ASYMPTOTIC PROPERTIES OF TRIGONOMETRIC TRANSFORMS VIA ALGEBRAIC THEORETICAL SIGNAL MODELS

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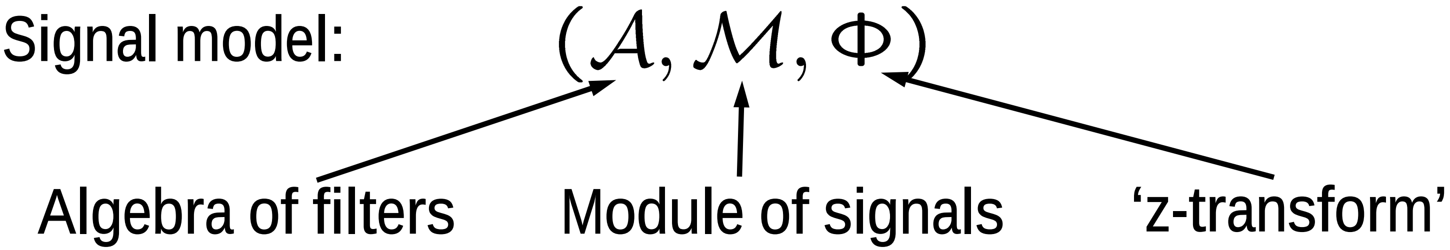
Outline

- Algebraic Signal Processing Theory
- From finite to infinite models: issue of convergence
- Termwise to normwise convergence
- Convergence using a direct approach
- Conclusion

Algebraic signal models

- Signal model: $(\mathcal{A}, \mathcal{M}, \Phi)$

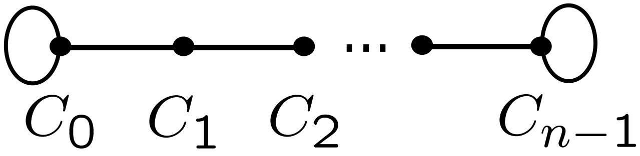


Algebra of filters

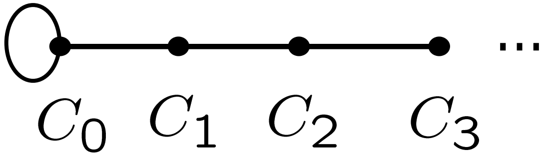
Module of signals

‘z-transform’

- Space models: visualization



Finite: left and right bc

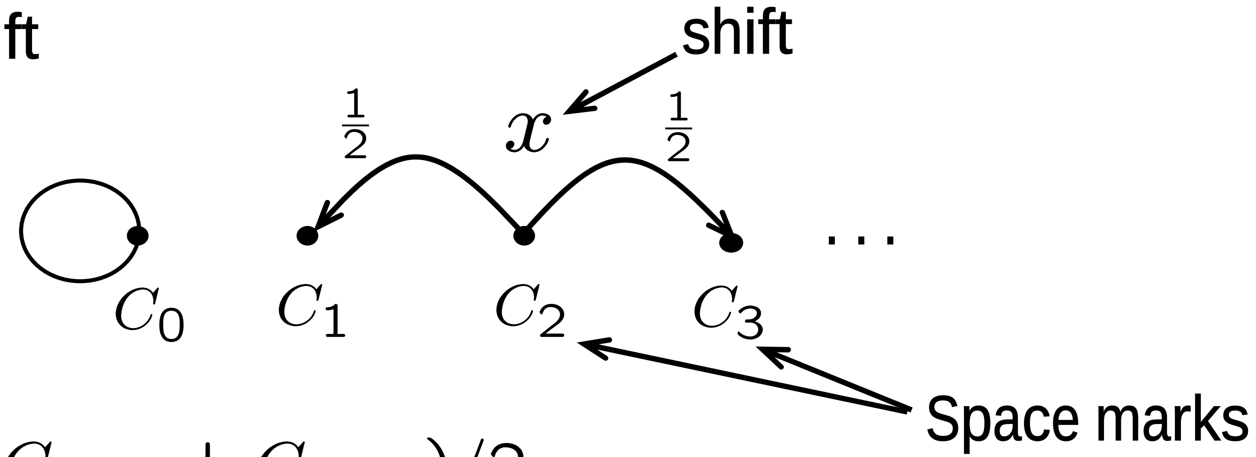


Infinite: only left bc

- As n $\to \infty$, does the finite model converge to infinite model? What does convergence mean?

Definition of a signal model

- Space marks, shift



$$x \cdot C_n = (C_{n-1} + C_{n+1})/2$$

- Shift matrix: captures operation of shift on basis

e.g., for DTTs, $\phi(x) = \frac{1}{2} \cdot \begin{bmatrix} \beta_1 & 1 & & & & \\ \beta_2 & 0 & 1 & & & \\ & 1 & 0 & . & & \\ & & 1 & . & 1 & \\ & & & . & 0 & \beta_3 \\ & & & & 1 & \beta_4 \end{bmatrix}$

Filter matrices

- If $h^{(n)} = \sum_{0 \le k < n} h_k x^k \qquad \phi_n(h^{(n)}) = \sum_{0 \le k < n} h_k \phi(x^k)$

- Key property: Fourier transform diagonalizes filter matrices

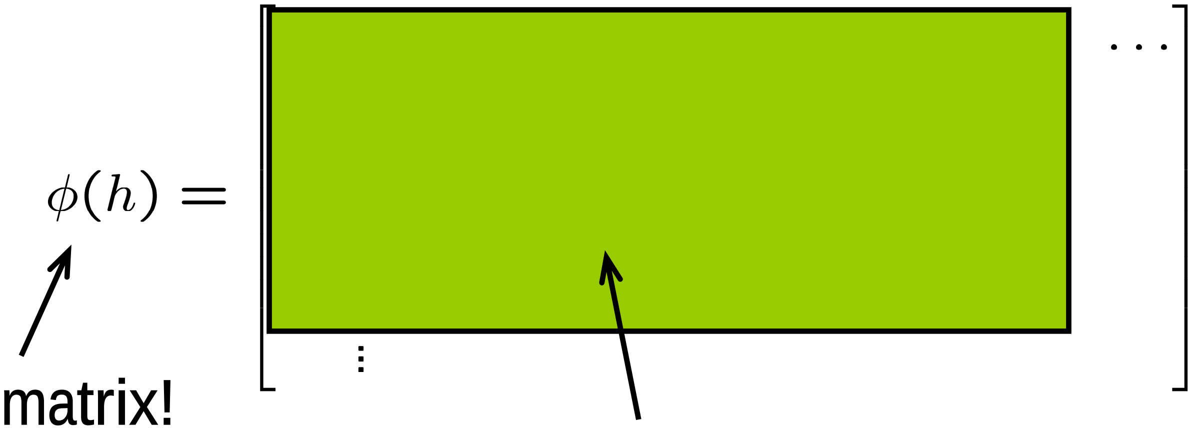
$$\mathcal{F}_n \cdot \phi_n(h^{(n)}) \cdot \mathcal{F}_n^{-1} = D$$

- Finite transform applied to infinite filter matrix
  Key question:

$$\mathcal{F}_n \cdot \phi(h) \cdot \mathcal{F}_n^{-1} \approx D \qquad \text{for large n?}$$

Finite sections of an infinite matrix

$$\mathcal{F}_n \cdot \phi(h) \cdot \mathcal{F}_n^{-1}$$



$$\phi(h) = \begin{bmatrix} \boxed{\tilde{\phi}_n(h)} & \cdots \\ \vdots & \end{bmatrix}$$

Infinite matrix!
But $\mathcal{F}_n$ finite

Finite section (nxn) $\tilde{\phi}_n(h)$

- Ideally, if transforms are asymptotically equivalent

$$\lim_{n \to \infty} \mathcal{F}_n \cdot \phi_n(h^{(n)}) \cdot \mathcal{F}_n^{-1} = \lim_{n \to \infty} \mathcal{F}_n \cdot \tilde{\phi}_n(h) \cdot \mathcal{F}_n^{-1}$$

Types of convergence

$$\lim_{n\to\infty} \mathcal{F}_n \cdot \phi_n(h^{(n)}) \cdot \mathcal{F}_n^{-1} = \lim_{n\to\infty} \mathcal{F}_n \cdot \tilde{\phi}_n(h) \cdot \mathcal{F}_n^{-1}$$

- Is termwise convergence enough?
- Performance degradation is function of weak norm
- For matrix $\mathbf{A} = (a_{ij})$, weak norm is defined as

$$|\mathbf{A}|^2 = \frac{1}{N} \sum_{i,j=0}^{N} |a_{ij}|^2 = \frac{1}{N} \|\mathbf{A}\|_F^2$$

- Example: $a_{ij} = 1/\sqrt[4]{N}$
  - Termwise: $\mathbf{A} \to \mathbf{0}$
  - Normwise: $|\mathbf{A}|_N \to \infty$ $(|\mathbf{A}|_N^2 = \sqrt{N})$

Normwise Convergence

- Modify criterion: use weak norm

$$\lim_{n\to\infty} \underbrace{\boxed{|\mathcal{F}_n}}_{\text{Fourier Transform}} \cdot \underbrace{\boxed{(\phi_n(h^{(n)}) - \tilde{\phi}_n(h)}}_{\text{Difference in filter matrices}}) \cdot \mathcal{F}_n^{-1}| = 0$$

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Termwise to normwise convergence

$$\alpha_{ij}^N \doteq [\mathbf{C}_N^H \mathbf{D}_N \mathbf{C}_N]_{kj} \qquad \alpha_{ij} \doteq \int_X P_k(x) P_j(x) f(x) d\mu(x)$$

- Given:
  - Termwise convergence $\alpha_{ij}^N \underset{N\to\infty}{\longrightarrow} \alpha_{ij}$
  - Decay of “Fourier coefficients” $\alpha_{ij} \longrightarrow 0, \quad i \longrightarrow \infty$ or $j \longrightarrow \infty$
- To show: $$\lim_{N\to\infty} \frac{1}{N} \sum_{i,j=0}^{N-1} |\alpha_{ij}^N - \alpha_{ij}|^2 = 0$$
- Equivalently: $$\lim_{N\to\infty} \frac{1}{N} \sum_{i,j=0}^{N-1} \alpha_{ij} \; |\alpha_{ij}^N - \alpha_{ij}| = 0$$

Key intuition for proof

- α_{ij} and $(\alpha_{ij}^{N_0} - \alpha_{ij})$ can be bounded separately, in different regions

Choose N_0 s.t.

$|\alpha_{ij}^{N} - \alpha_{ij}| < \epsilon$

ite matrix

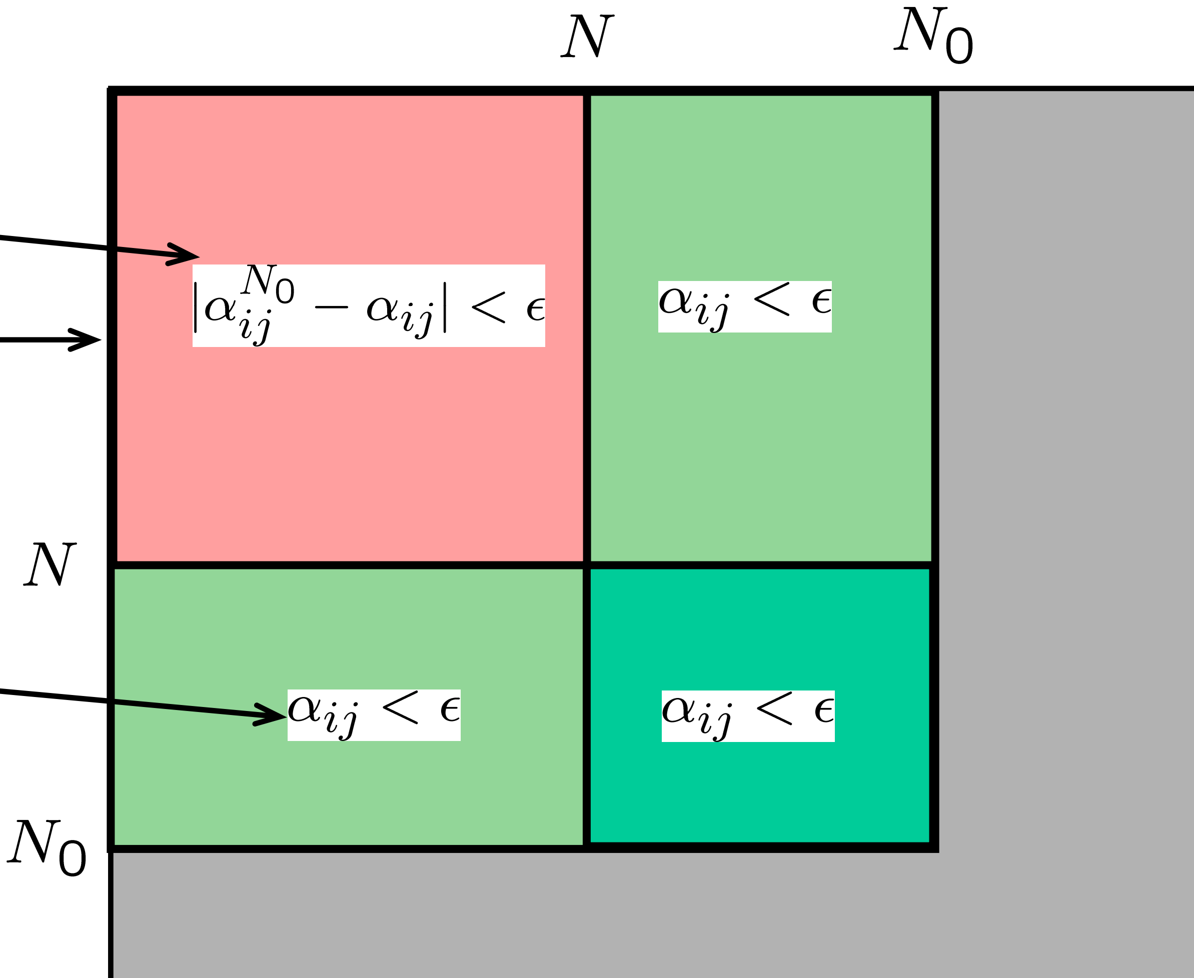
$\alpha_{ij}^{N} \xrightarrow[N\to\infty]{} \alpha_{ij}$ th entries

α_{ij} $|\alpha_{ij}^{N} - \alpha_{ij}|$

Choose N s.t.

$\alpha_{ij} < \epsilon$

$\alpha_{ij} \to 0$



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New Question

Q: Can we approximate infinite signal models by related, finite signal models?

A: In what sense?

Normwise Convergence

- Modify criterion: use weak norm

$$\lim_{n\to\infty} |\mathcal{F}_n \cdot (\phi_n(h^{(n)}) - \tilde{\phi}_n(h)) \cdot \mathcal{F}_n^{-1}| = 0$$

Fourier Transform

Difference in filter matrices

- "smoothness": $\sum_{k \geq 0} (k+1) h_k^2 < \infty$
- stronger than L_2
- neither stronger, nor weaker than L_1

Theorem Let $(\mathcal{A}, \mathcal{M}, \Phi)$ be the space sig model corresponding to the infinite *C*-transform, and $h \in \mathcal{A}^s$. For $n < \infty$, let $(\mathcal{A}_n, \mathcal{M}_n, \Phi_n)$ be the signal model corresponding to related finite *C*-transform, and $h^{(n)} \in \mathcal{A}_n$ be the "*n*-prefix" of h.

Then,

$$|\tilde{\phi}_n(h) - \phi_n(h^{(n)})| \to 0.$$

Filter matrix (infinite)

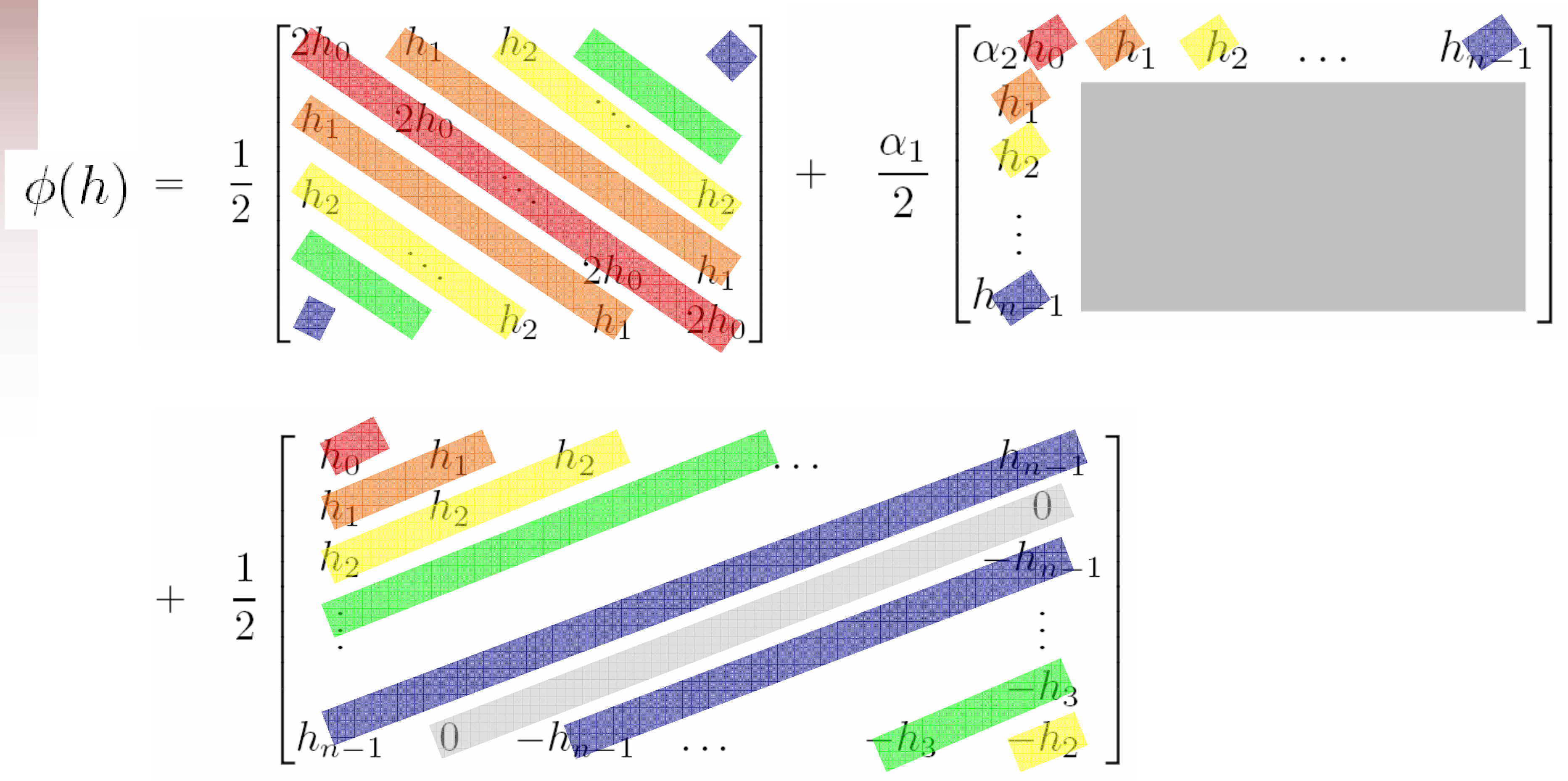
example: *T*-transform

$$\phi(h) = \frac{1}{2}\begin{bmatrix} 2h_0 & h_1 & h_2 & & \\ h_1 & 2h_0 & h_1 & \ddots & \\ h_2 & h_1 & 2h_0 & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & & & \end{bmatrix} + \frac{1}{2}\begin{bmatrix} h_0 & h_1 & h_2 & h_3 & \cdots \\ h_1 & h_2 & h_3 & \cdot & \\ h_2 & h_3 & \cdot & & \\ h_3 & \cdot & & & \\ \vdots & & & & \end{bmatrix}$$

$$+ \frac{\alpha_1}{2}\begin{bmatrix} \alpha_2 h_0 & h_1 & h_2 & h_3 & \cdots \\ h_1 & & & & \\ h_2 & & & & \\ h_3 & & & & \\ \vdots & & & & \end{bmatrix}$$

Filter matrix (finite)

example: DCT-3

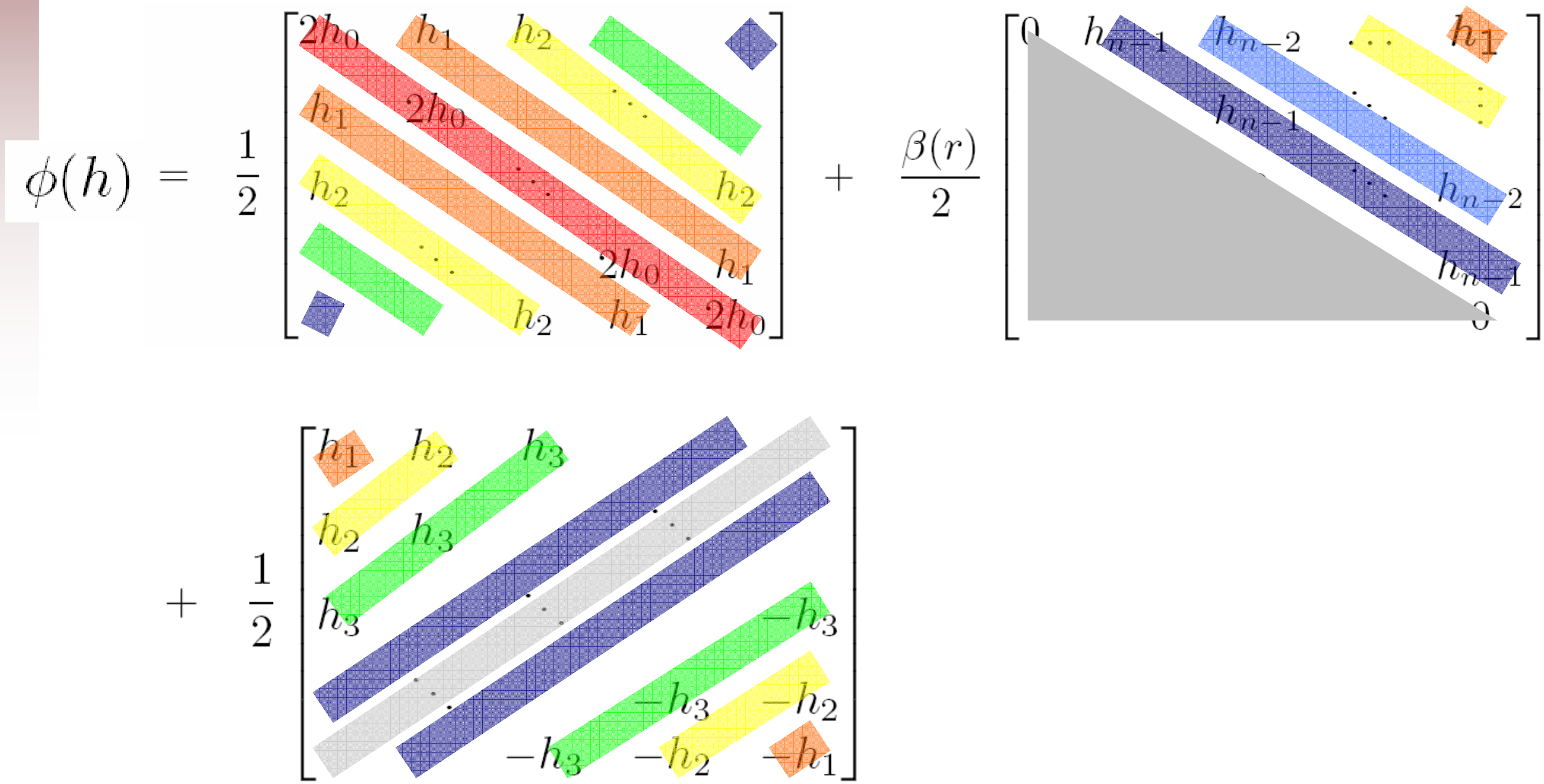


$$\phi(h) = \frac{1}{2}\begin{bmatrix} 2h_0 & h_1 & h_2 & \ddots & \\ h_1 & 2h_0 & & & \\ h_2 & & \ddots & & h_2 \\ & \ddots & & 2h_0 & h_1 \\ & & h_2 & h_1 & 2h_0 \end{bmatrix} + \frac{\alpha_1}{2}\begin{bmatrix} \alpha_2 h_0 & h_1 & h_2 & \dots & h_{n-1} \\ h_1 & & & & \\ h_2 & & & & \\ \vdots & & & & \\ h_{n-1} & & & & \end{bmatrix}$$

$$+ \frac{1}{2}\begin{bmatrix} h_0 & h_1 & h_2 & \dots & & h_{n-1} \\ h_1 & h_2 & & & & 0 \\ h_2 & & & & & -h_{n-1} \\ \vdots & & & & & \vdots \\ & & & & & -h_3 \\ h_{n-1} & 0 & -h_{n-1} & \dots & -h_3 & -h_2 \end{bmatrix}$$

Filter matrix (skew)

example: DCT-4(*r*)



$$\phi(h) = \frac{1}{2}\begin{bmatrix} 2h_0 & h_1 & h_2 & & \\ h_1 & 2h_0 & & \ddots & \\ h_2 & & \ddots & & h_2 \\ & \ddots & & 2h_0 & h_1 \\ & & h_2 & h_1 & 2h_0 \end{bmatrix} + \frac{\beta(r)}{2}\begin{bmatrix} 0 & h_{n-1} & h_{n-2} & \cdots & h_1 \\ & \ddots & h_{n-1} & \ddots & \vdots \\ & & \ddots & \ddots & h_{n-2} \\ & & & \ddots & h_{n-1} \\ & & & & 0 \end{bmatrix}$$

$$+ \frac{1}{2}\begin{bmatrix} h_1 & h_2 & h_3 & & & \\ h_2 & h_3 & & & \iddots & \\ h_3 & & & \iddots & & \\ & & \iddots & & & -h_3 \\ & & & & -h_3 & -h_2 \\ & & & -h_3 & -h_2 & -h_1 \end{bmatrix}$$

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Conclusions

- We investigated asymptotic decorrelation properties of DTTs / skew DTTs
  - fixed bug in old proof
- Adopted the Algebraic Signal Processing Theoretical point of view
  - identify signal model for transform:
    - filter algebra, signal module, “z-transform”, shift, b.c., etc.
- New perspective:

$$(\text{“nice” finite models})_{(n)} \xrightarrow[n \to \infty]{} \text{infinite models}$$

- How about models for other types of signal extension?
  - k-monomial, sparse polynomial

Thank you!