18-799F Algebraic Signal Processing Theory
Spring 2007
Assignment 4
Due Date: Feb. 21th 2:30pm (at the beginning of class)

1. *(40 pts)* Let $p(x) = \sum_{i=0}^{n} \beta_i x^i$, $\beta_n \neq 0$, be a polynomial of degree n.
   (a) Prove that the mapping
   $$\begin{array}{rrcl} \phi: & \mathbb{C}[x]/p(x) & \to & \mathbb{C}[x]/p(x) \\ & q(x) & \mapsto & xq(x) \pmod{p(x)} \end{array}$$
   is linear.
   (b) Represent this mapping as a matrix B_ϕ with respect to the canonical basis $\{1, x, x^2, \dots, x^{n-1}\}$.
   (c) Determine B_ϕ in the special case $p(x) = x^n - 1$. Do you know this matrix?

2. *(30 pts)* Let $p(x)$ as in question 1. Assume that it has pairwise distinct zeros; i.e. $p(x) = \beta_n \cdot \prod_{i=0}^{n-1}(x - \alpha_i)$, such that $i \neq j \Rightarrow \alpha_i \neq \alpha_j$. In the class you learned that the mapping
   $$\begin{array}{rrcl} \phi: & \mathbb{C}[x]/p(x) & \to & \bigoplus_{k=0}^{n-1} \mathbb{C}[x]/(x - \alpha_k) \\ & q(x) & \mapsto & (q(\alpha_0), \dots, q(\alpha_{n-1})) \end{array}$$
   is an isomorphism of algebras.
   (a) Represent this mapping as a matrix B_ϕ with respect to the canonical bases $b = \{1, x, x^2, \dots, x^{n-1}\}$ in $\mathbb{C}[x]/p(x)$ and $c = \{e_0, \dots, e_{n-1}\}$ in $\bigoplus_{k=0}^{n-1} \mathbb{C}[x]/(x - \alpha_k)$. Here, e_k is a canonical vector that has 0 in all positions except the k-th and 1 in the k-th position.
   (b) Determine B_ϕ in the special case $p(x) = x^n - 1$. Do you know the mapping $v \mapsto B_\phi v,\ v \in \mathbb{C}^n$?

3. *(30 pts)* Let $i = \sqrt{-1}$.
   (a) Determine the matrix representation of the mapping $\mathbb{C}[x]/(x^4 - 1) \to \bigoplus_{k=0}^{3} \mathbb{C}[x]/(x - i^k)$ with respect to canonical bases as in the previous problem. Explain your answer.
   (b) Determine the matrix representation of the mapping $\mathbb{C}[x]/(x^4 - 1) \to \mathbb{C}[x]/(x^2 - 1) \oplus \mathbb{C}[x]/(x^2 + 1)$, where in each summand on the right side we choose $\{1, x\}$ as a basis.

4. *(20 pts)* **Extra-credit** Let's continue the decomposition in 3(b).
   (a) Completely decompose both summands
   $$\mathbb{C}[x]/(x^2 - 1) \oplus \mathbb{C}[x]/(x^2 + 1) \to (\mathbb{C}[x]/(x - 1) \oplus \mathbb{C}[x]/(x + 1)) \oplus (\mathbb{C}[x]/(x - i) \oplus \mathbb{C}[x]/(x + i)).$$
   and determine the matrix representation of this mapping.
   (b) Use the results from parts (b) and (c) to obtain a factorization of the matrix in part (a) into a product of (sparse) matrices. Write this factorization as a (correct) matrix equation. Note: make sure you follow the order of the summands in the direct sum in 3(a).