Finite shift-invariant 2D signal models

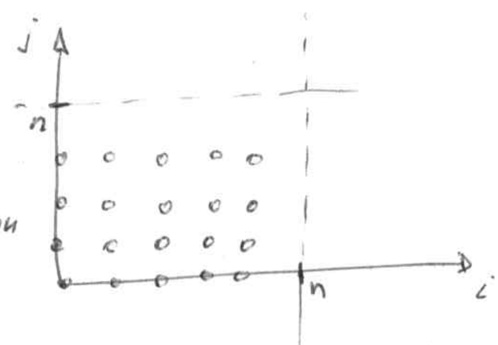
2D $\Rightarrow$ 2 shifts, e.g. x and y.

Then $\mathcal{A} = \mathbb{C}[x,y] / \langle ? \rangle = \mathbb{C}[x,y] / I$

- I is an ideal: $I = \langle p_1(x,y), \dots, p_k(x,y) \rangle$; $k \geq 2$ for finite-dimensional $\mathcal{A}$.

- $\mathbb{C}[x,y]$ is not a P.I.D., e.g. $I = \langle x^2, y^2 \rangle$.
- I can have arbitrary many generators; e.g. $I = \langle x^k y, x^{k-1} y^2, \dots, x y^{k-1} \rangle$.
- I needs ≥ 2 generators to have a finite basis; e.g. $\{x^i y^j\}$

$\mathbb{C}[x,y] / \langle x^n - 1, y^n - 1 \rangle$ has finite dimention dim $= n^2$



$\mathbb{C}[x,y] / (x^n - 1)$ still has infinite dimention

Assumption: We assume $I = \langle p(x,y), q(x,y) \rangle$, where

$$p(x,y) = x^n + \sum_{l,k=0}^{n-1} p_{k,l} x^k y^l$$

$$q(x,y) = y^n + \sum_{l,m=0}^{n-1} q_{m,l} x^m y^l$$

Then $\dim I = n^2$

Signal model

Assume finite 2D regular model:

$$\mathcal{A} = \mathcal{M} = \mathbb{C}[x,y]/\langle p(x,y), q(x,y)\rangle \;, \quad \text{basis } b = \{p_0(x,y), \ldots, p_{n^2-1}(x,y)\}$$

$$\Phi : \mathbb{C}^{n^2} \to \mathcal{M} \qquad (\mathbb{C}^{n^2} \text{ is a space of } n^2\times 1 \text{ } \underline{\text{vectors}})$$

$$\hat{s} \mapsto \sum_{k=0}^{n^2-1} s_k p_k(x,y)$$

Example (for the whole lecture):

$$\mathcal{A} = \mathcal{M} = \mathbb{C}[x,y]/\langle x^n-1, y^n-1\rangle \;; \quad b = \{x^i y^j\}_{0\le i,j\le n-1}$$

in general, $\Phi : \hat{s} \mapsto \sum_{k=0}^{n^2-1} s_k p_k(x,y) = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} s_{nk+l}\, x^k y^l$.

But in this case we consider $\hat{s}$ an $\underline{n\times n \text{ array}}$ $[s_{k,l}]_{0\le k,l\le n-1}$ (see example above); and space of signals is $\mathbb{C}^{n\times n}$ instead of $\mathbb{C}^{n^2}$.

Then $\quad \Phi : \hat{s} \mapsto \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} s_{k,l}\, x^k y^l$.

Filtering

Multiplication $\quad h(x,y)\cdot s(x,y) \mod \langle p(x,y), q(x,y)\rangle$

! This is well-defined only if I has a $\underline{\text{Groebner basis}}$. However, the above assumption ensures that $p(x,y), q(x,y)$ is a Groebner basis.

Example: $\quad h(x,y)\cdot s(x,y) \mod \langle x^n-1, y^n-1\rangle$

This is a 2D circular convolution.

Filtering in coordinates

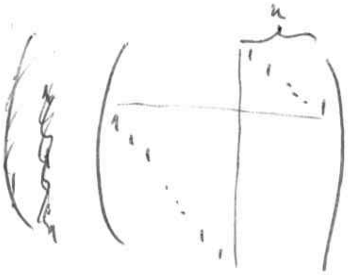
$\mathcal{M}$-basis b affords a representation φ of $\mathcal{A}$:

$$\varphi : \mathcal{A} \longrightarrow \mathbb{C}^{n^2 \times n^2}$$
$$h \longmapsto \varphi(h)$$

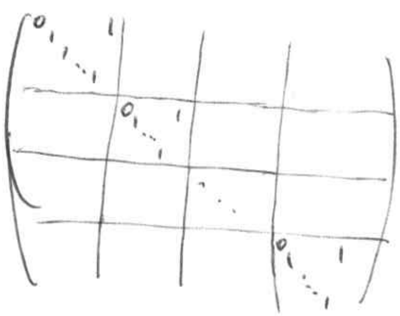
Then $h \cdot s \mod \langle \dots \rangle \iff \varphi(h) \cdot \hat{s}$

Example: $b = \{x^0y^0, x^0y^1, \dots, x^1y^0, x^1y^1, \dots, x^{n-1}y^{n-1}\} = \{p_{in+j}(x,y)\}$, $p_{in+j} = x^iy^j$

$$x \cdot p_{in+j} = x \cdot x^iy^j = x^{i+1}y^j = p_{(i+1)n+j} = p_{(in+j)+n} \Rightarrow$$

$\Rightarrow \varphi(x) =$  $= C_n \otimes I_n$, $C_n = \begin{pmatrix} 0 & & 1 \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ - circular shift

$$y \cdot p_{in+j} = y \cdot x^iy^j = x^iy^{j+1} = p_{in+j+1} \Rightarrow$$

$\Rightarrow \varphi(y) =$  $= I_n \otimes C_n$

Then $\varphi(h) = h(\varphi(x), \varphi(y)) = \sum h_{k,l} (C_n \otimes I_n)^k (I_n \otimes C_n)^l = \sum h_{k,l} (C_n^k \otimes I_n)(I_n \otimes C_n^l) =$

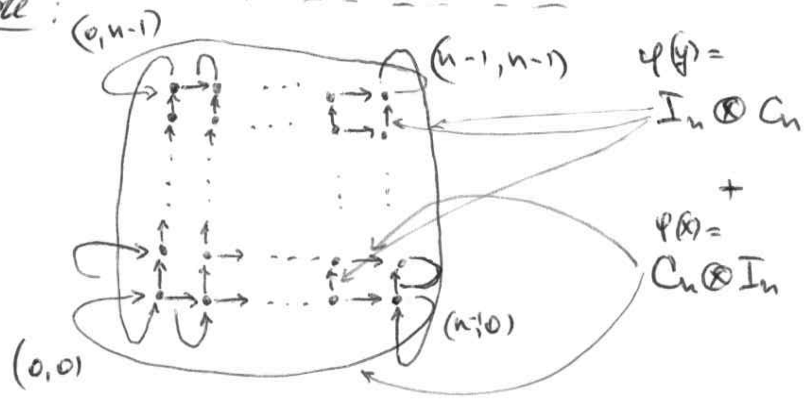
$$= \sum h_{k,l}\, C_n^k \otimes C_n^l .$$

(Recall that $(A \otimes B)(C \otimes D) = AC \otimes BD$).

Visualization

We define <u>visualization</u> of a signal model as a graph with adjacency matrix $\varphi(x)+\varphi(y)$.

Example: 2D time model



This is a torus (a "doughnut")

Spectrum and Fourier Transform

Assume that $p(x,y) = q(x,y) = 0$ has n^2 solutions $\alpha = \{(\alpha_k, \beta_k)\}_{0 \le k \le n^2-1}$

CRT:

$$\Delta : \mathcal{M} \longrightarrow \bigoplus_{k=0}^{n^2-1} \mathbb{C}[x,y] / \langle x-\alpha_k , y-\beta_k \rangle$$

$$s(x,y) \longmapsto \left(s(\alpha_0,\beta_0), \ldots, s(\alpha_{n^2-1}, \beta_{n^2-1}) \right)$$

Pure frequency: $f_k = \Delta^{-1}(e_k)$, $0 \le k \le n^2-1$

<u>In coordinates:</u>

Basis in $\mathcal{M}$: b (fixed by the signal model)
Basis in each spectral component: $\{1\}$

$$\mathcal{F} = \mathcal{P}_{b,\alpha} = [\, p_\ell(\alpha_k, \beta_k) \,]_{0 \le k,\ell \le n^2-1} \in \mathbb{C}^{n^2 \times n^2}$$

if bases other than $\{1\}$ $\Rightarrow$ $\mathcal{F}' = D \cdot \mathcal{F}$.

Example: $x^n - 1 = y^n - 1 = 0$ has n^2 solutions $\alpha = \{(\omega_n^k, \omega_n^\ell)\}_{0 \le k,\ell \le n-1}$

$$\alpha_{kn+\ell} = (\omega_n^k, \omega_n^\ell)$$

Basis $b = \{p_\ell\}_{0 \le \ell \le n^2-1}$, $p_{in+j} = x^i y^j$

Thus,

$$\mathcal{F} = \mathcal{P}_{b,\alpha} = [p_t(\alpha_m)]_{0 \le t,m \le n^2-1} = [p_{in+j}(\omega_n^k, \omega_n^\ell)]_{0 \le i,j,k,\ell \le n-1} =$$

(j, ℓ running first)

$$= [\omega_n^{ik+\ell j}]_{0 \le i,j,k,\ell \le n-1} =$$

$$= \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \\ 1 & \omega_n^2 & \omega_n^4 & & \\ \vdots & \vdots & & & \vdots \\ 1 & \omega_n^{n-1} & & & \\ 1 & & & & \\ 1 & \omega_n^{(n-1)} & \omega_n^{2(n-1)} & \cdots & \omega_n^{2(n-1)(n-1)} \end{pmatrix} = \begin{pmatrix} \omega_n^0 \cdot DFT_n & \omega_n^0 \cdot DFT_n & \cdots & \\ \omega_n^0 \cdot DFT_n & \omega_n^1 \cdot DFT_n & \cdots & \\ \vdots & \vdots & & \cdots \\ \omega_n^0 DFT_n & \omega_n^{n-1} \cdot DFT_n & \cdots & \omega_n^{(n-1)^2} DFT_n \end{pmatrix} =$$

$$= DFT_n \otimes DFT_n .$$

Frequency response

Each spectral component $M_k = \mathbb{C}[x,y] / \langle x - \alpha_k, y - \beta_k \rangle$ with basis $\{1\}$ affords an irreducible representation φ_k of $\mathcal{A}$:

$$\varphi_k : \mathcal{A} \to \mathbb{C}^{1 \times 1}$$

$$h(x,y) \mapsto h(\alpha_k, \beta_k) \quad (\equiv h(x,y) \cdot 1 \mod \langle x - \alpha_k, y - \beta_k \rangle)$$

$(h(\alpha_k, \beta_k))_{0 \le k \le n^2-1}$ is the <u>frequency response</u> of $h(x,y)$.

it implies

$$\mathcal{F} \varphi(h) \mathcal{F}^{-1} = \operatorname*{diag}_{0 \le k \le n^2-1} (h(\alpha_k, \beta_k))$$

Commutative diagram:

$$\begin{array}{ccc}
\mathbb{C}[x,y]/\langle p,q\rangle & \xrightarrow{\mathcal{F}} & \bigoplus \mathbb{C}[x,y]/\langle x-\alpha_k, y-\beta_k\rangle \\
\downarrow \varphi(h) & & \downarrow \operatorname{diag}_k \left(h(\alpha_k,\beta_k)\right) \\
\mathbb{C}[x,y]/\langle p,q\rangle & \xrightarrow{\mathcal{F}} & \bigoplus \mathbb{C}[x,y]/\langle x-\alpha_k, y-\beta_k\rangle
\end{array}$$

Example:

$$\mathcal{F} = DFT_n \otimes DFT_n$$

$\mathcal{F}$ diagonalizes $\varphi(x)$: $\mathcal{F}\varphi(x)\mathcal{F}^{-1} = (DFT_n \otimes DFT_n)(C_n \otimes I_n)(DFT_n \otimes DFT_n)^{-1} =$

$$= (DFT_n \otimes DFT_n)(C_n \otimes I_n)(DFT_n^{-1} \otimes DFT_n^{-1}) =$$

$$= DFT_n C_n DFT_n^{-1} \otimes DFT_n I_n DFT_n^{-1} = \operatorname*{diag}_{0 \le k \le n-1}(\omega_n^k) \otimes I_n .$$

$\mathcal{F}$ diagonalizes $\varphi(y)$:

$$\mathcal{F}\varphi(y)\mathcal{F}^{-1} = (DFT_n \otimes DFT_n)(I_n \otimes C_n)(DFT_n^{-1} \otimes DFT_n^{-1}) =$$

$$= DFT_n I_n DFT_n \otimes DFT_n C_n DFT_n^{-1} =$$

$$= I_n \otimes \operatorname*{diag}_{0 \le k \le n-1}(\omega_n^k) .$$

Hence, $\mathcal{F}$ diagonalizes $\varphi(x) + \varphi(y)$.

__Note__ : $\mathcal{F}$ diagonalizes $\varphi(x) + \varphi(y)$ is a "weaker" statement than
$\mathcal{F}$ diagonalizes $\varphi(x)$ __and__ $\varphi(y)$.

Separable vs non-separable SP

In our example: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x,y]/\langle p(x), p(y) \rangle$, $p(x) = x^n - 1$

This is a separable case $\Rightarrow \mathcal{F}$ is a tensor product of corresponding 1D $\mathcal{F}$.

In general, $\mathcal{A} = \mathcal{M} = \mathbb{C}[x,y]/\langle p(x,y), q(x,y) \rangle$ – non-separable.