_n - A$ i.e. A

case 2: $(I_n - A)^{-1}(I_n - A)^{-T}$ i.e. $(I_n - A^T)(I_n - A)$

Example: discrete space (non-causal)

Consider the GMRF (non-causal, homogeneous)

$$s_k = a(s_{k-1} + s_{k+1}) + \nu_k$$

Requires b.c.'s s_{-1} and s_n. No clear guideline what to choose. We draw from ASP, i.e., import b.c.'s from the finite space models.

For example $s_{-1} = s_0$, $s_1 = s_{n-1}$ $\iff$ $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/(x-1)U_{n-1}$

Φ = finite V-transform

$$\Rightarrow A = a \cdot \begin{pmatrix} 1 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 1 \end{pmatrix}$$

Determine KLT.

$I_n - A$ is symmetric, but pos. semidef.? Need Eigenvalues.

- Write $I_n - A$ as filter matrix $\varphi(h)$

$$\varphi(x) = \frac{1}{2}\begin{pmatrix} 1 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 1 \end{pmatrix} \Rightarrow I_n - A = \varphi(1 - 2ax)$$

- General theorem:

$$\mathcal{F} \cdot \varphi(h) \cdot \mathcal{F}^{-1} = \operatorname{diag}_{k}(h(\alpha_k))$$

here: $\mathcal{F}$ = DCT-2

$$\alpha_k = \cos\frac{k\pi}{n}, \quad 0 \le k < n$$

$\Rightarrow$ Evalues of $I_n - A$: $1 - 2a\cos\frac{k\pi}{n}$

Result: $I_n - A$ pos. semidef. $\iff$ $|a| \le \frac{1}{2}$

Now KLT:

Case 1: $|a| \le \frac{1}{2}$

KLT diagonalizes $\varphi(1-2ax) \Leftrightarrow$ diagonalizes $\varphi(x)$

hence KLT = (orth.) DCT-2, (unique) up to an orthogonal diagonal D: KLT = $D \cdot$ orth. DCT-2

Case 2: $|a| > \frac{1}{2}$

KLT diagonalizes $(I_n - A^T)(I_n - A) = \varphi(1-2ax)^2$
$= \varphi((1-2ax)^2)$

- orth. DCT-2 is a KLT
- but there are more choices if the squaring produces multiple Eigenvalues
  $\Leftrightarrow (1-2ax)^2$ ~~map~~ is not injective on $\cos\frac{k\pi}{n}$

What other GMRFs have DCT-2 as KLT?

Matrices diagonalized by DCT-2 $\Leftrightarrow \varphi(h)$

fix $m < n \Rightarrow h = a_0 T_0 + 2a_1 T_1 + \dots + 2a_m T_m$

corresponds to GMRF: $s_k = a_0 s_k + a_1(s_{k-1} + s_{k+1}) + \dots + a_m(s_{k-m} + s_{k+m}) + \nu_k$

with the b.c.'s associated with the algebraic model

<u>Theorem</u>: Let $1 \le m < n$. Every (orth.) DTT_n is a KLT for the homogeneous mth order GMRF

$$s_k = a_1(s_{k-1} + s_{k+1}) + \dots + a_m(s_{k-m} + s_{k+m}) + \nu_k$$

with the b.c.'s given by the signal extension of the algebraic model associated with the respective DTT.

On the Equivalence of GMRFs and signal models

Theorem: Consider a symmetric regular signal model $(\mathcal{A}, \mathcal{M}, \Phi)$ with diagonalizable shift matrix $\varphi(x) = A$ and a GMRF defined by this A.

Case 1: if $I_n - A$ is pos. semidef. then every KLT for the GMRF is an orth. FT for the algebraic model and vice-versa.

Case 2: Every orth. FT is a KLT. The converse holds if and only if $x \mapsto (1-x)^2$ is injective on the zeros of $p(x)$.

Proof:

Case 1: KLT diagonalizes $I_n - A$

$\Leftrightarrow$ " A

$\Leftrightarrow$ KLT is orth FT

Case 2: orth FT diagonalizes A

$\Rightarrow$ " $(I - A)^2$

$\mathcal{F}(I-A)^2\mathcal{F}^{-1} = \mathrm{diag}\left((1-\alpha_0)^2, (1-\alpha_1)^2, \dots, (1-\alpha_{n-1})^2\right)$

if 2 of these are equal then there are more choices for the KLT.